2001 AMC 12 Solutions

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1. The sum of two numbers is S. Suppose 3 is added to each number and then each of the resulting numbers is doubled. What is the sum of the final two numbers?

- A $2S + 3$
- B $3S + 2$
- C $3S + 6$
- D $2S + 6$
- E $2S + 12$

Solution:

Adding 3 to each number raises the sum from S to $S + 6$. Doubling each number doubles the sum, giving

$$2(S + 6) = 2S + 12.$$

Thus, the correct answer is **E**.

2. Let $P(n)$ and $S(n)$ denote the product and the sum, respectively, of the digits of the integer n. For example, $P(23) = 6$ and $S(23) = 5$. Suppose N is a two-digit number such that $N = P(N) + S(N)$. What is the units digit of N?

A 2

B 3

C 6

D 8

E 9

Solution:

Write $N = 10a + b$. Then $P(N) = ab$ and $S(N) = a + b$, so

$$10a + b = ab + a + b.$$

This reduces to $9a = ab$. Since $a \neq 0$, we can divide by a to get $b = 9$.

The units digit of N is 9.

Thus, the correct answer is **E**.

3. The state income tax where Kristin lives is levied at the rate of $p\%$ of the first $\$28000$ of annual income plus $(p+2)\%$ of any amount above $\$28000.$ Kristin noticed that the state income tax she paid amounted to $(p+0.25)\%$ of her annual income. What was her annual income?

- A $\$28000$
- B $\$32000$
- C $\$35000$
- D $\$42000$
- E $\$56000$

Solution:

Let her income be $x > 28000$ dollars. Writing the tax with both descriptions and multiplying by $100,$

$$p \cdot 28000 + (p+2)(x-28000) = (p+0.25)x.$$

Expanding, every term containing p cancels, leaving

$$2x - 56000 = 0.25x,$$

so $1.75x = 56000$ and $x = 32000.$

Thus, the correct answer is **B**.

4. The mean of three numbers is 10 more than the least of the numbers and 15 less than the greatest. The median of the three numbers is 5. What is their sum?

A 5

B 20

C 25

D 30

E 36

Solution:

Let m be the mean. The least number is $m-10$, the greatest is $m+15$, and the middle number is the median 5. Their sum is $3m$, so

$$(m-10)+5+(m+15)=3m.$$

This gives $m=10$, so the sum of the three numbers is $3m=30$.

Thus, the correct answer is **D**.

5. What is the product of all positive odd integers less than $10{,}000$?

A $\frac{10000!}{(5000!)^2}$

B $\frac{10000!}{2^{5000}}$

C $\frac{9999!}{2^{5000}}$

D $\frac{10000!}{2^{5000} \cdot 5000!}$

E $\frac{5000!}{2^{5000}}$

Solution:

The product of every integer from 1 to 10000 is $10000!$, so the product of the odd ones is $10000!$ divided by the product of the even ones.

The even numbers factor as

$$2 \cdot 4 \cdots 10000 = 2^{5000}(1 \cdot 2 \cdots 5000) = 2^{5000} \cdot 5000!.$$

Therefore the product of the odd integers is

$$\frac{10000!}{2^{5000} \cdot 5000!}.$$

Thus, the correct answer is **D**.

6. A telephone number has the form $ABC - DEF - GHIJ$, where each letter represents a different digit. The digits in each part of the number are in decreasing order; that is, $A > B > C$, $D > E > F$, and $G > H > I > J$. Furthermore, D, E, and F are consecutive even digits; G, H, I, and J are consecutive odd digits; and $A + B + C = 9$. Find A.

A 4

B 5

C 6

D 7

E 8

Solution:

The four consecutive decreasing odd digits $GHIJ$ are either 9753 or 7531, leaving one odd digit (1 or 9) for ABC.

Since $A + B + C = 9$ and the other two digits of ABC are even, the odd digit must be 1 (a 9 would force the two even digits to sum to 0). So the two even digits sum to 8.

