2002 AMC 12A Solutions

Typeset by: LIVE by Po-Shen Loh
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1. Compute the sum of all the roots of

$$(2x+3)(x-4)+(2x+3)(x-6)=0.$$

A $\frac{7}{2}$

B 4

C 5

D 7

E 13

Solution:

Factoring out $2x+3$ gives $(2x+3)\left[(x-4)+(x-6)\right]=(2x+3)(2x-10)=0$.

The roots are $-\frac{3}{2}$ and 5, whose sum is $-\frac{3}{2}+5=\frac{7}{2}$.

Thus, the correct answer is **A**.

2. Cindy was asked by her teacher to subtract 3 from a certain number and then divide the result by $9.$ Instead, she subtracted 9 and then divided the result by $3,$ giving an answer of $43.$ What would her answer have been had she worked the problem correctly?

A 15

B 34

C 43

D 51

E 138

Solution:

Let x be the number. Cindy computed $\dfrac{x-9}{3} = 43,$ so $x - 9 = 129$ and $x = 138.$

The correct computation gives $\dfrac{138-3}{9} = \dfrac{135}{9} = 15.$

Thus, the correct answer is **A**.

3. According to the standard convention for exponentiation,

$$2^{2^{2^2}} = 2^{\left(2^{\left(2^2\right)}\right)} = 2^{16} = 65{,}536.$$

If the order in which the exponentiations are performed is changed, how many other values are possible?

A 0

B 1

C 2

D 3

E 4

Solution:

The five parenthesizations of $2^{2^{2^2}}$ give

$$\left(\left(2^2\right)^2\right)^2 = 2^8, \quad \left(2^{2^2}\right)^2 = 2^8, \quad \left(2^2\right)^{2^2} = 2^8,$$

$$2^{\left(2^2\right)^2} = 2^{16}, \quad 2^{2^{2^2}} = 2^{16}.$$

So the only values are $2^{16} = 65{,}536$ and $2^8 = 256$. Besides the standard value there is exactly 1 other.

Thus, the correct answer is **B**.

4. Find the degree measure of an angle whose complement is 25% of its supplement.

A 48

B 60

C 75

D 120

E 150

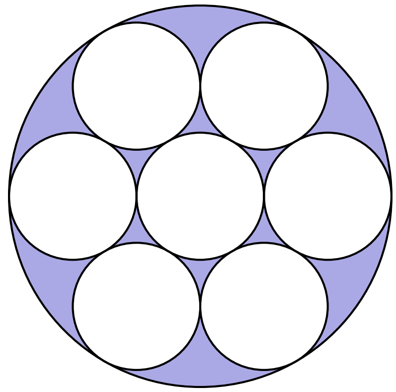
Solution:

Let the angle be x. Then $90 - x = \frac{1}{4}(180 - x)$, so $360 - 4x = 180 - x$.

This gives $3x = 180$, so $x = 60$.

Thus, the correct answer is **B**.

5. Each of the small circles in the figure has radius one. The innermost circle is tangent to the six circles that surround it, and each of those circles is tangent to the large circle and to its small-circle neighbors. Find the area of the shaded region.



A π

B 1.5π

C 2π

D 3π

E 3.5π

Solution:

Each of the six outer unit circles is tangent to the central unit circle, so its center is 2 units from the center. Adding one more radius, the large circle has radius 3 and area 9π.

The seven unit circles have total area 7π, so the shaded region has area $9\pi - 7\pi = 2\pi$.

Thus, the correct answer is **C**.

6. For how many positive integers m does there exist at least one positive integer n such that $m \cdot n \leq m + n$?

A 4

B 6

C 9

D 12

E infinitely many

Solution:

Taking $n = 1$, the inequality becomes $m \leq m + 1$, which holds for every positive integer m.

So every positive integer m works, and there are infinitely many.

Thus, the correct answer is **E**.

