2003 AMC 12A Solutions

Typeset by: LIVE by Po-Shen Loh
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1. What is the difference between the sum of the first 2003 even counting numbers and the sum of the first 2003 odd counting numbers?

   A 0

   B 1

   C 2

   D 2003

   E 4006

Solution:

The kth even counting number is $2k$ and the kth odd counting number is $2k-1$, which differ by 1.

Summing this difference over all 2003 pairs gives $2003 \cdot 1 = 2003$.

Thus, the correct answer is **D**.

2. Members of the Rockham Soccer League buy socks and T-shirts. Socks cost $\$4$ per pair and each T-shirt costs $\$5$ more than a pair of socks. Each member needs one pair of socks and a shirt for home games and another pair of socks and a shirt for away games. If the total cost is $\$2366,$ how many members are in the League?

A 77

B 91

C 143

D 182

E 286

Solution:

A T-shirt costs $4 + 5 = 9$ dollars.

Each member needs two pairs of socks and two shirts, costing $2(4) + 2(9) = 26$ dollars.

The number of members is $2366 \div 26 = 91.$

Thus, the correct answer is **B**.

3. A solid box is 15 cm by 10 cm by 8 cm. A new solid is formed by removing a cube 3 cm on a side from each corner of this box. What percent of the original volume is removed?

- A 4.5
- B 9
- C 12
- D 18
- E 24

Solution:

The original volume is $15 \cdot 10 \cdot 8 = 1200.$

Eight corner cubes are removed, each of volume $3^3 = 27,$ totaling $8 \cdot 27 = 216.$

The fraction removed is $\frac{216}{1200} = 0.18,$ which is $18\%.$

Thus, the correct answer is **D**.

4. It takes Mary 30 minutes to walk uphill 1 km from her home to school, but it takes her only 10 minutes to walk from school to home along the same route. What is her average speed, in km/hr, for the round trip?

- A 3
- B 3.125
- C 3.5
- D 4
- E 4.5

Solution:

Mary walks a total of 2 km in $30 + 10 = 40$ minutes, which is $\frac{2}{3}$ hour.

Her average speed is $2 \div \frac{2}{3} = 3$ km/hr.

Thus, the correct answer is **A**.

5. The sum of the two 5-digit numbers $\overline{AMC10}$ and $\overline{AMC12}$ is 123422. What is $A + M + C$?

- A 10
- B 11
- C 12
- D 13
- E 14

Solution:

Write $\overline{AMC10} = 100 \cdot \overline{AMC} + 10$ and $\overline{AMC12} = 100 \cdot \overline{AMC} + 12.$

Their sum is $200 \cdot \overline{AMC} + 22 = 123422,$ so $\overline{AMC} = 617.$

Then $A + M + C = 6 + 1 + 7 = 14.$

Thus, the correct answer is **E**.

6. Define $x \heartsuit y$ to be $|x - y|$ for all real numbers x and y. Which of the following statements is *not* true?

- A $x \heartsuit y = y \heartsuit x$ for all x and y
- B $2(x \heartsuit y) = (2x) \heartsuit (2y)$ for all x and y
- C $x \heartsuit 0 = x$ for all x
- D $x \heartsuit x = 0$ for all x
- E $x \heartsuit y > 0$ if $x \neq y$

Solution:

Since $x \heartsuit 0 = |x - 0| = |x|$, statement (C) claims $|x| = x$ for all x, which fails when $x < 0$. For example, $(-1) \heartsuit 0 = 1 \neq -1$.

Every other statement follows directly from properties of the absolute value.

Thus, the correct answer is **C**.

7. How many non-congruent triangles with perimeter 7 have integer side lengths?

- A 1
- **B** 2
- C 3
- D 4
- E 5

Solution:

Let the sides be $a \le b \le c$ with $a + b + c = 7$. The triangle inequality requires $c < a + b$, so $c < 3.5$, forcing $c = 3$.

Then $a + b = 4$ with $a \le b \le 3$, giving the triangles $1\text{-}3\text{-}3$ and $2\text{-}2\text{-}3$.

Thus, the correct answer is **B**.

8. What is the probability that a randomly drawn positive factor of 60 is less than 7?

A $\frac{1}{10}$

B $\frac{1}{6}$

C $\frac{1}{4}$

D $\frac{1}{3}$

E $\frac{1}{2}$

Solution:

The number 60 has 12 positive factors: $1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60.$

Six of them are less than 7, namely $1, 2, 3, 4, 5, 6$, so the probability is $\frac{6}{12} = \frac{1}{2}$.

