2003 AMC 12B Solutions

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1. Which of the following is the same as

$$\frac{2-4+6-8+10-12+14}{3-6+9-12+15-18+21}?$$

A -1

B $-\frac{2}{3}$

C $\frac{2}{3}$

D 1

E $\frac{14}{3}$

Solution:

Factor 2 from the numerator and 3 from the denominator:

$$\frac{2(1-2+3-4+5-6+7)}{3(1-2+3-4+5-6+7)}.$$

The equal parenthesized sums cancel, leaving $\frac{2}{3}$.

Thus, the correct answer is **C**.

2. Al gets the disease algebritis and must take one green pill and one pink pill each day for two weeks. A green pill costs $\$1$ more than a pink pill, and Al's pills cost a total of $\$546$ for the two weeks. How much does one green pill cost?

A $\$7$

B $\$14$

C $\$19$

D $\$20$

E $\$39$

Solution:

Over 14 days the daily cost of the two pills is

$$\frac{546}{14} = 39.$$

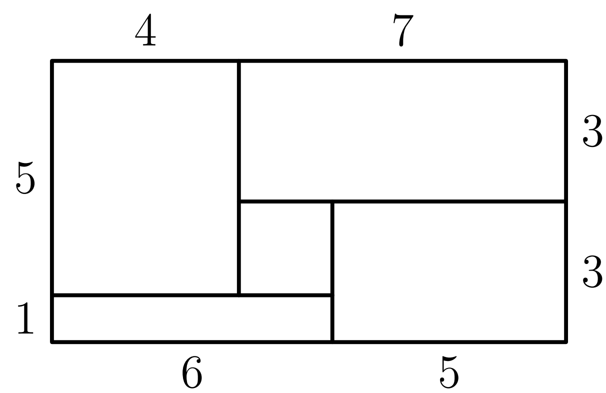
Let g be the cost of a green pill. The pink pill costs $g - 1$, so

$$g + (g - 1) = 39,$$

giving $g = 20.$

Thus, the correct answer is **D**.

3. Rose fills each of the rectangular regions of her rectangular flower bed with a different type of flower. The lengths, in feet, of the rectangular regions in her flower bed are as shown in the figure. She plants one flower per square foot in each region. Asters cost $\$1$ each, begonias $\$1.50$ each, cannas $\$2$ each, dahlias $\$2.50$ each, and Easter lilies $\$3$ each. What is the least possible cost, in dollars, for her garden?



A 108

B 115

C 132

D 144

E 156

Solution:

The five regions have areas $4, 6, 15, 20,$ and 21 square feet.

To minimize the cost, plant the most expensive flowers in the smallest regions. The least possible cost is

$$3(4) + 2.5(6) + 2(15) + 1.5(20) + 1(21) = 108.$$

Thus, the correct answer is **A**.

4. Moe uses a mower to cut his rectangular 90-foot by 150-foot lawn. The swath he cuts is 28 inches wide, but he overlaps each cut by 4 inches to make sure that no grass is missed. He walks at the rate of 5000 feet per hour while pushing the mower. Which of the following is closest to the number of hours it will take Moe to mow his lawn?

- A 0.75
- B 0.8
- C 1.35
- D 1.5
- E 3

Solution:

Because of the overlap, each pass adds a strip $28-4=24$ inches $=2$ feet wide. So each foot Moe walks mows 2 square feet, that is, $2\cdot 5000=10000$ square feet per hour.

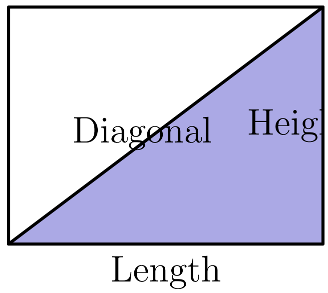
The lawn has area $90\cdot 150=13500$ square feet, so the time is

$$\frac{13500}{10000}=1.35$$

hours.

Thus, the correct answer is **C**.

5. Many television screens are rectangles that are measured by the length of their diagonals. The ratio of the horizontal length to the height in a standard television screen is $4:3$. The horizontal length of a 27-inch television screen is closest, in inches, to which of the following?