7. If an arc of 45° on circle A has the same length as an arc of 30° on circle B, then the ratio of the area of circle A to the area of circle B is

A $\frac{4}{9}$

B $\frac{2}{3}$

C $\frac{5}{6}$

D $\frac{3}{2}$

E $\frac{9}{4}$

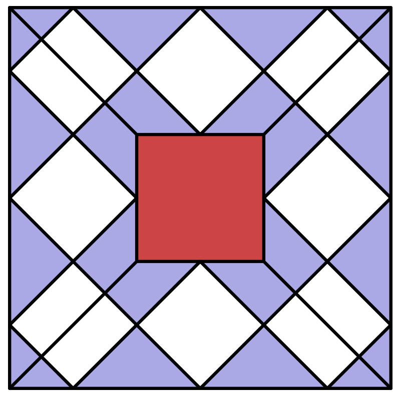
Solution:

Equal arc lengths give $\frac{45}{360} \cdot 2\pi R_A = \frac{30}{360} \cdot 2\pi R_B$, so $\frac{R_A}{R_B} = \frac{30}{45} = \frac{2}{3}$.

The ratio of areas is $\left(\frac{R_A}{R_B}\right)^2 = \frac{4}{9}$.

Thus, the correct answer is **A**.

8. Betsy designed a flag using blue triangles, small white squares, and a red center square, as shown. Let B be the total area of the blue triangles, W the total area of the white squares, and R the area of the red square. Which of the following is correct?



- A $B = W$
- B $W = R$
- C $B = R$
- D $3B = 2R$
- E $2R = W$

Solution:

Drawing the diagonals shown, the entire flag is tiled by congruent triangles. Counting, there are 24 triangles in the blue region, 24 in the white region, and 16 in the red square.

Since the blue and white regions contain the same number of triangles, $B = W$.

Thus, the correct answer is **A**.

9. Jamal wants to store 30 computer files on floppy disks, each of which has a capacity of 1.44 megabytes (mb). Three of his files require 0.8 mb of memory each, 12 more require 0.7 mb each, and the remaining 15 require 0.4 mb each. No file can be split between floppy disks. What is the minimal number of floppy disks that will hold all the files?

- A 12
- B 13
- C 14
- D 15
- E 16

Solution:

The files need $3(0.8) + 12(0.7) + 15(0.4) = 16.8$ mb, so at least $\dfrac{16.8}{1.44} = 11\frac{2}{3}$ disks by volume alone.

A disk containing a 0.8-mb file has room for only one more 0.4-mb file, leaving at least 0.24 mb unused. Across the three 0.8-mb files this wastes at least $3(0.24) = 0.72$ mb, over half a disk, forcing at least 13 disks.

Thirteen suffice: six disks each hold two 0.7-mb files, three disks each hold one 0.8-mb file plus one 0.4-mb file, and four disks each hold three 0.4-mb files.

Thus, the correct answer is **B**.

10. Sarah pours four ounces of coffee into an eight-ounce cup and four ounces of cream into a second cup of the same size. She then transfers half the coffee from the first cup to the second and, after stirring thoroughly, transfers half the liquid in the second cup back to the first. What fraction of the liquid in the first cup is now cream?

A $\frac{1}{4}$

B $\frac{1}{3}$

C $\frac{3}{8}$

D $\frac{2}{5}$

E $\frac{1}{2}$

Solution:

After transferring half the coffee, cup 1 has 2 oz coffee and cup 2 has 2 oz coffee and 4 oz cream, a total of 6 oz.

Transferring half of cup 2 back moves 1 oz coffee and 2 oz cream. Cup 1 then holds $2+1=3$ oz coffee and 2 oz cream.

The fraction that is cream is $\dfrac{2}{2+3}=\dfrac{2}{5}$.

Thus, the correct answer is **D**.