The three consecutive decreasing even digits DEF are $864, 642$, or 420, leaving the even pairs $\{2, 0\}$, $\{8, 0\}$, or $\{8, 6\}$ for ABC. Only $\{8, 0\}$ sums to 8, so $ABC = 810$ and $A = 8$.

Thus, the correct answer is **E**.

7. A charity sells 140 benefit tickets for a total of $\$2001.$ Some tickets sell for full price (a whole dollar amount), and the rest sell for half price. How much money is raised by the full-price tickets?

A $\$782$

B $\$986$

C $\$1158$

D $\$1219$

E $\$1449$

Solution:

Let n tickets sell at full price p dollars. Then

$$np + (140 - n)\frac{p}{2} = 2001,$$

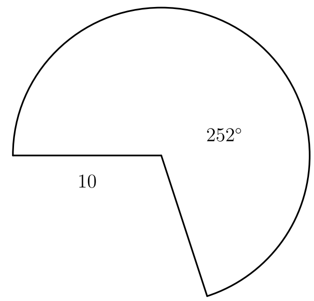
so $p(n + 140) = 4002 = 2 \cdot 3 \cdot 23 \cdot 29.$

Since $0 \leq n \leq 140,$ we need a factor of 4002 with $140 \leq n + 140 \leq 280.$ The only such factor is $174 = 2 \cdot 3 \cdot 29,$ giving $n = 34$ and $p = 23.$

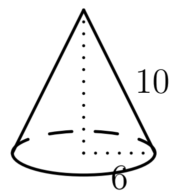
The full-price tickets raise $34 \cdot 23 = 782$ dollars.

Thus, the correct answer is **A**.

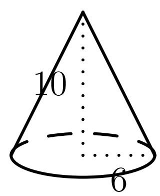
8. Which of the cones below can be formed from a 252° sector of a circle of radius 10 by aligning the two straight sides?



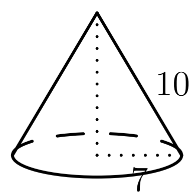
A



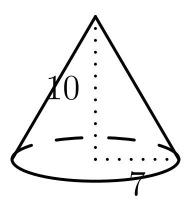
B



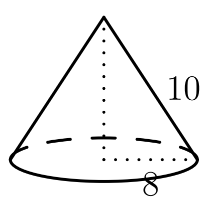
C



D



E



Solution:

When the sector is rolled into a cone, its radius 10 becomes the slant height, and its arc becomes the base circle.

The arc length is

$$\frac{252}{360} \cdot 2\pi(10) = \frac{7}{10} \cdot 20\pi = 14\pi,$$

so the base circumference is 14π and the base radius is $7.$

The cone therefore has base radius 7 and slant height $10,$ which is choice C.

Thus, the correct answer is **C**.

9. Let f be a function satisfying $f(xy) = \dfrac{f(x)}{y}$ for all positive real numbers x and $y.$ If $f(500) = 3,$ what is the value of $f(600)$?

A 1

B 2

C $\dfrac{5}{2}$

D 3

E $\dfrac{18}{5}$

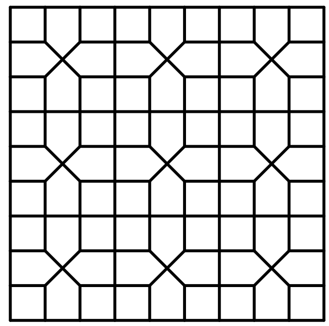
Solution:

Choose $x = 500$ and $y = \dfrac{6}{5}$ so that $xy = 600.$ Then

$$f(600) = \frac{f(500)}{6/5} = \frac{3}{6/5} = \frac{5}{2}.$$

Thus, the correct answer is **C**.

10. The plane is tiled by congruent squares and congruent pentagons as indicated. The percent of the plane that is enclosed by the pentagons is closest to



A 50

B 52

C 54

D 56

E 58

Solution:

The pattern repeats over a 3×3 block of nine unit squares. Four of these nine squares are not covered by pentagons; the rest of the area belongs to the pentagons.

So the pentagons enclose

$$1 - \frac{4}{9} = \frac{5}{9} = 55.5\overline{5}\%,$$

which is closest to 56.