A 20

B 20.5

C 21

D 21.5

E 22

Solution:

A rectangle with side ratio $4:3$ has height, length, and diagonal in ratio $3:4:5$. With diagonal 27, the horizontal length is

$$\frac{4}{5}(27) = 21.6,$$

which is closest to 21.5.

Thus, the correct answer is **D**.

6. The second and fourth terms of a geometric sequence are 2 and 6. Which of the following is a possible first term?

A $-\sqrt{3}$

B $-\frac{2\sqrt{3}}{3}$

C $-\frac{\sqrt{3}}{3}$

D $\sqrt{3}$

E 3

Solution:

Let the first term be a and the common ratio r. Then $ar = 2$ and $ar^3 = 6$, so $r^2 = 3$ and $r = \pm\sqrt{3}$.

The first term is

$$a = \frac{2}{r} = \pm\frac{2}{\sqrt{3}} = \pm\frac{2\sqrt{3}}{3}.$$

The choice $-\frac{2\sqrt{3}}{3}$ appears among the options.

Thus, the correct answer is **B**.

7. Penniless Pete's piggy bank has no pennies in it, but it has 100 coins, all nickels, dimes, and quarters, whose total value is $\$8.35$. It does not necessarily contain coins of all three types. What is the difference between the largest and smallest number of dimes that could be in the bank?

A 0

B 13

C 37

D 64

E 83

Solution:

Let n, d, q be the numbers of nickels, dimes, quarters. Then $n + d + q = 100$ and $n + 2d + 5q = 167$ (dividing the value equation by 5).

Subtracting gives $d + 4q = 67$, so $d = 67 - 4q$.

The largest d is at $q = 0$, giving $d = 67$ (with $n = 33$). The smallest occurs at $q = 16$, giving $d = 3$ (with $n = 81$). The difference is $67 - 3 = 64$.

Thus, the correct answer is **D**.

8. Let $\clubsuit(x)$ denote the sum of the digits of the positive integer x. For example, $\clubsuit(8) = 8$ and $\clubsuit(123) = 1 + 2 + 3 = 6$. For how many two-digit values of x is $\clubsuit(\clubsuit(x)) = 3$?

- A 3
- B 4
- C 6
- D 9
- E 10

Solution:

Let $y = \clubsuit(x)$. Since $x \leq 99$, we have $y \leq 18$. Then $\clubsuit(y) = 3$ requires $y = 3$ or $y = 12$.

The two-digit numbers with digit sum 3 are $12, 21, 30$ (3 values), and those with digit sum 12 are $39, 48, 57, 66, 75, 84, 93$ (7 values), for 10 in all.

Thus, the correct answer is **E**.

9. Let f be a linear function for which $f(6) - f(2) = 12$. What is $f(12) - f(2)$?

- A 12
- B 18
- C 24
- D 30
- E 36

Solution:

The slope of f is

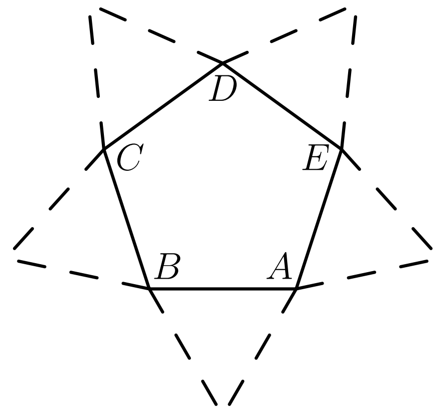
$$\frac{f(6) - f(2)}{6 - 2} = \frac{12}{4} = 3.$$

Therefore

$$f(12) - f(2) = 3(12 - 2) = 30.$$

Thus, the correct answer is **D**.

10. Several figures can be made by attaching two equilateral triangles to the regular pentagon $ABCDE$ in two of the five positions shown. How many non-congruent figures can be constructed in this way?



A 1

B 2

C 3

D 4

E 5

Solution:

Assume one triangle is attached to side AB. The second triangle can be attached to a side that is one step away or two steps away from AB.

