

2000 AMC 12 Solutions

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1. In the year 2001, the United States will host the International Mathematical Olympiad. Let I , M , and O be distinct positive integers such that the product $I \cdot M \cdot O = 2001$. What is the largest possible value of the sum $I + M + O$?

A 23

B 55

C 99

D 111

E 671

Solution:

Factoring gives $2001 = 3 \cdot 23 \cdot 29$.

To maximize the sum of three distinct positive integers with this product, take $I = 1$, $M = 3$, and $O = 23 \cdot 29 = 667$.

The largest sum is $1 + 3 + 667 = 671$.

Thus, the correct answer is **E**.

2. What is $2000(2000^{2000})$?

A 2000^{2001}

B 4000^{2000}

C 2000^{4000}

D $4,000,000^{2000}$

E $2000^{4,000,000}$

Solution:

Writing $2000 = 2000^1$, we get

$$2000^1 \cdot 2000^{2000} = 2000^{1+2000} = 2000^{2001}.$$

All of the other options are larger than this.

Thus, the correct answer is **A**.

3. Each day, Jenny ate 20% of the jellybeans that were in her jar at the beginning of that day. At the end of the second day, 32 remained. How many jellybeans were in the jar originally?

- A 40
- B 50**
- C 55
- D 60
- E 75

Solution:

Since 20% is eaten each day, 80% remains at the end of each day. If x is the original number, then

$$(0.8)^2x = 32.$$

Solving gives $0.64x = 32$, so $x = 50$.

Thus, the correct answer is **B**.

4. The Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, 21, \dots$ starts with two 1s, and each term afterwards is the sum of its two predecessors. Which one of the ten digits is the last to appear in the units position of a number in the Fibonacci sequence?

A 0

B 4

C 6

D 7

E 9

Solution:

The sequence of units digits begins

$1, 1, 2, 3, 5, 8, 3, 1, 4, 5, 9, 4, 3, 7, 0, 7, 7, 4, 1, 5, 6, \dots$

Scanning this list, the digit 6 is the last of the ten digits to appear.

Thus, the correct answer is **C**.

5. If $|x - 2| = p$, where $x < 2$, then what is $x - p$?

A -2

B 2

C $2 - 2p$

D $2p - 2$

E $|2p - 2|$

Solution:

Since $x < 2$, we have $|x - 2| = 2 - x = p$, so $x = 2 - p$.

Then

$$x - p = (2 - p) - p = 2 - 2p.$$

Thus, the correct answer is **C**.

6. Two different prime numbers between 4 and 18 are chosen. When their sum is subtracted from their product, which of the following numbers could be obtained?

- A 21
- B 60
- C 119
- D 180
- E 231

Solution:

The primes between 4 and 18 are 5, 7, 11, 13, and 17.

For two such primes, $xy - (x + y) = (x - 1)(y - 1) - 1$ is a product of two even numbers minus 1, hence odd. Among the choices, only 119 is odd.

Indeed, $11 \cdot 13 - (11 + 13) = 143 - 24 = 119$.

Thus, the correct answer is **C**.

7. How many positive integers b have the property that $\log_b 729$ is a positive integer?

- A 0
- B 1
- C 2
- D 3
- E 4

Solution:

If $\log_b 729 = n$, then $b^n = 729 = 3^6$, so n must be a positive divisor of 6.

The possibilities $n = 1, 2, 3, 6$ give $b = 3^6, 3^3, 3^2, 3^1$, that is, 729, 27, 9, and 3.

There are 4 such values of b .

Thus, the correct answer is **E**.

8. Figures 0, 1, 2, and 3 consist of 1, 5, 13, and 25 nonoverlapping unit squares, respectively. If the pattern were continued, how many nonoverlapping unit squares would there be in figure 100?

