1999 AMC 12 Solutions

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1.

$$1 - 2 + 3 - 4 + \cdots - 98 + 99 = ?$$

A -50

B -49

C 0

D 49

E 50

Solution:

Pairing consecutive terms gives

$$(1 - 2) + (3 - 4) + \cdots + (97 - 98) + 99.$$

There are 49 pairs, each equal to -1, so the sum is $-49 + 99 = 50$.

Thus, the correct answer is **E**.

2. Which one of the following statements is false?

- A All equilateral triangles are congruent to each other.
- B All equilateral triangles are convex.
- C All equilateral triangles are equiangular.
- D All equilateral triangles are regular polygons.
- E All equilateral triangles are similar to each other.

Solution:

Equilateral triangles with side lengths 1 and 2 have the same shape but different sizes, so they are similar but not congruent. Every equilateral triangle is convex, equiangular (all angles 60°), and a regular polygon, so the only false statement is that they are all congruent.

Thus, the correct answer is **A**.

3. The number halfway between $\frac{1}{8}$ and $\frac{1}{10}$ is

A $\frac{1}{80}$

B $\frac{1}{40}$

C $\frac{1}{18}$

D $\frac{1}{9}$

E $\frac{9}{80}$

Solution:

The halfway point is the average

$$\frac{1}{2}\left(\frac{1}{8}+\frac{1}{10}\right)=\frac{1}{2}\cdot\frac{18}{80}=\frac{9}{80}.$$

Thus, the correct answer is **E**.

4. Find the sum of all prime numbers between 1 and 100 that are simultaneously 1 greater than a multiple of 4 and 1 less than a multiple of 5.

- A 118
- B 137
- C 158
- D 187
- E 245

Solution:

A number that is 1 less than a multiple of 5 ends in 4 or 9, and one that is 1 greater than a multiple of 4 is odd. Together these give numbers $\equiv 9 \pmod{20}$, namely $9, 29, 49, 69, 89.$

Among these, only 29 and 89 are prime, and their sum is $29 + 89 = 118.$

Thus, the correct answer is **A**.

5. The marked price of a book was 30% less than the suggested retail price. Alice purchased the book for half the marked price at a Fiftieth Anniversary sale. What percent of the suggested retail price did Alice pay?

A 25%

B 30%

C 35%

D 60%

E 65%

Solution:

If the suggested retail price is P, then the marked price is $0.7P$. Alice pays half of this, $0.35P$, which is 35% of the suggested retail price.

Thus, the correct answer is **C**.

6. What is the sum of the digits of the decimal form of the product $2^{1999} \cdot 5^{2001}$?

- A 2
- B 4
- C 5
- D 7
- E 10

Solution:

Write

$$2^{1999} \cdot 5^{2001} = 2^{1999} \cdot 5^{1999} \cdot 5^2 = 25 \cdot 10^{1999},$$

which is 25 followed by 1999 zeros. The sum of the digits is $2 + 5 = 7.$

Thus, the correct answer is **D**.

7. What is the largest number of acute angles that a convex hexagon can have?

- A 2
- B 3
- C 4
- D 5
- E 6

Solution:

Each acute interior angle corresponds to an exterior angle greater than 90°. Since the exterior angles of a convex polygon sum to 360°, at most three of them can exceed 90°. Hence there are at most three acute angles, and a hexagon achieving three acute angles exists.

Thus, the correct answer is **B**.

8. At the end of 1994 Walter was half as old as his grandmother. The sum of the years in which they were born is 3838. How old will Walter be at the end of 1999?

- A 48
- B 49
- C 53
- D 55
- E 101

Solution:

Let Walter be w years old at the end of 1994, so his grandmother is $2w$. Their birth years are $1994 - w$ and $1994 - 2w$, and

$$(1994 - w) + (1994 - 2w) = 3838.$$

This gives $3988 - 3w = 3838$, so $w = 50$.

At the end of 1999, Walter will be $50 + 5 = 55$.

Thus, the correct answer is **D**.

9. Before Ashley started a three-hour drive, her car's odometer reading was $29792,$ a palindrome. (A palindrome is a number that reads the same way from left to right as it does from right to left.) At her destination, the odometer reading was another palindrome. If Ashley never exceeded the speed limit of 75 miles per hour, which of the following was her greatest possible average speed?