Thus, the correct answer is **E**.

9. A set S of points in the xy-plane is symmetric about the origin, both coordinate axes, and the line $y = x$. If $(2, 3)$ is in S, what is the smallest number of points in S?

A 1

B 2

C 4

D 8

E 16

Solution:

Reflecting across $y = x$ gives $(3, 2)$, and reflecting across the axes gives all points $(\pm 2, \pm 3)$ and $(\pm 3, \pm 2)$.

There are 8 such points, and this set is already symmetric about the origin, both axes, and $y = x$.

Thus, the correct answer is **D**.

10. Al, Bert, and Carl are the winners of a school drawing for a pile of Halloween candy, which they are to divide in a ratio of $3:2:1$, respectively. Due to some confusion they come at different times to claim their prizes, and each assumes he is the first to arrive. If each takes what he believes to be his correct share of candy, what fraction of the candy goes unclaimed?

A $\frac{1}{18}$

B $\frac{1}{6}$

C $\frac{2}{9}$

D $\frac{5}{18}$

E $\frac{5}{12}$

Solution:

The shares are $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$ of the pile.

Each person assumes he is first, so Al leaves $\frac{1}{2}$, Bert leaves $\frac{2}{3}$, and Carl leaves $\frac{5}{6}$ of the candy present when he arrives.

The unclaimed fraction is $\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{5}{6} = \frac{5}{18}$, regardless of the order.

Thus, the correct answer is **D**.

11. A square and an equilateral triangle have the same perimeter. Let A be the area of the circle circumscribed about the square and B be the area of the circle circumscribed about the triangle. Find A/B.

A $\frac{9}{16}$

B $\frac{3}{4}$

C $\frac{27}{32}$

D $\frac{3\sqrt{6}}{8}$

E 1

Solution:

Let the common perimeter be 12, so the square has side 3 and the triangle has side 4.

The square's circumradius is $\frac{3\sqrt{2}}{2}$, so $A = \pi\left(\frac{3\sqrt{2}}{2}\right)^2 = \frac{9\pi}{2}$.

The triangle's circumradius is $\frac{4}{\sqrt{3}}$, so $B = \pi\left(\frac{4}{\sqrt{3}}\right)^2 = \frac{16\pi}{3}$.

Then $\frac{A}{B} = \frac{9/2}{16/3} = \frac{27}{32}$.

Thus, the correct answer is **C**.

12. Sally has five red cards numbered 1 through 5 and four blue cards numbered 3 through $6.$ She stacks the cards so that the colors alternate and so that the number on each red card divides evenly into the number on each neighboring blue card. What is the sum of the numbers on the middle three cards?

- A 8
- B 9
- C 10
- D 11
- E 12

Solution:

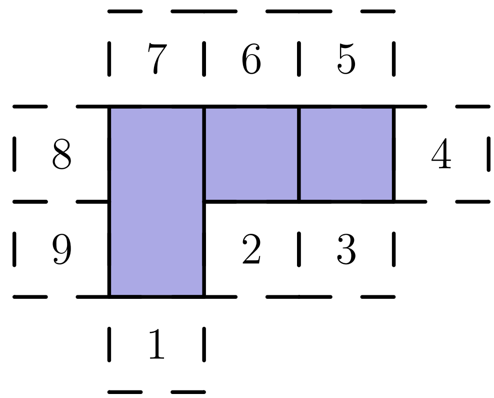
Since 4 divides only 4 and 5 divides only 5 among $3, 4, 5, 6,$ the two ends must be $R4, B4, \ldots, B5, R5.$

Because 2 divides only 4 and $6,$ the next card is $R2, B6,$ and since 3 divides only 3 and $6,$ the full stack is $R4, B4, R2, B6, R3, B3, R1, B5, R5.$

The middle three cards are $6, 3, 3,$ which sum to $12.$

Thus, the correct answer is **E**.

13. The polygon enclosed by the solid lines in the figure consists of 4 congruent squares joined edge-to-edge. One more congruent square is attached to an edge at one of the nine positions indicated. How many of the nine resulting polygons can be folded to form a cube with one face missing?