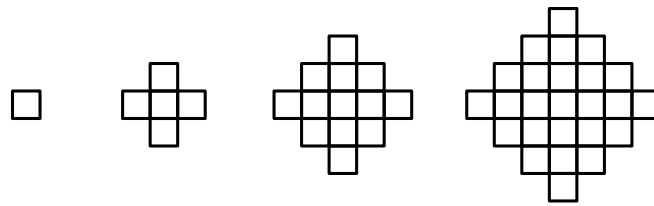


figure 0 figure 1 figure 2 figure 3

- A 10401
- B 19801
- C 20201
- D 39801
- E 40801

Solution:

Figure n is a diamond whose row lengths increase through the odd numbers and back down, giving a total of $n^2 + (n + 1)^2$ unit squares. This matches 1, 5, 13, 25 for $n = 0, 1, 2, 3$.

Therefore figure 100 has

$$100^2 + 101^2 = 10000 + 10201 = 20201$$

unit squares.

Thus, the correct answer is **C**.

9. Mrs. Walter gave an exam in a mathematics class of five students. She entered the scores in random order into a spreadsheet, which recalculated the class average after each score was entered. Mrs. Walter noticed that after each score was entered, the average was always an integer. The scores (listed in ascending order) were 71, 76, 80, 82, and 91. What was the last score Mrs. Walter entered?

A 71

B 76

C 80

D 82

E 91

Solution:

The total is $71 + 76 + 80 + 82 + 91 = 400$, which is divisible by 5. The sum of the first three scores must be divisible by 3.

Modulo 3, the scores are 2, 1, 2, 1, 1. The only triple summing to a multiple of 3 is $76 + 82 + 91 = 249$, so these are the first three (with 91 third, since the first two, 76 and 82, must have equal parity).

Since $249 \equiv 1 \pmod{4}$, the fourth score must be $\equiv 3 \pmod{4}$, which is 71. That leaves 80 as the last score entered.

Thus, the correct answer is **C**.

10. The point $P = (1, 2, 3)$ is reflected in the xy -plane, then its image Q is rotated by 180° about the x -axis to produce R , and finally, R is translated by 5 units in the positive y direction to produce S . What are the coordinates of S ?

A $(1, 7, -3)$

B $(-1, 7, -3)$

C $(-1, -2, 8)$

D $(-1, 3, 3)$

E $(1, 3, 3)$

Solution:

Reflecting $(1, 2, 3)$ in the xy -plane gives $Q = (1, 2, -3)$.

Rotating 180° about the x -axis negates y and z , giving $R = (1, -2, 3)$.

Translating 5 units in the positive y direction gives $S = (1, 3, 3)$.

Thus, the correct answer is **E**.

11. Two non-zero real numbers, a and b , satisfy $ab = a - b$. Find a possible value of

$$\frac{a}{b} + \frac{b}{a} - ab.$$

A -2

B $-\frac{1}{2}$

C $\frac{1}{3}$

D $\frac{1}{2}$

E 2

Solution:

Combining over a common denominator,

$$\frac{a}{b} + \frac{b}{a} - ab = \frac{a^2 + b^2 - (ab)^2}{ab}.$$

Replacing ab with $a - b$ in the numerator,

$$a^2 + b^2 - (a - b)^2 = a^2 + b^2 - (a^2 - 2ab + b^2) = 2ab.$$

Therefore the expression equals $\frac{2ab}{ab} = 2$.

Thus, the correct answer is **E**.

12. Let A , M , and C be nonnegative integers such that $A + M + C = 12$. What is the maximum value of

$$A \cdot M \cdot C + A \cdot M + M \cdot C + C \cdot A?$$

- A 62
- B 72
- C 92
- D 102
- E 112

Solution:

Observe that

$$AMC + AM + MC + CA = (A + 1)(M + 1)(C + 1) - (A + M + C) - 1.$$

Since $A + M + C = 12$, this equals $(A + 1)(M + 1)(C + 1) - 13$. The three factors sum to 15, so their product is maximized when each equals 5, giving $5^3 = 125$.