11. Mr. Earl E. Bird leaves his house for work at exactly 8:00 A.M. every morning. When he averages 40 miles per hour, he arrives at his workplace three minutes late. When he averages 60 miles per hour, he arrives three minutes early. At what average speed, in miles per hour, should Mr. Bird drive to arrive at his workplace precisely on time?

A 45

B 48

C 50

D 55

E 58

Solution:

Let t be the time in hours to arrive on time. Since three minutes is 0.05 hours, $40(t + 0.05) = 60(t - 0.05)$.

This gives $40t + 2 = 60t - 3,$ so $t = 0.25.$ The distance is $40(0.25 + 0.05) = 12$ miles, and the required speed is $\dfrac{12}{0.25} = 48$ miles per hour.

Thus, the correct answer is **B**.

12. Both roots of the quadratic equation

$$x^2 - 63x + k = 0$$

are prime numbers. The number of possible values of k is

A 0

B 1

C 2

D 4

E more than four

Solution:

If the roots are primes p and q, then by Vieta's formulas $p + q = 63$ and $pq = k$.

Since 63 is odd, one prime must be even, namely 2, and the other is 61. Both are prime, so $k = 2 \cdot 61 = 122$ is the only possible value.

Thus, the correct answer is **B**.

13. Two different positive numbers a and b each differ from their reciprocals by 1. What is $a+b$?

- A 1
- B 2
- C $\sqrt{5}$
- D $\sqrt{6}$
- E 3

Solution:

A positive number x differs from its reciprocal by 1 when $x-\dfrac{1}{x}=1$ or $x-\dfrac{1}{x}=-1$, i.e. $x^2-x-1=0$ or $x^2+x-1=0$.

The positive roots are $\dfrac{1+\sqrt{5}}{2}$ and $\dfrac{-1+\sqrt{5}}{2}$, which are reciprocals of each other. Their sum is $a+b=\sqrt{5}$.

Thus, the correct answer is **C**.

14. For all positive integers n, let $f(n) = \log_{2002} n^2$. Let

$$N = f(11) + f(13) + f(14).$$

Which of the following relations is true?

A $N > 1$

B $N = 1$

C $1 < N < 2$

D $N = 2$

E $N > 2$

Solution:

Using $\log a^2 = 2 \log a$ and adding logs,

$$N = \log_{2002} 11^2 + \log_{2002} 13^2 + \log_{2002} 14^2 = \log_{2002}(11 \cdot 13 \cdot 14)^2.$$

Since $11 \cdot 13 \cdot 14 = 2002$, this is $\log_{2002} 2002^2 = 2.$

Thus, the correct answer is **D**.

15. The mean, median, unique mode, and range of a collection of eight integers are all equal to 8. The largest integer that can be an element of this collection is

A 11

B 12

C 13

D 14

E 15

Solution:

The collection $6, 6, 6, 8, 8, 8, 8, 14$ has mean, median, unique mode, and range all equal to 8, so 14 is attainable.

Suppose the largest were 15. The range 8 forces the smallest to be 7, and the median 8 fixes the two middle values as $8, 8$. Then $7 + 8 + 8 + 15 = 38$, so the remaining four values sum to $64 - 38 = 26$, averaging 6.5. At least one would be below 7, contradicting the minimum. So 15 is impossible.

Thus, the correct answer is **D**.

16. Tina randomly selects two distinct numbers from the set $\{1, 2, 3, 4, 5\}$, and Sergio randomly selects a number from the set $\{1, 2, \ldots, 10\}$. The probability that Sergio's number is larger than the sum of the two numbers chosen by Tina is

A $\frac{2}{5}$

B $\frac{9}{20}$

C $\frac{1}{2}$

D $\frac{11}{20}$

E $\frac{24}{25}$

Solution:

Tina's ten pairs have sums $3, 4, 5, 5, 6, 6, 7, 7, 8, 9$. For a sum s, Sergio's number exceeds it with probability $\frac{10 - s}{10}$.

