2004 AMC 12A Solutions

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1. Alicia earns $\$20$ per hour, of which 1.45% is deducted to pay local taxes. How many cents per hour of Alicia's wages are used to pay local taxes?

A 0.0029

B 0.029

C 0.29

D 2.9

E 29

Solution:

Since $\$20$ is 2000 cents, the tax is

$$0.0145 \times 2000 = 29$$

cents per hour.

Thus, the correct answer is **E**.

2. On the AMC 12, each correct answer is worth 6 points, each incorrect answer is worth 0 points, and each problem left unanswered is worth 2.5 points. If Charlyn leaves 8 of the 25 problems unanswered, how many of the remaining problems must she answer correctly in order to score at least 100?

- A 11
- B 13
- C 14
- D 16
- E 17

Solution:

The 8 unanswered problems are worth $2.5 \times 8 = 20$ points, so Charlyn needs at least $100 - 20 = 80$ more points from correct answers.

Each correct answer is worth 6 points, and the smallest multiple of 6 that is at least 80 is $84 = 6 \times 14$. So she needs at least 14 correct answers.

Thus, the correct answer is **C**.

3. For how many ordered pairs of positive integers (x, y) is $x + 2y = 100$?

- A 33
- B 49
- C 50
- D 99
- E 100

Solution:

Writing $x = 100 - 2y$, the value of x is a positive integer precisely when y is a positive integer with $1 \leq y \leq 49$.

This gives 49 valid ordered pairs.

Thus, the correct answer is **B**.

4. Bertha has 6 daughters and no sons. Some of her daughters have 6 daughters, and the rest have none. Bertha has a total of 30 daughters and granddaughters, and no great-granddaughters. How many of Bertha's daughters and granddaughters have no daughters?

- A 22
- B 23
- C 24
- D 25
- E 26

Solution:

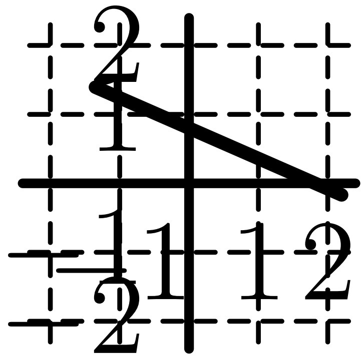
Bertha has $30 - 6 = 24$ granddaughters, none of whom have daughters.

These granddaughters are the children of $24/6 = 4$ of Bertha's daughters, so exactly 4 women have daughters.

Therefore the number of women with no daughters is $30 - 4 = 26.$

Thus, the correct answer is **E**.

5. The graph of a line $y = mx + b$ is shown. Which of the following is true?



A $mb < -1$

B $-1 < mb < 0$

C $mb = 0$

D $0 < mb < 1$

E $mb > 1$

Solution:

The y-intercept of the line is between 0 and 1, so $0 < b < 1.$

The slope is negative and shallow, between -1 and 0, so $-1 < m < 0.$

The product mb is therefore negative with absolute value less than 1, giving $-1 < mb < 0.$

Thus, the correct answer is **B**.

6. Let $U = 2 \cdot 2004^{2005}$, $V = 2004^{2005}$, $W = 2003 \cdot 2004^{2004}$, $X = 2 \cdot 2004^{2004}$, $Y = 2004^{2004}$ and $Z = 2004^{2003}$. Which of the following is largest?

A $U - V$

B $V - W$

C $W - X$

D $X - Y$

E $Y - Z$

Solution:

Compute each difference by factoring:

$U - V = 2004^{2005}$, $V - W = 2004^{2004}$, $W - X = 2001 \cdot 2004^{2004}$, $X - Y = 2004^{2004}$, and $Y - Z = 2003 \cdot 2004^{2003}$.

Since $2004^{2005} = 2004 \cdot 2004^{2004}$ exceeds each of the others, none of which reaches 2004^{2005}, the difference $U - V$ is the largest.

Thus, the correct answer is **A**.

7. A game is played with tokens according to the following rule. In each round, the player with the most tokens gives one token to each of the other players and also places one token into a discard pile. The game ends when some player runs out of tokens. Players $A, B,$ and C start with $15, 14,$ and 13 tokens, respectively. How many rounds will there be in the game?

A 36

B 37

C 38

D 39

E 40

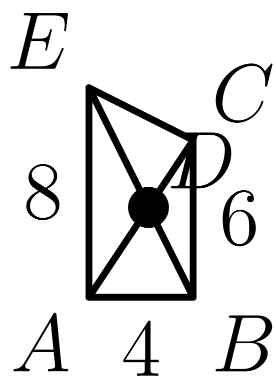
Solution:

After three rounds the players $A, B,$ and C have $14, 13,$ and 12 tokens, respectively. Every subsequent three rounds reduces each player's supply by one token.