A 2

B 3

C 4

D 5

E 6

Solution:

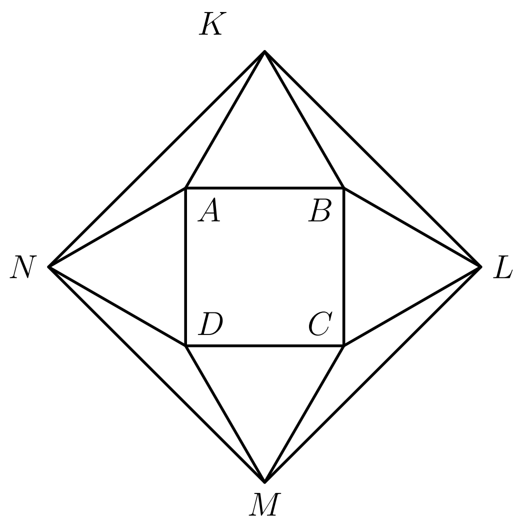
A cube missing one face has 5 faces, so the fifth square must complete a foldable arrangement.

Folding the four-square piece wraps it around four faces of a cube, identifying two of its edges. The fifth square then folds up onto a face exactly when it is attached along one of the free edges.

Of the nine indicated positions, 6 of them work.

Thus, the correct answer is **E**.

14. Points $K, L, M,$ and N lie in the plane of the square $ABCD$ so that $AKB, BLC,$ $CMD,$ and DNA are equilateral triangles. If $ABCD$ has an area of $16,$ find the area of $KLMN.$



A 32

B $16 + 16\sqrt{3}$

C 48

D $32 + 16\sqrt{3}$

E 64

Solution:

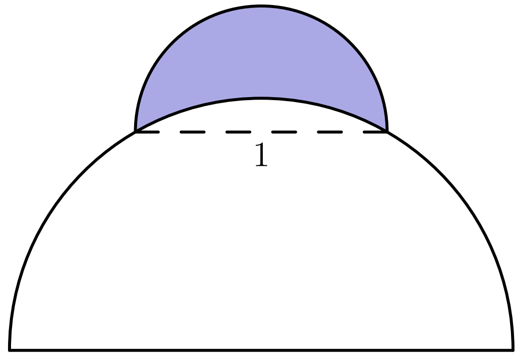
The square $ABCD$ has side $4.$ By the 90° rotational symmetry, $KLMN$ is also a square.

Each equilateral triangle on a side of length 4 has height $2\sqrt{3},$ so the diagonal $KM = 4 + 2(2\sqrt{3}) = 4 + 4\sqrt{3}.$

A square with diagonal d has area $\frac{1}{2}d^2$, so $[KLMN] = \frac{1}{2}(4 + 4\sqrt{3})^2 = 32 + 16\sqrt{3}.$

Thus, the correct answer is **D**.

15. A semicircle of diameter 1 sits at the top of a semicircle of diameter $2,$ as shown. The shaded area inside the smaller semicircle and outside the larger semicircle is called a *lune*. Determine the area of this lune.



A $\frac{1}{6}\pi - \frac{\sqrt{3}}{4}$

B $\frac{\sqrt{3}}{4} - \frac{1}{12}\pi$

C $\frac{\sqrt{3}}{4} - \frac{1}{24}\pi$

D $\frac{\sqrt{3}}{4} + \frac{1}{24}\pi$

E $\frac{\sqrt{3}}{4} + \frac{1}{12}\pi$

Solution:

The small semicircle's diameter is a chord of length 1 in the large circle of radius $1,$ so it subtends a 60° angle at the center.

The region bounded by that chord and the small arc is an equilateral triangle of area $\frac{\sqrt{3}}{4}$ topped by the small semicircle of area $\frac{1}{2}\pi\left(\frac{1}{2}\right)^2 = \frac{\pi}{8}.$

Subtracting the 60° sector of the large circle, $\frac{1}{6}\pi(1)^2 = \frac{\pi}{6}$, gives the lune area

$$\frac{\sqrt{3}}{4} + \frac{\pi}{8} - \frac{\pi}{6} = \frac{\sqrt{3}}{4} - \frac{\pi}{24}.$$

Thus, the correct answer is **C**.

16. A point P is chosen at random in the interior of equilateral triangle ABC. What is the probability that $\triangle ABP$ has a greater area than each of $\triangle ACP$ and $\triangle BCP$?