Attaching it to BC or CD gives two figures; attaching it to AE or DE gives figures that are mirror images of these across the pentagon's axis of symmetry.

So there are only 2 non-congruent figures.

Thus, the correct answer is **B**.

11. Cassandra sets her watch to the correct time at noon. At the actual time of 1:00 PM, she notices that her watch reads 12:57 and 36 seconds. Assuming that her watch loses time at a constant rate, what will be the actual time when her watch first reads 10:00 PM?

- A 10:22 PM and 24 seconds
- B 10:24 PM
- C 10:25 PM
- D 10:27 PM
- E 10:30 PM

Solution:

In 60 real minutes the watch advances only 57 minutes 36 seconds $= 57.6$ minutes. So when the watch shows t minutes past noon, the real elapsed time is $\frac{60}{57.6}t = \frac{25}{24}t$ minutes.

The watch reads 10:00 PM after 600 recorded minutes, so the real elapsed time is

$$\frac{25}{24}(600) = 625$$

minutes $= 10$ hours 25 minutes past noon. The actual time is 10:25 PM.

Thus, the correct answer is **C**.

12. What is the largest integer that is a divisor of

$$(n+1)(n+3)(n+5)(n+7)(n+9)$$

for all positive even integers n?

A 3

B 5

C 11

D 15

E 165

Solution:

For even $n,$ the five factors are consecutive odd numbers. Among any five consecutive odd numbers, at least one is divisible by 3 and exactly one by $5,$ so the product is always divisible by $15.$

No larger divisor always works: the products for $n=2$ and $n=10$ are $3 \cdot 5 \cdot 7 \cdot 9 \cdot 11$ and $11 \cdot 13 \cdot 15 \cdot 17 \cdot 19,$ whose greatest common divisor is $15.$

Thus, the correct answer is **D**.

13. An ice cream cone consists of a sphere of vanilla ice cream and a right circular cone that has the same diameter as the sphere. If the ice cream melts, it will exactly fill the cone. Assume that the melted ice cream occupies 75% of the volume of the frozen ice cream. What is the ratio of the cone's height to its radius?

- A $2:1$
- B $3:1$
- C $4:1$
- D $16:3$
- E $6:1$

Solution:

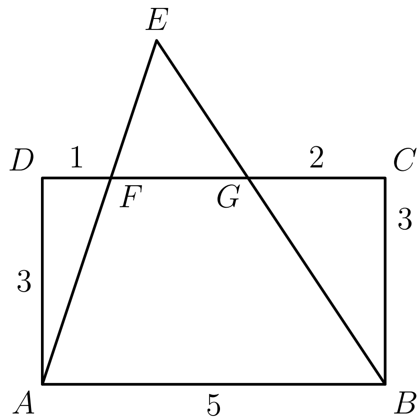
Let r be the common radius and h the cone's height. The melted ice cream fills the cone, so

$$\frac{3}{4} \cdot \frac{4}{3}\pi r^3 = \frac{1}{3}\pi r^2 h.$$

This simplifies to $\pi r^3 = \frac{1}{3}\pi r^2 h$, so $h = 3r$, a ratio of $3:1$.

Thus, the correct answer is **B**.

14. In rectangle $ABCD$, $AB = 5$ and $BC = 3$. Points F and G are on $\overline{CD}$ so that $DF = 1$ and $GC = 2$. Lines AF and BG intersect at E. Find the area of $\triangle AEB$.



A 10

B $\frac{21}{2}$

C 12

D $\frac{25}{2}$

E 15

Solution:

Since $FG = 5 - 1 - 2 = 2$ and $\overline{FG} \parallel \overline{AB}$, triangles FEG and AEB are similar with ratio $\frac{FG}{AB} = \frac{2}{5}$.