The corresponding values of $10 - s$ are $7, 6, 5, 5, 4, 4, 3, 3, 2, 1$, totaling 40. The overall probability is $\frac{40}{10 \cdot 10} = \frac{2}{5}$.

Thus, the correct answer is **A**.

17. Several sets of prime numbers, such as $\{7, 83, 421, 659\}$, use each of the nine nonzero digits exactly once. What is the smallest possible sum such a set of primes could have?

A 193

B 207

C 225

D 252

E 477

Solution:

The even digits $4, 6, 8$ cannot be the units digit of a multi-digit prime, so each must appear in a tens place or higher, contributing at least $40 + 60 + 80 = 180$. The other six digits contribute at least $1 + 2 + 3 + 5 + 7 + 9 = 27$, so the sum is at least 207.

This bound is achieved, for example by $\{2, 3, 5, 41, 67, 89\}$, whose sum is 207.

Thus, the correct answer is **B**.

18. Let C_1 and C_2 be circles defined by

$$(x-10)^2+y^2=36$$

and

$$(x+15)^2+y^2=81,$$

respectively. What is the length of the shortest line segment $\overline{PQ}$ that is tangent to C_1 at P and to C_2 at Q?

A 15

B 18

C 20

D 21

E 24

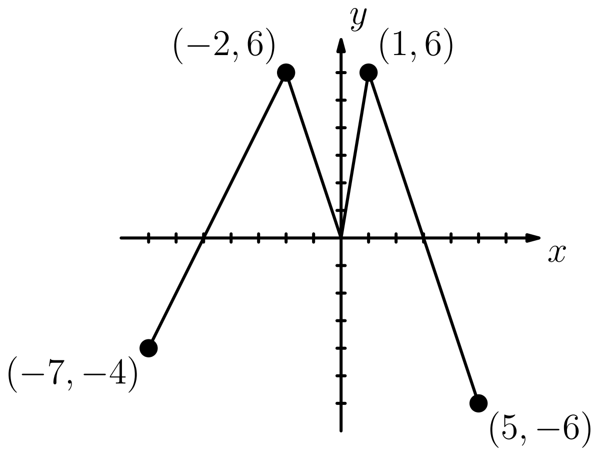
Solution:

The centers are $A=(10,0)$ and $B=(-15,0)$, with radii 6 and 9, so $AB=25$. The shortest tangent is the internal one, meeting $\overline{AB}$ at a point D that splits it in the ratio $6:9$, giving $D=(0,0)$.

The right triangles APD and BQD are similar with ratio $2:3$. Then $PD=\sqrt{10^2-6^2}=8$ and $QD=\sqrt{15^2-9^2}=12$, so $PQ=8+12=20$.

Thus, the correct answer is **C**.

19. The graph of the function f is shown below. How many solutions does the equation $f(f(x)) = 6$ have?



A 2

B 4

C 5

D 6

E 7

Solution:

The graph reaches 6 at $x = -2$ and $x = 1$, so $f(f(x)) = 6$ requires $f(x) = -2$ or $f(x) = 1$.

The horizontal line $y = -2$ meets the graph twice, and $y = 1$ meets it four times, giving $2 + 4 = 6$ solutions.

Thus, the correct answer is **D**.

20. Suppose that a and b are digits, not both nine and not both zero, and the repeating decimal $0.\overline{ab}$ is expressed as a fraction in lowest terms. How many different denominators are possible?

A 3

B 4

C 5

D 8

E 9

Solution:

Since $0.\overline{ab} = \dfrac{\overline{ab}}{99}$, the reduced denominator divides $99 = 3^2 \cdot 11$. The divisors are $1, 3, 9, 11, 33, 99.$

The denominator 1 would require $\overline{ab} = 99$, i.e. $a = b = 9$, which is excluded. Each of $3, 9, 11, 33, 99$ is achievable, giving 5 possible denominators.

Thus, the correct answer is **C**.