After 36 rounds they have $3, 2,$ and 1 tokens. In the 37th round player $A,$ who has the most, gives away all three of their tokens and runs out, ending the game.

Thus, the correct answer is **B**.

8. In the figure, $\angle EAB$ and $\angle ABC$ are right angles, $AB = 4, BC = 6, AE = 8,$ and $\overline{AC}$ and $\overline{BE}$ intersect at $D.$ What is the difference between the areas of $\triangle ADE$ and $\triangle BDC$?



A 2

B 4

C 5

D 8

E 9

Solution:

Let $x, y,$ and z be the areas of $\triangle ADE, \triangle BDC,$ and $\triangle ABD,$ respectively.

Then $\triangle ABE$ has area $\frac{1}{2} \cdot 4 \cdot 8 = 16 = x + z,$ and $\triangle ABC$ has area $\frac{1}{2} \cdot 4 \cdot 6 = 12 = y + z.$

The requested difference is

$$x - y = (x + z) - (y + z) = 16 - 12 = 4.$$

Thus, the correct answer is **B**.

9. A company sells peanut butter in cylindrical jars. Marketing research suggests that using wider jars will increase sales. If the diameter of the jars is increased by 25% without altering the volume, by what percent must the height be decreased?

- A 10
- B 25
- C 36
- D 50
- E 60

Solution:

Multiplying the diameter by $\frac{5}{4}$ multiplies the base area by $\left(\frac{5}{4}\right)^2 = \frac{25}{16}$.

To keep the volume fixed, the height must be multiplied by $\frac{16}{25} = 0.64$. That is a decrease of $1 - 0.64 = 0.36$, or 36%.

Thus, the correct answer is **C**.

10. The sum of 49 consecutive integers is 7^5. What is their median?

A 7

B 7^2

C 7^3

D 7^4

E 7^5

Solution:

The sum of a set of consecutive integers equals the number of terms times their mean, and for consecutive integers the mean equals the median.

So the median is

$$\frac{7^5}{49} = \frac{7^5}{7^2} = 7^3 = 343.$$

Thus, the correct answer is **C**.

11. The average value of all the pennies, nickels, dimes, and quarters in Paula's purse is 20 cents. If she had one more quarter, the average value would be 21 cents. How many dimes does she have in her purse?

- A 0
- B 1
- C 2
- D 3
- E 4

Solution:

If Paula has n coins, their total value is $20n$ cents. Adding a quarter gives $n+1$ coins worth $20n+25$ cents, which must also equal $21(n+1)$ cents.

So $20n+25=21(n+1)$, giving $n=4$.

Four coins totalling 80 cents must be three quarters and one nickel, so the number of dimes is 0.

Thus, the correct answer is **A**.

12. Let $A = (0, 9)$ and $B = (0, 12)$. Points A' and B' are on the line $y = x$, and $\overline{AA'}$ and $\overline{BB'}$ intersect at $C = (2, 8)$. What is the length of $\overline{A'B'}$?

- A 2
- B $2\sqrt{2}$
- C 3
- D $2 + \sqrt{2}$
- E $3\sqrt{2}$

Solution:

Line AC passes through $(0, 9)$ with slope $\frac{8-9}{2-0} = -\frac{1}{2}$, so its equation is $y = -\frac{1}{2}x + 9$. Setting $y = x$ gives $A' = (6, 6)$.

Line BC passes through $(0, 12)$ with slope -2, so $y = -2x + 12$. Setting $y = x$ gives $B' = (4, 4)$.

Then

$$A'B' = \sqrt{(6-4)^2 + (6-4)^2} = 2\sqrt{2}.$$

Thus, the correct answer is **B**.

13. Let S be the set of points (a, b) in the coordinate plane, where each of a and b may be $-1, 0,$ or 1. How many distinct lines pass through at least two members of S?

A 8

B 20

C 24

D 27

E 36

Solution:

There are $\binom{9}{2} = 36$ pairs of points, and each pair determines a line.

However, there are three horizontal, three vertical, and two diagonal lines that each pass through three collinear points of S. Each such line is counted 3 times, an overcount of 2 per line.

With 8 such lines, the number of distinct lines is $36 - 2 \cdot 8 = 20$.

Thus, the correct answer is **B**.