Let the distance from E to line CD be k. Then the distance from E to AB is $k + 3$, and

$$\frac{k}{k+3} = \frac{2}{5},$$

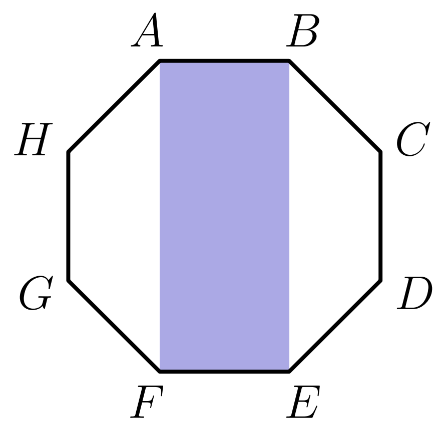
giving $k = 2$.

The height of $\triangle AEB$ is $k + 3 = 5,$ so its area is

$$\frac{1}{2}(5)(5) = \frac{25}{2}.$$

Thus, the correct answer is **D**.

15. A regular octagon $ABCDEFGH$ has an area of one square unit. What is the area of the rectangle $ABEF$?



A $1 - \frac{\sqrt{2}}{2}$

B $\frac{\sqrt{2}}{4}$

C $\sqrt{2} - 1$

D $\frac{1}{2}$

E $\frac{1 + \sqrt{2}}{4}$

Solution:

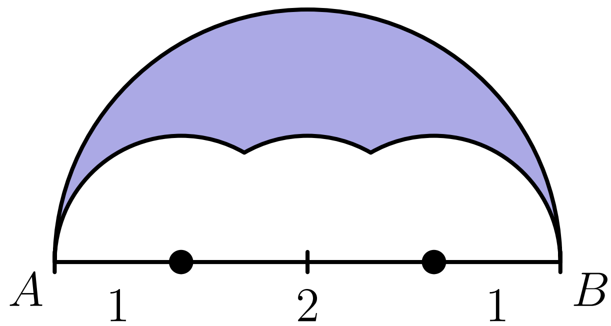
Let O be the center of the octagon. Joining O to the vertices splits the octagon into 8 congruent triangles, so $\triangle AOB$ has area $\frac{1}{8}$.

Since O is the midpoint of $\overline{AE}$, triangles AOB and BOE have equal areas, so $\triangle ABE$ has area $\frac{1}{4}$.

The rectangle $ABEF$ is split by diagonal $\overline{BE}$ into two congruent triangles, so $\triangle ABE$ is half of it. Hence $ABEF$ has area $\frac{1}{2}$.

Thus, the correct answer is **D**.

16. Three semicircles of radius 1 are constructed on diameter $\overline{AB}$ of a semicircle of radius 2. The centers of the small semicircles divide $\overline{AB}$ into four line segments of equal length, as shown. What is the area of the shaded region that lies within the large semicircle but outside the smaller semicircles?



A $\pi - \sqrt{3}$

B $\pi - \sqrt{2}$

C $\dfrac{\pi + \sqrt{2}}{2}$

D $\dfrac{\pi + \sqrt{3}}{2}$

E $\dfrac{7}{6}\pi - \dfrac{\sqrt{3}}{2}$

Solution:

The large semicircle has area $\frac{1}{2}\pi(2)^2 = 2\pi$.

Where the small semicircles overlap, adjacent ones meet at points a distance 1 from two centers, forming equilateral triangles. The region removed from the large semicircle consists of five congruent 60° sectors of radius 1, each of area $\frac{\pi}{6}$, together with two equilateral triangles of side 1, each of area $\frac{\sqrt{3}}{4}$.

The shaded area is

$$2\pi - 5 \cdot \frac{\pi}{6} - 2 \cdot \frac{\sqrt{3}}{4} = \frac{7}{6}\pi - \frac{\sqrt{3}}{2}.$$

Thus, the correct answer is **E**.

17. If $\log(xy^3) = 1$ and $\log(x^2y) = 1$, what is $\log(xy)$?

A $-\frac{1}{2}$

B 0

C $\frac{1}{2}$

D $\frac{3}{5}$

E 1

Solution:

Let $X = \log x$ and $Y = \log y$. Then

$$X + 3Y = 1 \quad \text{and} \quad 2X + Y = 1.$$

Solving gives $X = \frac{2}{5}$ and $Y = \frac{1}{5}$, so

$$\log(xy) = X + Y = \frac{3}{5}.$$

Thus, the correct answer is **D**.