21. Consider the sequence of numbers $4, 7, 1, 8, 9, 7, 6, \ldots$ For $n > 2$, the nth term of the sequence is the units digit of the sum of the two previous terms. Let S_n denote the sum of the first n terms of this sequence. The smallest value of n for which $S_n > 10{,}000$ is

A 1992

B 1999

C 2001

D 2002

E 2004

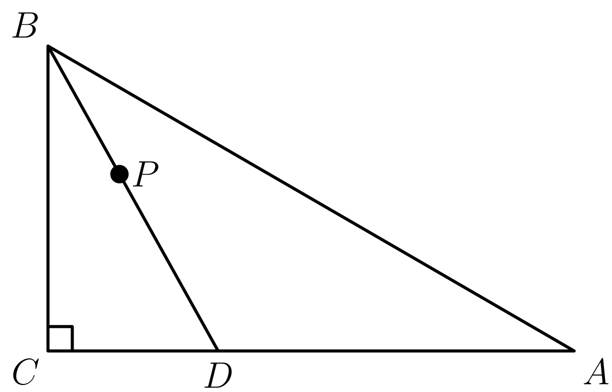
Solution:

Continuing the sequence gives $4, 7, 1, 8, 9, 7, 6, 3, 9, 2, 1, 3, 4, 7, 1, \ldots,$ which repeats with period 12. Each block of 12 terms sums to 60.

The largest k with $60k \leq 10{,}000$ is $k = 166$, giving $S_{12 \cdot 166} = 9960$. Adding the next terms $4, 7, 1, 8, 9, 7, 6$ contributes 42, pushing the total past $10{,}000$. So $n = 12 \cdot 166 + 7 = 1999$.

Thus, the correct answer is **B**.

22. Triangle ABC is a right triangle with $\angle ACB$ as its right angle, $m\angle ABC = 60^\circ$, and $AB = 10$. Let P be randomly chosen inside $\triangle ABC$, and extend $\overline{BP}$ to meet $\overline{AC}$ at D. What is the probability that $BD > 5\sqrt{2}$?



A $\dfrac{2-\sqrt{2}}{2}$

B $\dfrac{1}{3}$

C $\dfrac{3-\sqrt{3}}{3}$

D $\dfrac{1}{2}$

E $\dfrac{5-\sqrt{5}}{5}$

Solution:

Since $AB = 10$ and $\angle ABC = 60^\circ$, the 30-60-90 triangle has $BC = 5$ and $AC = 5\sqrt{3}$.

Place E on $\overline{AC}$ with $CE = 5$; then $BE = \sqrt{5^2 + 5^2} = 5\sqrt{2}$. As D moves along $\overline{AC}$, $BD = \sqrt{25 + CD^2}$ exceeds $5\sqrt{2}$ exactly when $CD > 5$, i.e. when D lies beyond E, which happens iff P is inside $\triangle ABE$.

The probability is $\dfrac{[ABE]}{[ABC]} = \dfrac{EA}{CA} = \dfrac{5\sqrt{3}-5}{5\sqrt{3}} = \dfrac{3-\sqrt{3}}{3}$.

Thus, the correct answer is **C**.

23. In triangle ABC, side $\overline{AC}$ and the perpendicular bisector of $\overline{BC}$ meet in point D, and $\overline{BD}$ bisects $\angle ABC$. If $AD = 9$ and $DC = 7$, what is the area of triangle ABD?

- A 14
- B 21
- C 28
- D $14\sqrt{5}$
- E $28\sqrt{5}$

Solution:

Since D lies on the perpendicular bisector of $\overline{BC}$, $DB = DC = 7$. The angle bisector $\overline{BD}$ gives $\dfrac{AB}{BC} = \dfrac{AD}{DC} = \dfrac{9}{7}$, so write $AB = 9x$ and $BC = 7x$.