Thus, the correct answer is **D**.

11. A box contains exactly five chips, three red and two white. Chips are randomly removed one at a time without replacement until all the red chips are drawn or all the white chips are drawn. What is the probability that the last chip drawn is white?

A $\frac{3}{10}$

B $\frac{2}{5}$

C $\frac{1}{2}$

D $\frac{3}{5}$

E $\frac{7}{10}$

Solution:

Imagine continuing until all five chips are removed. The process actually stops on a white chip exactly when the whites run out before the reds, i.e. when the last chip in the full ordering is red.

The last of the five chips is equally likely to be any chip, so it is red with probability $\frac{3}{5}$.

Thus, the correct answer is **D**.

12. How many positive integers not exceeding 2001 are multiples of 3 or 4 but not 5?

A 768

B 801

C 934

D 1067

E 1167

Solution:

Multiples of 3 or 4 up to 2001 number

$$667 + 500 - 166 = 1001,$$

using $\lfloor 2001/3 \rfloor = 667$, $\lfloor 2001/4 \rfloor = 500$, and $\lfloor 2001/12 \rfloor = 166$.

Among these, the ones divisible by 5 are multiples of 15 or 20:

$$133 + 100 - 33 = 200,$$

using $\lfloor 2001/15 \rfloor = 133$, $\lfloor 2001/20 \rfloor = 100$, and $\lfloor 2001/60 \rfloor = 33$.

The count is $1001 - 200 = 801$.

Thus, the correct answer is **B**.

13. The parabola with equation $y = ax^2 + bx + c$ and vertex (h, k) is reflected about the line $y = k$. This results in the parabola with equation $y = dx^2 + ex + f$. Which of the following equals $a + b + c + d + e + f$?

A $2b$

B $2c$

C $2a + 2b$

D $2h$

E $2k$

Solution:

The value $a + b + c$ is the first parabola at $x = 1$, and $d + e + f$ is the reflected parabola at $x = 1$.

Reflecting the curve about $y = k$ replaces each height y by $2k - y$. So at $x = 1$ the two heights sum to

$$(a + b + c) + (d + e + f) = 2k.$$

Thus, the correct answer is **E**.

14. Given the nine-sided regular polygon $A_1A_2A_3A_4A_5A_6A_7A_8A_9$, how many distinct equilateral triangles in the plane of the polygon have at least two vertices in the set $\{A_1, A_2, \ldots, A_9\}$?

A 30

B 36

C 63

D 66

E 72

Solution:

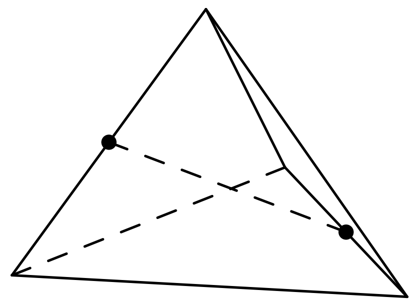
Each of the $\binom{9}{2} = 36$ pairs of vertices is a side of exactly two equilateral triangles, giving 72 triangles counted with multiplicity.

The triangles $A_1A_4A_7$, $A_2A_5A_8$, and $A_3A_6A_9$ have all three vertices in the set, so each is counted three times instead of once, an overcount of 2 apiece.

The number of distinct triangles is $72 - 3 \cdot 2 = 66.$

Thus, the correct answer is **D**.

15. An insect lives on the surface of a regular tetrahedron with edges of length 1. It wishes to travel on the surface of the tetrahedron from the midpoint of one edge to the midpoint of the opposite edge. What is the length of the shortest such trip? (Note: Two edges of a tetrahedron are *opposite* if they have no common endpoint.)



A $\frac{1}{2}\sqrt{3}$

B 1

C $\sqrt{2}$

D $\frac{3}{2}$

E 2

Solution:

Unfold the two faces the insect crosses into the plane. They form a rhombus of side 1 made of two equilateral triangles.

The two opposite-edge midpoints become the midpoints of opposite sides of this rhombus, which are exactly 1 unit apart along a straight segment. Folding back preserves the length, so the shortest trip is 1.