The maximum value is $125 - 13 = 112$.

Thus, the correct answer is **E**.

13. One morning each member of Angela's family drank an 8-ounce mixture of coffee with milk. The amounts of coffee and milk varied from cup to cup, but were never zero. Angela drank a quarter of the total amount of milk and a sixth of the total amount of coffee. How many people are in the family?

- A 3
- B 4
- C 5
- D 6
- E 7

Solution:

Measure amounts in 8-ounce cups, so Angela's cup holds c coffee and m milk with $c + m = 1$.

Since Angela drank a sixth of the coffee, the total coffee is $6c$; since she drank a quarter of the milk, the total milk is $4m$. The number of people equals the total number of cups,

$$6c + 4m = 6c + 4(1 - c) = 4 + 2c.$$

This is an integer only when $2c$ is an integer, and since $0 < c < 1$ this forces $c = \frac{1}{2}$, giving $4 + 1 = 5$ people.

Thus, the correct answer is **C**.

14. When the mean, median, and mode of the list

$$10, 2, 5, 2, 4, 2, x$$

are arranged in increasing order, they form a non-constant arithmetic progression. What is the sum of all possible real values of x ?

- A 3
- B 6
- C 9
- D 17
- E 20**

Solution:

The six fixed numbers sum to 25, so the mean is $\frac{25 + x}{7}$, and the mode is 2. If $x \leq 2$, then 2 is both median and mode, forcing a constant progression, so $x > 2$.

Case $2 < x < 4$: the median is x . Requiring $2, x, \frac{25 + x}{7}$ to form an arithmetic progression yields $x = 3$ as the only value in this range.

Case $x \geq 4$: the median is 4, and the progression 2, 4, 6 forces the mean to be 6, so $\frac{25 + x}{7} = 6$, giving $x = 17$.

The sum of all possible values is $3 + 17 = 20$.

Thus, the correct answer is **E**.

15. Let f be a function for which $f\left(\frac{x}{3}\right) = x^2 + x + 1$. Find the sum of all values of z for which $f(3z) = 7$.

A $-\frac{1}{3}$

B $-\frac{1}{9}$

C 0

D $\frac{5}{9}$

E $\frac{5}{3}$

Solution:

Setting $\frac{x}{3} = 3z$ gives $x = 9z$, so

$$f(3z) = (9z)^2 + 9z + 1 = 81z^2 + 9z + 1 = 7.$$

This rearranges to $81z^2 + 9z - 6 = 0$.

By the sum-of-roots formula, the sum of the values of z is $-\frac{9}{81} = -\frac{1}{9}$.

Thus, the correct answer is **B**.

16. A checkerboard of 13 rows and 17 columns has a number written in each square, beginning in the upper left corner, so that the first row is numbered 1, 2, ..., 17, the second row 18, 19, ..., 34, and so on down the board. If the board is renumbered so that the left column, top to bottom, is 1, 2, ..., 13, the second column 14, 15, ..., 26 and so on across the board, some squares have the same numbers in both numbering systems. Find the sum of the numbers in these squares (under either system).

A 222

B 333

C 444

D 555

E 666

Solution:

The square (m, n) is numbered $17(m - 1) + n$ originally and $13(n - 1) + m$ after renumbering. Setting these equal gives

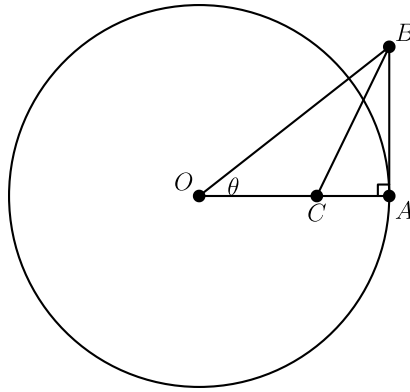
$$4m - 3n = 1.$$

The solutions with $1 \leq m \leq 13$ and $1 \leq n \leq 17$ are $(1, 1)$, $(4, 5)$, $(7, 9)$, $(10, 13)$, and $(13, 17)$.