18. Let x and y be positive integers such that $7x^5 = 11y^{13}$. The minimum possible value of x has a prime factorization $a^c b^d$. What is $a + b + c + d$?

A 30

B 31

C 32

D 33

E 34

Solution:

For the minimum x, neither x nor y has prime factors other than 7 and 11. Write $x = 7^c 11^d$, so $7x^5 = 7^{5c+1} 11^{5d}$. Writing $y = 7^m 11^n$, we need $7^{5c+1} 11^{5d} = 7^{13m} 11^{13n+1}$.

Matching exponents: $5c + 1 \equiv 0 \pmod{13}$ gives the least $c = 5$, and $5d \equiv 1 \pmod{13}$ gives the least $d = 8$. So $a = 7, b = 11$, and

$$a + b + c + d = 7 + 11 + 5 + 8 = 31.$$

Thus, the correct answer is **B**.

19. Let S be the set of permutations of the sequence $1, 2, 3, 4, 5$ for which the first term is not 1. A permutation is chosen randomly from S. The probability that the second term is 2, in lowest terms, is a/b. What is $a + b$?

A 5

B 6

C 11

D 16

E 19

Solution:

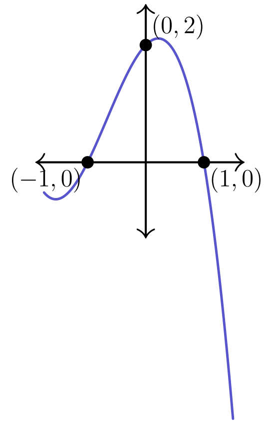
The set S contains $4 \cdot 4! = 96$ permutations, since the first term has 4 choices and the remaining four terms can be arranged in $4!$ ways.

For the second term to be 2, the first term must be $3, 4,$ or 5 (not 1, not 2), giving 3 choices, and the remaining three terms can be arranged in $3!$ ways: $3 \cdot 3! = 18$.

The probability is $\frac{18}{96} = \frac{3}{16}$, so $a + b = 3 + 16 = 19$.

Thus, the correct answer is **E**.

20. Part of the graph of $f(x) = ax^3 + bx^2 + cx + d$ is shown. What is b?



A -4

B -2

C 0

D 2

E 4

Solution:

The graph passes through $(-1, 0)$, $(1, 0)$, and $(0, 2)$. So $f(0) = d = 2$.

Adding

$$f(1) + f(-1) = (a + b + c + d) + (-a + b - c + d) = 2b + 2d = 0,$$

so $b = -d = -2$.

Thus, the correct answer is **B**.

21. An object moves 8 cm in a straight line from A to B, turns at an angle α, measured in radians and chosen at random from the interval $(0, \pi)$, and moves 5 cm in a straight line to C. What is the probability that $AC < 7$?

A $\frac{1}{6}$

B $\frac{1}{5}$

C $\frac{1}{4}$

D $\frac{1}{3}$

E $\frac{1}{2}$

Solution:

Let $\beta = \pi - \alpha$ be the interior angle of $\triangle ABC$ at B. By the Law of Cosines,

$$AC^2 = 8^2 + 5^2 - 2(8)(5)\cos\beta = 89 - 80\cos\beta.$$

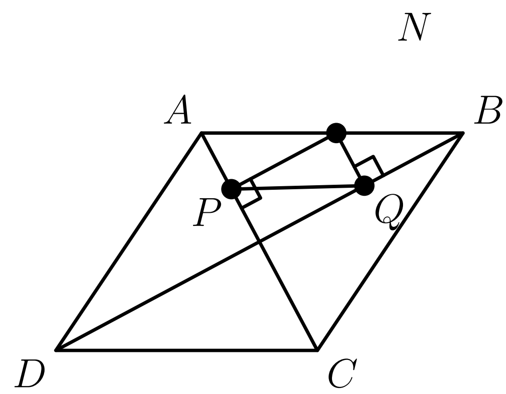
Then $AC < 7$ means $89 - 80\cos\beta < 49$, i.e. $\cos\beta > \frac{1}{2}$, i.e. $\beta < \frac{\pi}{3}$.