14. A sequence of three real numbers forms an arithmetic progression with a first term of 9. If 2 is added to the second term and 20 is added to the third term, the three resulting numbers form a geometric progression. What is the smallest possible value for the third term of the geometric progression?

A 1

B 4

C 36

D 49

E 81

Solution:

The arithmetic progression is $9, 9+d, 9+2d$. After the additions, the geometric progression is $9, 11+d, 29+2d$.

The geometric condition gives $(11+d)^2 = 9(29+2d)$, which simplifies to $d^2 + 4d - 140 = 0$, so $d = 10$ or $d = -14$.

The corresponding third terms $29 + 2d$ are 49 and 1, so the smallest possible value is 1.

Thus, the correct answer is **A**.

15. Brenda and Sally run in opposite directions on a circular track, starting at diametrically opposite points. They first meet after Brenda has run 100 meters. They next meet after Sally has run 150 meters past their first meeting point. Each girl runs at a constant speed. What is the length of the track in meters?

- A 250
- B 300
- C 350
- D 400
- E 500

Solution:

Starting at opposite ends, when they first meet they have together run half the track. Between the first and second meetings, they together run a full track length.

Since Brenda runs at a constant speed and covered 100 meters before the first meeting, she covers $2 \cdot 100 = 200$ meters between the two meetings.

Adding Sally's 150 meters over that same interval gives a track length of $200 + 150 = 350$ meters.

Thus, the correct answer is **C**.

16. The set of all real numbers x for which

$$\log_{2004}(\log_{2003}(\log_{2002}(\log_{2001} x)))$$

is defined is $\{x \mid x > c\}$. What is the value of c?

- A 0
- B 2001^{2002}
- C 2002^{2003}
- D 2003^{2004}
- E $2001^{2002^{2003}}$

Solution:

The expression is defined if and only if $\log_{2003}(\log_{2002}(\log_{2001} x)) > 0$, that is, $\log_{2002}(\log_{2001} x) > 1$.

This holds if and only if $\log_{2001} x > 2002$, which is equivalent to $x > 2001^{2002}$.

Therefore $c = 2001^{2002}$.

Thus, the correct answer is **B**.

17. Let f be a function with the following properties:

(i) $f(1) = 1$, and

(ii) $f(2n) = n \cdot f(n)$ for any positive integer n.

What is the value of $f(2^{100})$?

A 1

B 2^{99}

C 2^{100}

D 2^{4950}

E 2^{9999}

Solution:

Applying $f(2n) = n \cdot f(n)$ with $n = 2^k$, we get $f(2^{k+1}) = 2^k \cdot f(2^k)$.

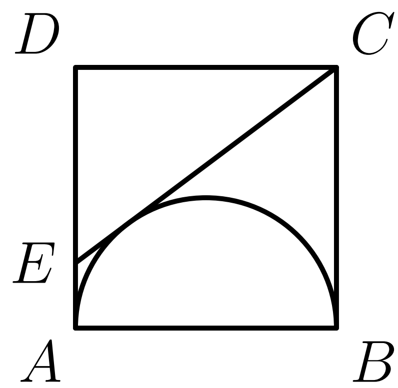
Unwinding from $f(2^1) = f(2) = 1 \cdot f(1) = 2^0$, the exponents accumulate:

$$f(2^n) = 2^{0+1+2+\cdots+(n-1)} = 2^{n(n-1)/2}.$$

Therefore $f(2^{100}) = 2^{100 \cdot 99/2} = 2^{4950}$.

Thus, the correct answer is **D**.

18. Square $ABCD$ has side length $2.$ A semicircle with diameter $\overline{AB}$ is constructed inside the square, and the tangent to the semicircle from C intersects side $\overline{AD}$ at $E.$ What is the length of $\overline{CE}$?



A $\dfrac{2+\sqrt{5}}{2}$

B $\sqrt{5}$

C $\sqrt{6}$

D $\dfrac{5}{2}$

E $5-\sqrt{5}$

Solution:

Let F be the point where CE touches the semicircle. Since CB and CF are both tangents from C, we have $CF = CB = 2.$ Similarly, with $x = AE$, the tangents from E give $EF = EA = x.$

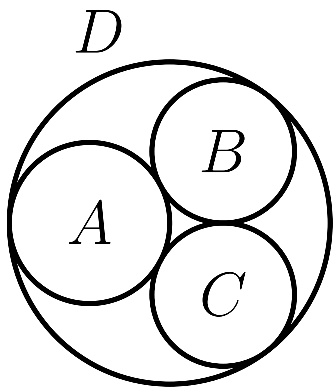
Thus $CE = CF + FE = 2 + x.$ In right triangle CDE, where $CD = 2$ and $DE = 2 - x$,

$$(2-x)^2 + 2^2 = (2+x)^2.$$

Expanding gives $8x = 4$, so $x = \frac{1}{2}$ and $CE = 2 + \frac{1}{2} = \frac{5}{2}.$

Thus, the correct answer is **D**.