A $\frac{1}{6}$

B $\frac{1}{4}$

C $\frac{1}{3}$

D $\frac{1}{2}$

E $\frac{2}{3}$

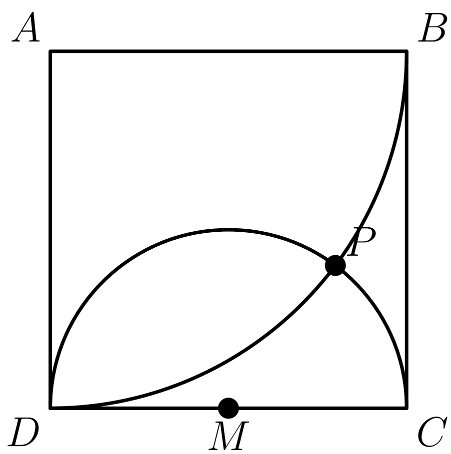
Solution:

The triangles $\triangle ABP$, $\triangle ACP$, and $\triangle BCP$ have equal bases (the sides of the equilateral triangle), so their areas are proportional to the distances from P to those sides.

By the threefold symmetry of the equilateral triangle, $\triangle ABP$ is the largest with the same probability as each of the other two, so that probability is $\frac{1}{3}$.

Thus, the correct answer is **C**.

17. Square $ABCD$ has sides of length 4, and M is the midpoint of $\overline{CD}$. A circle with radius 2 and center M intersects a circle with radius 4 and center A at points P and D. What is the distance from P to $\overline{AD}$?



A 3

B $\frac{16}{5}$

C $\frac{13}{4}$

D $2\sqrt{3}$

E $\frac{7}{2}$

Solution:

Place $D = (0,0), C = (4,0)$, and $A = (0,4)$. The circle centered at $M = (2,0)$ is $(x-2)^2 + y^2 = 4$, and the circle centered at A is $x^2 + (y-4)^2 = 16$.

Solving these equations gives the intersection $P = \left(\frac{16}{5}, \frac{8}{5}\right)$.

Since $\overline{AD}$ lies on the y-axis, the distance from P to $\overline{AD}$ is its x-coordinate, $\frac{16}{5}$.

Thus, the correct answer is **B**.

18. Let n be a 5-digit number, and let q and r be the quotient and remainder, respectively, when n is divided by 100. For how many values of n is $q + r$ divisible by 11?

A 8180

B 8181

C 8182

D 9000

E 9090

Solution:

Since $n = 100q + r = (q + r) + 99q$ and 99 is divisible by 11, we have $q + r \equiv n \pmod{11}$.

So $11 \mid (q + r)$ exactly when $11 \mid n$.

Among the 5-digit numbers, the count of multiples of 11 is $\left\lfloor \frac{99999}{11} \right\rfloor - \left\lfloor \frac{9999}{11} \right\rfloor = 9090 - 909 = 8181$.

Thus, the correct answer is **B**.

19. A parabola with equation $y = ax^2 + bx + c$ is reflected about the x-axis. The parabola and its reflection are translated horizontally five units in opposite directions to become the graphs of $y = f(x)$ and $y = g(x)$, respectively. Which of the following describes the graph of $y = (f + g)(x)$?

A a parabola tangent to the x-axis

B a parabola not tangent to the x-axis

C a horizontal line

D a non-horizontal line

E the graph of a cubic function

Solution:

Write the parabola in vertex form $y = a(x - h)^2 + k$. Its reflection about the x-axis is $y = -a(x - h)^2 - k$.

Shifting in opposite directions gives $f(x) = a(x - h + 5)^2 + k$ and $g(x) = -a(x - h - 5)^2 - k$.

Adding, the squared terms cancel and $(f + g)(x) = 20a(x - h)$, which is a non-horizontal line since $a \neq 0$.

Thus, the correct answer is **D**.

20. How many 15-letter arrangements of 5 A's, 5 B's, and 5 C's have no A's in the first 5 letters, no B's in the next 5 letters, and no C's in the last 5 letters?

- A $\displaystyle\sum_{k=0}^{5}\binom{5}{k}^{3}$
- B $3^5 \cdot 2^5$
- C 2^{15}
- D $\dfrac{15!}{(5!)^3}$
- E 3^{15}

Solution:

Suppose the first block holds k B's and $5-k$ C's. The remaining k C's must go in the second block (since the third has no C's), forcing $5-k$ A's there.