These squares hold the numbers 1, 56, 111, 166, and 221, whose sum is 555.

Thus, the correct answer is **D**.

17. A circle centered at O has radius 1 and contains the point A . Segment AB is tangent to the circle at A and $\angle AOB = \theta$. If point C lies on \overline{OA} and BC bisects $\angle ABO$, then what is OC ?



- A $\sec^2 \theta - \tan \theta$
- B $\frac{1}{2}$
- C $\frac{\cos^2 \theta}{1 + \sin \theta}$
- D $\frac{1}{1 + \sin \theta}$**
- E $\frac{\sin \theta}{\cos^2 \theta}$

Solution:

Because $OA = 1$ and AB is tangent at A , angle OAB is right, so

$$BA = \tan \theta, \quad OB = \sec \theta.$$

Since BC bisects $\angle ABO$, the angle bisector theorem gives $\frac{OC}{CA} = \frac{OB}{BA}$. Using $OC + CA = OA = 1$,

$$OC = \frac{OB}{OB + BA} = \frac{\sec \theta}{\sec \theta + \tan \theta}.$$

Multiplying numerator and denominator by $\cos \theta$ gives $OC = \frac{1}{1 + \sin \theta}$.

Thus, the correct answer is **D**.

18. In year N , the 300th day of the year is a Tuesday. In year $N + 1$, the 200th day is also a Tuesday. On what day of the week did the 100th day of year $N - 1$ occur?

A Thursday

B Friday

C Saturday

D Sunday

E Monday

Solution:

From day 300 of year N to day 200 of year $N + 1$, the number of days is $365 - 300 + 200 = 265$ if N is not a leap year. But $265 = 7 \cdot 37 + 6$, which would land on a Monday, not a Tuesday.

So year N is a leap year, and the gap is $266 = 7 \cdot 38$ days, giving a Tuesday as stated. It follows that year $N - 1$ is not a leap year.

The 100th day of year $N - 1$ precedes the Tuesday on day 300 of year N by $365 - 100 + 300 = 565$ days. Since $565 = 7 \cdot 80 + 5$, that day is 5 weekdays before Tuesday, namely Thursday.

Thus, the correct answer is **A**.

19. In triangle ABC , $AB = 13$, $BC = 14$, and $AC = 15$. Let D denote the midpoint of \overline{BC} and let E denote the intersection of \overline{BC} with the bisector of angle BAC . Which of the following is closest to the area of triangle ADE ?

- A 2
- B 2.5
- C 3
- D 3.5
- E 4

Solution:

By Heron's formula, the area of $\triangle ABC$ is $\sqrt{21 \cdot 8 \cdot 7 \cdot 6} = 84$, so the altitude from A to BC is $\frac{2 \cdot 84}{14} = 12$.

The midpoint D is 7 from B . The bisector from A meets BC at E with $BE : EC = AB : AC = 13 : 15$, so $BE = 14 \cdot \frac{13}{28} = 6.5$.

Both D and E lie on BC , so $\triangle ADE$ has base $DE = 7 - 6.5 = 0.5$ and altitude 12, giving area

$$\frac{1}{2} \cdot 0.5 \cdot 12 = 3.$$

Thus, the correct answer is **C**.

20. If x , y , and z are positive numbers satisfying

$$x + \frac{1}{y} = 4, \quad y + \frac{1}{z} = 1, \quad \text{and} \quad z + \frac{1}{x} = \frac{7}{3},$$

then what is xyz ?