A $33\frac{1}{3}$

B $53\frac{1}{3}$

C $66\frac{2}{3}$

D $70\frac{1}{3}$

E $74\frac{1}{3}$

Solution:

The palindromes after 29792 are $29892,$ $29992,$ $30003,$ and $30103.$ In three hours Ashley can drive at most $3 \cdot 75 = 225$ miles.

Reaching 30103 would require $30103 - 29792 = 311$ miles, which is too far. Reaching 30003 requires $30003 - 29792 = 211$ miles, giving average speed $\dfrac{211}{3} = 70\frac{1}{3}$ miles per hour.

Thus, the correct answer is **D**.

10. A sealed envelope contains a card with a single digit on it. Three of the following statements are true, and the other is false.

I. The digit is $1.$

II. The digit is not $2.$

III. The digit is $3.$

IV. The digit is not $4.$

Which one of the following must necessarily be correct?

A I is true.

B I is false.

C II is true.

D III is true.

E IV is false.

Solution:

Statements I and III cannot both be true, so the single false statement is one of them. Therefore statements II and IV are both true, which makes "II is true" necessarily correct.

The digit is thus 1 or $3.$ If it were $1,$ then (B) and (D) are false; if it were $3,$ then (A) is false; and (E) is always incorrect. Only (C) is guaranteed.

Thus, the correct answer is **C**.

11. The student lockers at Olympic High are numbered consecutively beginning with locker number 1. The plastic digits used to number the lockers cost 2 cents apiece. Thus, it costs 2 cents to label locker number 9 and 4 cents to label locker number 10. If it costs $\$137.94$ to label all the lockers, how many lockers are there at the school?

A 2001

B 2010

C 2100

D 2726

E 6897

Solution:

Labeling costs $\$137.94/\$0.02 = 6897$ digits. Lockers 1-9 use 9 digits, lockers 10-99 use $2 \cdot 90 = 180$ digits, and lockers 100-999 use $3 \cdot 900 = 2700$ digits.

The remaining digits number $6897 - 2700 - 180 - 9 = 4008$, which label $4008/4 = 1002$ four-digit lockers. In all there are $999 + 1002 = 2001$ lockers.

Thus, the correct answer is **A**.

12. What is the maximum number of points of intersection of the graphs of two different fourth degree polynomial functions $y = p(x)$ and $y = q(x)$, each with leading coefficient 1?

A 1

B 2

C 3

D 4

E 8

Solution:

The x-coordinates of the intersection points are the roots of $p(x) - q(x)$. Because both leading coefficients are 1, the x^4 terms cancel, so $p(x) - q(x)$ has degree at most 3 and therefore at most 3 roots. Three intersections are achievable.

Thus, the correct answer is **C**.

13. Define a sequence of real numbers $a_1, a_2, a_3, ...$ by $a_1 = 1$ and $a_{n+1}^3 = 99a_n^3$ for all $n \geq 1$. Then a_{100} equals

A 33^{33}

B 33^{99}

C 99^{33}

D 99^{99}

E none of these

Solution:

Taking cube roots, $a_{n+1} = \sqrt[3]{99}\, a_n$, so the sequence is geometric with first term 1 and ratio $\sqrt[3]{99}$. Then

$$a_{100} = \left(\sqrt[3]{99}\right)^{99} = 99^{33}.$$

Thus, the correct answer is **C**.

14. Four girls — Mary, Alina, Tina, and Hanna — sang songs in a concert as trios, with one girl sitting out each time. Hanna sang 7 songs, which was more than any other girl, and Mary sang 4 songs, which was fewer than any other girl. How many songs did these trios sing?

- A 7
- B 8
- C 9
- D 10
- E 11

Solution:

If N songs are sung, the total number of girl-appearances is $3N$. Alina and Tina each sang strictly between 4 and 7, so each sang 5 or 6.

Then $3N = 7 + 4 + (\text{Alina}) + (\text{Tina})$, which is $21, 22$, or 23. Only 21 is a multiple of 3, so $N = 7$.

Thus, the correct answer is **A**.

15. Let x be a real number such that $\sec x - \tan x = 2$. Then $\sec x + \tan x = ?$

A 0.1

B 0.2

C 0.3

D 0.4

E 0.5

Solution:

Since $\sec^2 x - \tan^2 x = 1$, we have

$$(\sec x - \tan x)(\sec x + \tan x) = 1.$$

With $\sec x - \tan x = 2$, it follows that $\sec x + \tan x = \frac{1}{2} = 0.5$.