19. Circles $A, B,$ and C are externally tangent to each other and internally tangent to circle D. Circles B and C are congruent. Circle A has radius 1 and passes through the center of D. What is the radius of circle B?



A $\frac{2}{3}$

B $\frac{\sqrt{3}}{2}$

C $\frac{7}{8}$

D $\frac{8}{9}$

E $\frac{1+\sqrt{3}}{3}$

Solution:

Circle A has radius 1 and passes through the center of D while being internally tangent to D, so D has radius 2.

Place the center of D at the origin, with A centered at $(-1, 0)$. By symmetry, B has center (x, y) and radius r, with C its mirror image across the horizontal axis, so the two congruent circles touch on that axis and $y = r$.

Internal tangency to D gives $x^2 + y^2 = (2 - r)^2$, and external tangency to A gives $(x + 1)^2 + y^2 = (1 + r)^2$.

Subtracting and using $y = r$ yields $x = \frac{2}{3}$ and $r = \frac{8}{9}$. The radius of circle B is $\frac{8}{9}$.

Thus, the correct answer is **D**.

20. Select numbers a and b between 0 and 1 independently and at random, and let c be their sum. Let $A, B,$ and C be the results when $a, b,$ and $c,$ respectively, are rounded to the nearest integer. What is the probability that $A + B = C$?

A $\frac{1}{4}$

B $\frac{1}{3}$

C $\frac{1}{2}$

D $\frac{2}{3}$

E $\frac{3}{4}$

Solution:

Represent the choices as a point (a, b) in the unit square. Each of a and b rounds to 0 if below $\frac{1}{2}$ and to 1 otherwise, while $c = a + b$ rounds based on $\frac{1}{2}$ and $\frac{3}{2}$.

The equation $A + B = C$ holds in these regions:

if $a + b < \frac{1}{2}$ then $A = B = C = 0$; if exactly one of a, b is at least $\frac{1}{2}$ and $a + b < \frac{3}{2}$ then that variable rounds to $1 = C$; and if $a + b \geq \frac{3}{2}$ then $A = B = 1$ and $C = 2$.

These regions consist of two corner triangles of area $\frac{1}{8}$ each and two central strips, with combined area $\frac{3}{4}$. Since the square has area 1, the probability is $\frac{3}{4}$.

Thus, the correct answer is **E**.

21. If

$$\sum_{n=0}^{\infty} \cos^{2n}\theta = 5,$$

what is the value of $\cos 2\theta$?

A $\frac{1}{5}$

B $\frac{2}{5}$

C $\frac{\sqrt{5}}{5}$

D $\frac{3}{5}$

E $\frac{4}{5}$

Solution:

The series is geometric with first term 1 and ratio $\cos^2\theta$, so its sum is $\dfrac{1}{1-\cos^2\theta} = \dfrac{1}{\sin^2\theta} = 5.$
Thus $\sin^2\theta = \frac{1}{5}$, and

$$\cos 2\theta = 1 - 2\sin^2\theta = 1 - \tfrac{2}{5} = \tfrac{3}{5}.$$

Thus, the correct answer is **D**.

22. Three mutually tangent spheres of radius 1 rest on a horizontal plane. A sphere of radius 2 rests on them. What is the distance from the plane to the top of the larger sphere?

A $3+\dfrac{\sqrt{30}}{2}$

B $3+\dfrac{\sqrt{69}}{3}$

C $3+\dfrac{\sqrt{123}}{4}$

D $\dfrac{52}{9}$

E $3+2\sqrt{2}$

Solution:

Let the centers of the three unit spheres be A, B, C, forming an equilateral triangle of side 2 at height 1 above the plane, and let E be the center of the large sphere directly above the centroid D of $\triangle ABC$.

The distance from a vertex to the centroid is $AD = \frac{2\sqrt{3}}{3}$, and $AE = 1+2 = 3$, so

$$DE = \sqrt{3^2 - \left(\tfrac{2\sqrt{3}}{3}\right)^2} = \sqrt{9-\tfrac{4}{3}} = \frac{\sqrt{69}}{3}.$$

Since D is 1 unit above the plane and the top of the large sphere is 2 units above E, the total height is

$$1+\frac{\sqrt{69}}{3}+2 = 3+\frac{\sqrt{69}}{3}.$$

Thus, the correct answer is **B**.