Thus, the correct answer is **B**.

16. A spider has one sock and one shoe for each of its eight legs. In how many different orders can the spider put on its socks and shoes, assuming that, on each leg, the sock must be put on before the shoe?

- A $8!$
- B $2^8\, 8!$
- C $(8!)^2$
- D $\frac{16!}{2^8}$
- E $16!$

Solution:

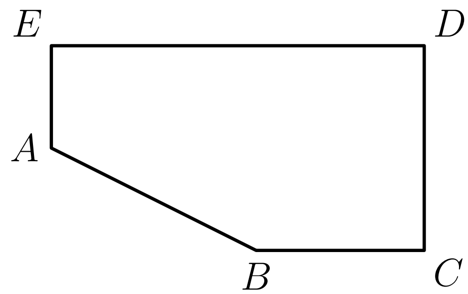
Think of the 16 items (8 socks and 8 shoes) arranged in some order: there are $16!$ arrangements.

For each leg, the sock comes before the shoe in exactly half of all arrangements. Imposing this on all eight legs independently divides by 2^8, giving

$$\frac{16!}{2^8}.$$

Thus, the correct answer is **D**.

17. A point P is selected at random from the interior of the pentagon with vertices $A = (0, 2), B = (4, 0)$, $C = (2\pi + 1, 0), D = (2\pi + 1, 4)$, and $E = (0, 4)$. What is the probability that $\angle APB$ is obtuse?



A $\frac{1}{5}$

B $\frac{1}{4}$

C $\frac{5}{16}$

D $\frac{3}{8}$

E $\frac{1}{2}$

Solution:

$\angle APB = 90^\circ$ when P is on the circle with diameter AB, centered at $(2, 1)$ with radius $\frac{|AB|}{2} = \frac{\sqrt{20}}{2} = \sqrt{5}$. The angle is obtuse when P is inside this circle.

The relevant half-disk lies wholly within the pentagon, with area $\frac{1}{2}\pi(\sqrt{5})^2 = \frac{5\pi}{2}$.

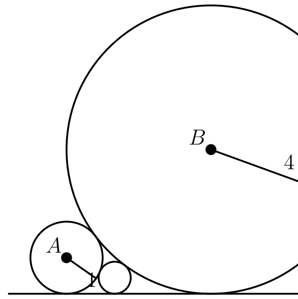
The pentagon is the rectangle with corners $(0, 0), C, D, E$ minus triangle OAB, so its area is

$$4(2\pi + 1) - \frac{1}{2}(2)(4) = 8\pi.$$

The probability is $\frac{5\pi/2}{8\pi} = \frac{5}{16}$.

Thus, the correct answer is **C**.

18. A circle centered at A with a radius of 1 and a circle centered at B with a radius of 4 are externally tangent. A third circle is tangent to the first two and to one of their common external tangents as shown. The radius of the third circle is



A $\frac{1}{3}$

B $\frac{2}{5}$

C $\frac{5}{12}$

D $\frac{4}{9}$

E $\frac{1}{2}$

Solution:

When two mutually tangent circles of radii r and s both rest on a line, the distance between their points of tangency is $2\sqrt{rs}$.

The big circles' contact points are $2\sqrt{1 \cdot 4} = 4$ apart. Placing the small circle of radius x between them, its two tangent distances add up:

$$2\sqrt{1 \cdot x} + 2\sqrt{4 \cdot x} = 4.$$

Then $6\sqrt{x} = 4,$ so $\sqrt{x} = \frac{2}{3}$ and $x = \frac{4}{9}$.

Thus, the correct answer is **D**.

19. The polynomial $P(x) = x^3 + ax^2 + bx + c$ has the property that the mean of its zeros, the product of its zeros, and the sum of its coefficients are all equal. If the y-intercept of the graph of $y = P(x)$ is 2, what is b?

A -11

B -10

C -9

D 1

E 5

Solution:

The y-intercept is $P(0) = c = 2$. By Vieta's formulas the product of the zeros is $-c = -2$, the mean of the zeros is $-\frac{a}{3}$, and the sum of the coefficients is $P(1) = 1 + a + b + c$.