A $\frac{2}{3}$

B 1

C $\frac{4}{3}$

D 2

E $\frac{7}{3}$

Solution:

Adding the three equations gives

$$\left(x + \frac{1}{y}\right) + \left(y + \frac{1}{z}\right) + \left(z + \frac{1}{x}\right) = 4 + 1 + \frac{7}{3} = \frac{22}{3}.$$

Multiplying them gives

$$4 \cdot 1 \cdot \frac{7}{3} = \frac{28}{3}.$$

Expanding the product,

$$\left(x + \frac{1}{y}\right) \left(y + \frac{1}{z}\right) \left(z + \frac{1}{x}\right) = xyz + \left(x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) + \frac{1}{xyz}.$$

The middle group is the sum $\frac{22}{3}$, so $xyz + \frac{1}{xyz} = \frac{28}{3} - \frac{22}{3} = 2$.

Hence $(xyz - 1)^2 = 0$, so $xyz = 1$.

Thus, the correct answer is **B**.

21. Through a point on the hypotenuse of a right triangle, lines are drawn parallel to the legs of the triangle so that the triangle is divided into a square and two smaller right triangles. The area of one of the two small right triangles is m times the area of the square. What is the ratio of the area of the other small right triangle to the area of the square?

A $\frac{1}{2m + 1}$

B m

C $1 - m$

D $\frac{1}{4m}$

E $\frac{1}{8m^2}$

Solution:

Let the square have side 1. The small triangle sharing one side of the square has a perpendicular leg r , so its area is $\frac{1}{2} \cdot 1 \cdot r = m$, giving $r = 2m$.

The two small triangles are similar, so the other triangle's leg along the square is $\frac{1}{r}$, and its area is

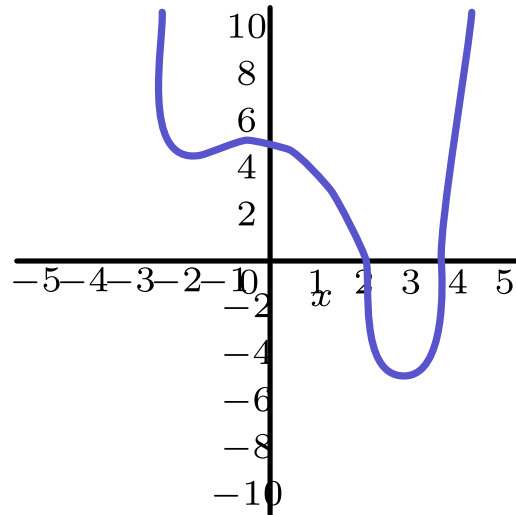
$$\frac{1}{2} \cdot 1 \cdot \frac{1}{r} = \frac{1}{2r} = \frac{1}{4m}.$$

Thus, the correct answer is **D**.

22. The graph below shows a portion of the curve defined by the quartic polynomial

$$P(x) = x^4 + ax^3 + bx^2 + cx + d.$$

Which of the following is the smallest?



- A $P(-1)$
- B The product of the zeros of P
- C The product of the non-real zeros of P**
- D The sum of the coefficients of P
- E The sum of the real zeros of P

Solution:

The graph crosses the x -axis exactly twice, both times at positive values, so P has two real zeros and two non-real (complex conjugate) zeros.

Reading off the graph: the sum of the coefficients is $P(1) > 3$; $P(-1) > 4$; the sum of the real zeros is greater than 4.5; and the product of all zeros is d , the y -intercept, which is less than 6.

The product of the real zeros is greater than 4.5, so the product of the non-real zeros, equal to $\frac{\text{product of all zeros}}{\text{product of real zeros}}$, is less than $\frac{6}{4.5} < 2$.

This is smaller than every other listed quantity.

Thus, the correct answer is **C**.

23. Professor Gamble buys a lottery ticket, which requires that he pick six different integers from 1 through 46, inclusive. He chooses his numbers so that the sum of the base-ten logarithms of his six numbers is an integer. It so happens that the integers on the winning ticket have the same property -- the sum of the base-ten logarithms is an integer. What is the probability that Professor Gamble holds the winning ticket?