23. A polynomial

$$P(x) = c_{2004}x^{2004} + c_{2003}x^{2003} + \cdots + c_1x + c_0$$

has real coefficients with $c_{2004} \neq 0$ and 2004 distinct complex zeros $z_k = a_k + b_k i$, $1 \leq k \leq 2004$ with a_k and b_k real, $a_1 = b_1 = 0$, and

$$\sum_{k=1}^{2004} a_k = \sum_{k=1}^{2004} b_k.$$

Which of the following quantities can be a nonzero number?

A c_0

B c_{2003}

C $b_2 b_3 \ldots b_{2004}$

D $\sum_{k=1}^{2004} a_k$

E $\sum_{k=1}^{2004} c_k$

Solution:

Since $z_1 = a_1 + b_1 i = 0$ is a root, $c_0 = P(0) = 0$.

The nonreal zeros occur in conjugate pairs, so $\sum b_k = 0$, and the hypothesis then forces $\sum a_k = 0$. The coefficient c_{2003} equals $-c_{2004}$ times the sum of the roots $\sum a_k + i \sum b_k = 0$, so $c_{2003} = 0$.

Because the degree is even, at least one of $z_2, \ldots, z_{2004}$ is real, making one $b_k = 0$, so $b_2 b_3 \cdots b_{2004} = 0$. Thus (A) through (D) all must be 0.

On the other hand, $\sum_{k=1}^{2004} c_k = P(1)$, and a valid polynomial such as $P(x) = x(x - 2)(x - 3) \cdots (x - 2003) \left(x + \sum_{k=2}^{2003} k \right)$ has $P(1) \neq 0$. So only $\sum c_k$ can be nonzero.

Thus, the correct answer is **E**.

24. A plane contains points A and B with $AB = 1$. Let S be the union of all disks of radius 1 in the plane that cover $\overline{AB}$. What is the area of S?

A $2\pi + \sqrt{3}$

B $\frac{8\pi}{3}$

C $3\pi - \frac{\sqrt{3}}{2}$

D $\frac{10\pi}{3} - \sqrt{3}$

E $4\pi - 2\sqrt{3}$

Solution:

A radius-1 disk covers segment $\overline{AB}$ exactly when its center is within 1 of both A and B. That region R is the lens where the two unit circles centered at A and B overlap.

Each unit circle passes through the other's center, so the lens is bounded by two 120° arcs. Two 120° sectors of area $\frac{\pi}{3}$ overlap in two equilateral triangles of total area $\frac{\sqrt{3}}{2}$, giving R area $\frac{2\pi}{3} - \frac{\sqrt{3}}{2}$.

The set S consists of all points within 1 of R. Beyond R itself, this adds two 60° sectors of radius 1 (each area $\frac{\pi}{6}$) and two 120° annuli of outer radius 2 and inner radius 1 (each area π).

Therefore the area of S is

$$\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) + 2 \cdot \frac{\pi}{6} + 2\pi = 3\pi - \frac{\sqrt{3}}{2}.$$

Thus, the correct answer is **C**.

25. For each integer $n \geq 4$, let a_n denote the base-n number $0.\overline{133}_n$. The product $a_4 a_5 \ldots a_{99}$ can be expressed as $\frac{m}{n!}$, where m and n are positive integers and n is as small as possible. What is the value of m?

A 98

B 101

C 132

D 798

E 962

Solution:

Since $n^3 \cdot a_n = 133.\overline{133}_n = a_n + n^2 + 3n + 3$, we get

$$a_n = \frac{n^2 + 3n + 3}{n^3 - 1} = \frac{(n+1)^3 - 1}{n(n^3 - 1)}.$$

Writing $n^3 - 1 = (n-1)(n^2 + n + 1)$ and $(n+1)^3 - 1 = n\big((n+1)^2 + (n+1) + 1\big)$, the product $a_4 a_5 \cdots a_{99}$ telescopes to

$$\frac{3!}{99!} \cdot \frac{100^3 - 1}{6^3} = \frac{3!}{99!} \cdot \frac{99(100^2 + 100 + 1)}{63}.$$

This simplifies to $\frac{(2)(10101)}{(21)(98!)} = \frac{962}{98!}$, so $m = 962$ (with the smallest possible $n = 98$).

Thus, the correct answer is **E**.

Problems: https://live.poshenloh.com/past-contests/amc12/2004A