Let $\theta = \angle ABD = \angle DBC$. In isosceles $\triangle BDC$, the foot of the perpendicular is the midpoint M of $\overline{BC}$, so $\cos\theta = \dfrac{BM}{BD} = \dfrac{7x/2}{7} = \dfrac{x}{2}$.

Applying the Law of Cosines in $\triangle ABD$:

$$9^2 = (9x)^2 + 7^2 - 2(9x)(7) \cdot \frac{x}{2},$$

which simplifies to $81 = 18x^2 + 49$, so $x = \dfrac{4}{3}$ and $AB = 12$.

Now $\triangle ABD$ has sides $9, 7, 12$. By Heron's formula with $s = 14$, the area is $\sqrt{14 \cdot 5 \cdot 7 \cdot 2} = \sqrt{980} = 14\sqrt{5}$.

Thus, the correct answer is **D**.

24. Find the number of ordered pairs of real numbers (a, b) such that $(a+bi)^{2002} = a - bi$.

A 1001

B 1002

C 2001

D 2002

E 2004

Solution:

Let $z = a + bi$. The equation is $z^{2002} = \overline{z}$. Taking magnitudes, $|z|^{2002} = |z|$, so $|z|\left(|z|^{2001} - 1\right) = 0$, giving $|z| = 0$ or $|z| = 1$.

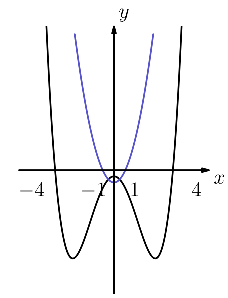
If $|z| = 0$, then $(a, b) = (0, 0)$, one solution. If $|z| = 1$, then $\overline{z} = \dfrac{1}{z}$, so $z^{2002} = \dfrac{1}{z}$, i.e. $z^{2003} = 1$, which has 2003 distinct roots.

Altogether there are $1 + 2003 = 2004$ ordered pairs.

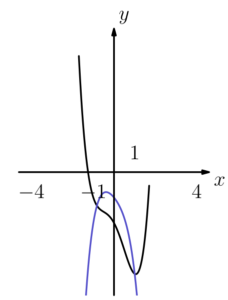
Thus, the correct answer is **E**.

25. The nonzero coefficients of a polynomial P with real coefficients are all replaced by their mean to form a polynomial Q. Which of the following could be a graph of $y = P(x)$ and $y = Q(x)$ over the interval $-4 \leq x \leq 4$?

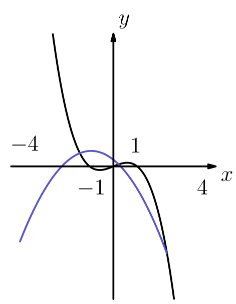
A



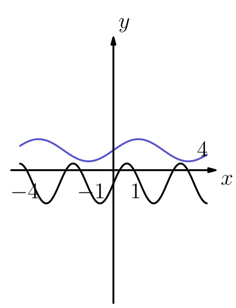
B



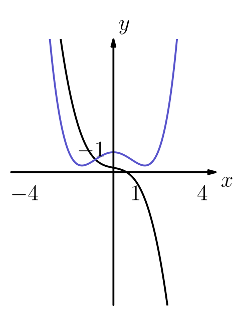
C



D



E



Solution:

Replacing the nonzero coefficients by their mean keeps the total of the coefficients unchanged, so P and Q have the same coefficient sum. Since $P(1)$ and $Q(1)$ each equal that sum, $P(1) = Q(1)$.

Therefore the graphs of $y = P(x)$ and $y = Q(x)$ must cross at $x = 1$. The only choice showing an intersection at $x = 1$ is graph B. (There, $P(x) = 2x^4 - 3x^2 - 3x - 4$ and $Q(x) = -2x^4 - 2x^2 - 2x - 2$.)

Thus, the correct answer is **B**.

Problems: https://live.poshenloh.com/past-contests/amc12/2002A