A $\frac{1}{5}$

B $\frac{1}{4}$

C $\frac{1}{3}$

D $\frac{1}{2}$

E 1

Solution:

The sum of the logarithms is an integer k exactly when the product of the six numbers is 10^k . Since $10 = 2 \cdot 5$, each chosen number must be of the form $2^a 5^b$, so it comes from

$$1, 2, 4, 5, 8, 10, 16, 20, 25, 32, 40.$$

For each, record the excess of factors of 2 over factors of 5:

0, 1, 2, -1, 3, 0, 4, 1, -2, 5, 2. The product is a power of 10 only if the six chosen values have equal totals of 2s and 5s, i.e. their excesses sum to 0.

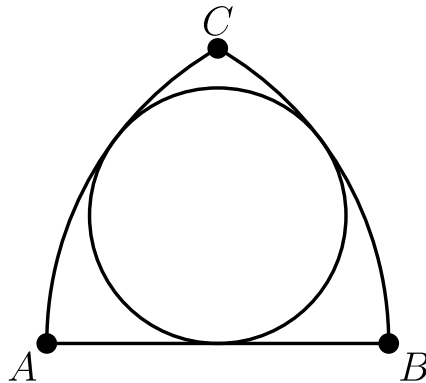
Working through the possibilities, exactly four valid tickets exist:

$\{1, 5, 10, 20, 25, 40\}$, $\{1, 2, 5, 10, 25, 40\}$, $\{1, 2, 4, 5, 10, 25\}$, and $\{1, 4, 5, 10, 20, 25\}$.

Professor Gamble holds one of these four, and only one matches the winning ticket, so the probability is $\frac{1}{4}$.

Thus, the correct answer is **B**.

24. If circular arcs AC and BC have centers at B and A , respectively, then there exists a circle tangent to both arc AC and arc BC , and to \overline{AB} . If the length of arc BC is 12, then what is the circumference of the circle?



- A 24
- B 25
- C 26
- D 27**
- E 28

Solution:

Each arc has radius AB , and C is at distance AB from both A and B , so $\triangle ABC$ is equilateral. Thus arc BC subtends 60° of a circle of radius AB , whose full circumference is $6 \cdot 12 = 72$.

Let the small circle have radius r and touch \overline{AB} at its midpoint D , where $AD = \frac{1}{2}AB$. By Power of a Point, $AD^2 = AB(AB - 2r)$, so

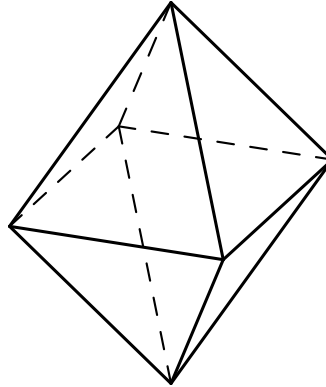
$$\frac{AB^2}{4} = AB^2 - 2r AB,$$

giving $2r = \frac{3}{4}AB$, hence $r = \frac{3}{8}AB$.

The circumferences are in the ratio of the radii, so the small circle's circumference is $\frac{3}{8} \cdot 72 = 27$.

Thus, the correct answer is **D**.

25. Eight congruent equilateral triangles, each of a different color, are used to construct a regular octahedron. How many distinguishable ways are there to construct the octahedron? (Two colored octahedrons are distinguishable if neither can be rotated to look just like the other.)



- A 210
- B 560
- C 840
- D 1260
- E 1680**

Solution:

There are $8!$ ways to assign the eight distinct colors to the eight faces. Two assignments give the same octahedron exactly when one is a rotation of the other.

The rotation group of a regular octahedron has **24** elements. Because all eight colors are different, no nontrivial rotation fixes a coloring, so each distinguishable octahedron corresponds to exactly **24** assignments.

Therefore the number of distinguishable octahedrons is

$$\frac{8!}{24} = \frac{40320}{24} = 1680.$$

Thus, the correct answer is **E**.

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