Then the third block contains the remaining k A's and $5-k$ B's.

For each k, the k B's in the first block, k C's in the second, and k A's in the third can be placed in $\binom{5}{k}^3$ ways, so the total is $\displaystyle\sum_{k=0}^{5}\binom{5}{k}^{3}$.

Thus, the correct answer is **A**.

21. The graph of the polynomial

$$P(x) = x^5 + ax^4 + bx^3 + cx^2 + dx + e$$

has five distinct x-intercepts, one of which is at $(0, 0)$. Which of the following coefficients cannot be zero?

- A a
- B b
- C c
- D d
- E e

Solution:

Since $(0, 0)$ is an intercept, $P(0) = e = 0$, so $P(x) = x\left(x^4 + ax^3 + bx^2 + cx + d\right).$

The four remaining intercepts are nonzero and distinct, and d equals their product, which is therefore nonzero.

Any of a, b, c can be zero for suitable choices of those roots, but $d \neq 0$.

Thus, the correct answer is **D**.

22. Objects A and B move simultaneously in the coordinate plane via a sequence of steps, each of length one. Object A starts at $(0, 0)$ and each of its steps is either right or up, both equally likely. Object B starts at $(5, 7)$ and each of its steps is either left or down, both equally likely. Which of the following is closest to the probability that the objects meet?

- A 0.10
- B 0.15
- C 0.20
- D 0.25
- E 0.30

Solution:

The objects are 12 steps apart, so they can only meet after each takes 6 steps, on the anti-diagonal $x + y = 6$.

Pairing A's six-step path with B's reversed six-step path matches meeting pairs one-to-one with the $\binom{12}{5}$ monotone walks from $(0, 0)$ to $(5, 7)$.

The probability is $\frac{\binom{12}{5}}{2^{12}} = \frac{792}{4096} \approx 0.19$, which is closest to 0.20.

Thus, the correct answer is **C**.

23. How many perfect squares are divisors of the product $1! \cdot 2! \cdot 3! \cdots 9!$?

A 504

B 672

C 864

D 936

E 1008

Solution:

The product is

$$1! \cdot 2! \cdots 9! = 2^{30} \cdot 3^{13} \cdot 5^5 \cdot 7^3.$$

A perfect-square divisor has the form $2^{2a}3^{2b}5^{2c}7^{2d}$ with $0 \leq a \leq 15, 0 \leq b \leq 6, 0 \leq c \leq 2$, and $0 \leq d \leq 1$.

The number of choices is $16 \cdot 7 \cdot 3 \cdot 2 = 672$.

Thus, the correct answer is **B**.

24. If $a \geq b > 1$, what is the largest possible value of $\log_a(a/b) + \log_b(b/a)$?

A -2

B 0

C 2

D 3

E 4

Solution:

Expand: $\log_a \dfrac{a}{b} + \log_b \dfrac{b}{a} = (1 - \log_a b) + (1 - \log_b a) = 2 - (\log_a b + \log_b a)\,.$

Let $c = \log_a b > 0$. Since $c + \dfrac{1}{c} \geq 2$ by AM-GM, the expression is at most 0.

Equality holds when $c = 1$, that is, when $a = b$, so the largest value is 0.

Thus, the correct answer is **B**.

25. Let $f(x) = \sqrt{ax^2 + bx}$. For how many real values of a is there at least one positive value of b for which the domain of f and the range of f are the same set?

- A 0
- B 1
- C 2
- D 3
- E infinitely many

Solution:

If $a = 0$, then $f(x) = \sqrt{bx}$ has domain and range both $[0, \infty)$, so $a = 0$ works.

If $a > 0$, the domain is unbounded but the range still starts at 0 and grows without bound in a way that cannot match the domain, so no such b exists.

If $a < 0$, the domain is $[0, -b/a]$ and the range is $\left[0, \frac{b}{2\sqrt{-a}}\right]$. Equating the right endpoints gives $-\frac{b}{a} = \frac{b}{2\sqrt{-a}}$, so $2\sqrt{-a} = -a$, giving $a = -4$.

Thus there are 2 values of a, and the correct answer is **C**.

Problems: https://live.poshenloh.com/past-contests/amc12/2003A