All three are equal to -2. From $-\frac{a}{3} = -2$ we get $a = 6$.

Then $1 + a + b + c = -2$ becomes $1 + 6 + b + 2 = -2$, so $b = -11$.

Thus, the correct answer is **A**.

20. Points $A = (3, 9), B = (1, 1), C = (5, 3)$, and $D = (a, b)$ lie in the first quadrant and are the vertices of quadrilateral $ABCD$. The quadrilateral formed by joining the midpoints of $\overline{AB}, \overline{BC}, \overline{CD}$, and $\overline{DA}$ is a square. What is the sum of the coordinates of point D?

- A 7
- B 9
- C 10
- D 12
- E 16

Solution:

The midpoints are $M = (2, 5)$ of $\overline{AB}$ and $N = (3, 2)$ of $\overline{BC}$.

For the midpoint quadrilateral to be a square, consecutive sides are perpendicular and equal. With $\overrightarrow{NM} = \langle -1, 3 \rangle$, the side $\overrightarrow{MQ}$ to the midpoint Q of $\overline{DA}$ must be $\langle 3, 1 \rangle$, so $Q = (5, 6)$.

Since Q is the midpoint of $\overline{DA}$ and $A = (3, 9)$, we get $D = 2Q - A = (7, 3)$. The sum of its coordinates is $7 + 3 = 10$.

Thus, the correct answer is **C**.

21. Four positive integers $a, b, c,$ and d have a product of $8!$ and satisfy

$$ab + a + b = 524,$$

$$bc + b + c = 146,$$

$$cd + c + d = 104.$$

What is $a - d$?

- A 4
- B 6
- C 8
- D 10
- E 12

Solution:

Adding 1 to each equation factors the left sides:

$$(a+1)(b+1) = 525 = 3 \cdot 5^2 \cdot 7, \quad (b+1)(c+1) = 147 = 3 \cdot 7^2, \quad (c+1)(d+1) = 105 = 3 \cdot 5 \cdot 7.$$

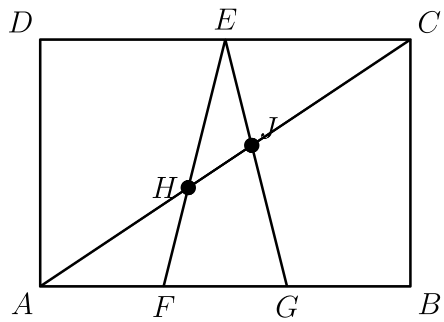
Since 525 has a factor of 25 while 147 is not divisible by 5, the factor $a+1$ must carry the 25. Among divisors of 525, only $a+1=25$ makes $a=24$ divide $8! = 40320$.

Then $b+1=21, c+1=7,$ and $d+1=15$, giving $b=20, c=6, d=14$. (Indeed $24 \cdot 20 \cdot 6 \cdot 14 = 40320 = 8!$.)

So $a - d = 24 - 14 = 10.$

Thus, the correct answer is **D**.

22. In rectangle $ABCD$, points F and G lie on $\overline{AB}$ so that $AF = FG = GB$ and E is the midpoint of $\overline{DC}$. Also, $\overline{AC}$ intersects $\overline{EF}$ at H and $\overline{EG}$ at J. The area of rectangle $ABCD$ is 70. Find the area of triangle EHJ.



A $\frac{5}{2}$

B $\frac{35}{12}$

C 3

D $\frac{7}{2}$

E $\frac{35}{8}$

Solution:

Triangle EFG has base $FG = \frac{1}{3}AB$ and height equal to the rectangle's height, so its area is $\frac{1}{6}(70) = \frac{35}{3}$.

Because $EC \parallel AF$, triangles AFH and CEH are similar with ratio $\frac{EC}{AF} = \frac{3}{2}$, so $\frac{EH}{EF} = \frac{3}{5}$. Likewise $\frac{EJ}{EG} = \frac{3}{7}$.