As α is uniform on $(0, \pi)$, so is β. The probability is

$$\frac{\pi/3}{\pi} = \frac{1}{3}.$$

Thus, the correct answer is **D**.

22. Let $ABCD$ be a rhombus with $AC = 16$ and $BD = 30$. Let N be a point on $\overline{AB}$, and let P and Q be the feet of the perpendiculars from N to $\overline{AC}$ and $\overline{BD}$, respectively. Which of the following is closest to the minimum possible value of PQ?



A 6.5

B 6.75

C 7

D 7.25

E 7.5

Solution:

Let O be the intersection of the diagonals. Then $\triangle AOB$ is right-angled at O with legs $OA = 8$ and $OB = 15$. Quadrilateral $OPNQ$ has right angles at O, P, and Q, so it is a rectangle and $PQ = ON$.

The minimum of ON is the altitude from O to $\overline{AB}$ in $\triangle AOB$. Since $AB = \sqrt{8^2 + 15^2} = 17$, equating the two area expressions gives

$$ON = \frac{OA \cdot OB}{AB} = \frac{8 \cdot 15}{17} = \frac{120}{17} \approx 7.06.$$

This is closest to 7.

Thus, the correct answer is **C**.

23. The number of x-intercepts on the graph of $y = \sin(1/x)$ in the interval $(0.0001, 0.001)$ is closest to

A 2900

B 3000

C 3100

D 3200

E 3300

Solution:

The intercepts occur where $1/x = k\pi$, that is $x = \dfrac{1}{k\pi}$ for a nonzero integer k.

The condition $0.0001 < \dfrac{1}{k\pi} < 0.001$ becomes

$$\frac{1000}{\pi} < k < \frac{10000}{\pi}.$$

The number of such integers is

$$\left\lfloor \frac{10000}{\pi} \right\rfloor - \left\lfloor \frac{1000}{\pi} \right\rfloor = 3183 - 318 = 2865,$$

closest to 2900.

Thus, the correct answer is **A**.

24. Positive integers a, b, and c are chosen so that $a < b < c$, and the system of equations

$$2x + y = 2003 \quad \text{and} \quad y = |x-a| + |x-b| + |x-c|$$

has exactly one solution. What is the minimum value of c?

- A 668
- B 669
- C 1002
- D 2003
- E 2004

Solution:

The function $y = |x-a| + |x-b| + |x-c|$ is piecewise linear with slopes $-3, -1, 1, 3$ and corners at $x = a, b, c$. The line $2x + y = 2003$ has slope -2.

A line of slope -2 meets this graph exactly once only if it passes through the leftmost corner $(a,\ b + c - 2a)$, where the graph's slope jumps from -3 to -1. Substituting,

$$2a + (b + c - 2a) = 2003,$$

so $b + c = 2003$.

Since $b < c$, we need $c > \dfrac{2003}{2}$, so the minimum is $c = 1002$ (with $b = 1001$).

Thus, the correct answer is **C**.

25. Three points are chosen randomly and independently on a circle. What is the probability that all three pairwise distances between the points are less than the radius of the circle?

A $\frac{1}{36}$

B $\frac{1}{24}$

C $\frac{1}{18}$

D $\frac{1}{12}$

E $\frac{1}{9}$

Solution:

A chord has length less than the radius exactly when the arc it subtends is less than 60°, since a chord of a 60° arc equals the radius.

All three pairwise chords are shorter than the radius precisely when the three points all lie within some arc of 60°.

The probability that n random points all lie within some arc of angle L is $n\left(\frac{L}{2\pi}\right)^{n-1}$.
With $n = 3$ and $L = \frac{\pi}{3}$ (that is, $\frac{L}{2\pi} = \frac{1}{6}$), the probability is

$$3\left(\frac{1}{6}\right)^2 = \frac{1}{12}.$$

Thus, the correct answer is **D**.

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