Then $\frac{[EHJ]}{[EFG]} = \frac{EH}{EF} \cdot \frac{EJ}{EG} = \frac{3}{5} \cdot \frac{3}{7} = \frac{9}{35}$, giving

$$[EHJ] = \frac{9}{35} \cdot \frac{35}{3} = 3.$$

Thus, the correct answer is **C**.

23. A polynomial of degree four with leading coefficient 1 and integer coefficients has two real zeros, both of which are integers. Which of the following can also be a zero of the polynomial?

A $\frac{1+i\sqrt{11}}{2}$

B $\frac{1+i}{2}$

C $\frac{1}{2}+i$

D $1+\frac{i}{2}$

E $\frac{1+i\sqrt{13}}{2}$

Solution:

Writing $P(x)=(x-r)(x-s)(x^2+\alpha x+\beta)$ with integer roots $r, s,$ matching coefficients forces α and β to be integers.

The other two zeros are

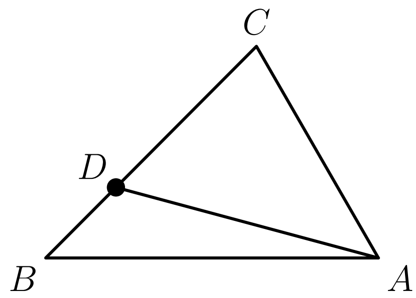
$$-\frac{\alpha}{2}\pm\frac{i\sqrt{4\beta-\alpha^2}}{2}.$$

A real part of $\frac{1}{2}$ requires $\alpha=-1,$ making the imaginary part $\frac{\sqrt{4\beta-1}}{2}.$

Choice A needs $\sqrt{4\beta-1}=\sqrt{11},$ i.e. $\beta=3,$ an integer, so it works. The other choices force a non-integer β (for example choice E needs $\beta=3.5,$ and choice D needs $\alpha=-2$ with $\beta=\frac{5}{4}$).

Thus, the correct answer is **A**.

24. In triangle ABC, $\angle ABC = 45°$. Point D is on $\overline{BC}$ so that $2 \cdot BD = CD$ and $\angle DAB = 15°$. Find $\angle ACB$.



A $54°$

B $60°$

C $72°$

D $75°$

E $90°$

Solution:

Let E be the foot of the perpendicular from C to line AD. The exterior angle of $\triangle ADB$ gives $\angle ADC = 15° + 45° = 60°$, so $\triangle CDE$ is a 30-60-90 triangle with $DE = \frac{1}{2}CD = BD$.

Then $\triangle BDE$ is isosceles with $\angle EBD = \angle BED = 30°$, and since $\angle ECB = 30°$ too, $\triangle BEC$ is isosceles with $BE = EC$.

Also $\angle ABE = 45° - 30° = 15° = \angle EAB$, so $\triangle ABE$ is isosceles with $AE = BE$. Hence $AE = BE = EC$, making right triangle AEC isosceles with $\angle ECA = 45°$.

Therefore $\angle ACB = \angle ECA + \angle ECD = 45° + 30° = 75°$.

Thus, the correct answer is **D**.

25. Consider sequences of positive real numbers of the form $x, 2000, y, \ldots,$ in which every term after the first is 1 less than the product of its two immediate neighbors. For how many different values of x does the term 2001 appear somewhere in the sequence?

A 1

B 2

C 3

D 4

E more than 4

Solution:

If a, b, c are consecutive terms then $b = ac - 1$, so $c = \dfrac{1+b}{a}$. Applying this repeatedly, the first five terms are

$$a,\ b,\ \frac{1+b}{a},\ \frac{1+a+b}{ab},\ \frac{1+a}{b},$$

after which a and b recur, so the sequence is periodic with period 5.

Here $b = 2000$ is the second term. The value 2001 can be placed in any one of the other four of the five distinct positions, and each choice determines $x = a$ uniquely and yields a valid sequence of positive reals.

So there are 4 values of x.

Thus, the correct answer is **D**.

Problems: https://live.poshenloh.com/past-contests/amc12/2001