Thus, the correct answer is **E**.

16. What is the radius of a circle inscribed in a rhombus with diagonals of length 10 and 24?

A 4

B $\frac{58}{13}$

C $\frac{60}{13}$

D 5

E 6

Solution:

The half-diagonals are 5 and $12,$ so each side of the rhombus is $\sqrt{5^2 + 12^2} = 13.$ One of the four right triangles formed by the diagonals has legs 5 and 12 and area $30.$

The altitude from the center to the side of length 13 is $\frac{2 \cdot 30}{13} = \frac{60}{13},$ which is the inscribed circle's radius.

Thus, the correct answer is **C**.

17. Let $P(x)$ be a polynomial such that when $P(x)$ is divided by $x-19$, the remainder is 99, and when $P(x)$ is divided by $x-99$, the remainder is 19. What is the remainder when $P(x)$ is divided by $(x-19)(x-99)$?

- A $-x+80$
- B $x+80$
- C $-x+118$
- D $x+118$
- E 0

Solution:

By the Remainder Theorem, $P(19)=99$ and $P(99)=19$. Write

$$P(x)=(x-19)(x-99)Q(x)+ax+b.$$

Then

$$19a+b=99, \qquad 99a+b=19.$$

Subtracting gives $80a=-80$, so $a=-1$ and $b=118$. The remainder is $-x+118$.

Thus, the correct answer is **C**.

18. How many zeros does $f(x) = \cos(\log x)$ have on the interval $0 < x < 1$?

A 0

B 1

C 2

D 10

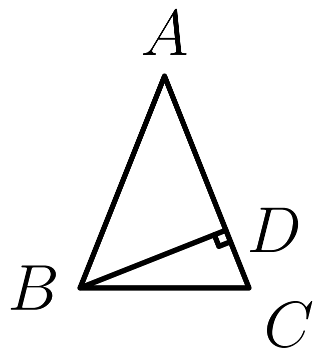
E infinitely many

Solution:

As x ranges over $(0, 1)$, $\log x$ ranges over all negative real numbers. The cosine function is zero at $\frac{\pi}{2} - n\pi$ for every positive integer n, all of which are negative, so f has infinitely many zeros.

Thus, the correct answer is **E**.

19. Consider all triangles ABC satisfying the following conditions: $AB = AC$, D is a point on $\overline{AC}$ for which $\overline{BD} \perp \overline{AC}$, AD and CD are integers, and $BD^2 = 57$. Among all such triangles, the smallest possible value of AC is



A 9

B 10

C 11

D 12

E 13

Solution:

Let $AD = n$ and $CD = m$. Since $\triangle ADB$ is right-angled at D, $AB^2 = n^2 + 57$. Also $AB = AC = m + n$, so

$$(m+n)^2 = n^2 + 57,$$

which simplifies to $m(m+2n) = 57$.

The positive integer solutions are $m = 1, n = 28$ (giving $AC = 29$) and $m = 3, n = 8$ (giving $AC = 11$). The smallest possible value of AC is 11.

Thus, the correct answer is **C**.

20. The sequence $a_1, a_2, a_3, \ldots$ satisfies $a_1 = 19, a_9 = 99,$ and, for all $n \geq 3, a_n$ is the arithmetic mean of the first $n - 1$ terms. Find a_2.

- A 29
- B 59
- C 79
- D 99
- E 179

Solution:

For $n \geq 3, (n-1)a_n = a_1 + \cdots + a_{n-1}$. Then

$$a_{n+1} = \frac{(n-1)a_n + a_n}{n} = a_n,$$

so the sequence is constant from a_3 onward. Hence $a_3 = a_9 = 99$.

Since $a_3 = \dfrac{a_1 + a_2}{2} = \dfrac{19 + a_2}{2} = 99$, we get $a_2 = 179$.

Thus, the correct answer is **E**.

21. A circle is circumscribed about a triangle with sides $20, 21,$ and $29,$ thus dividing the interior of the circle into four regions. Let $A, B,$ and C be the areas of the non-triangular regions, with C being the largest. Then

A $A + B = C$

B $A + B + 210 = C$

C $A^2 + B^2 = C^2$

D $20A + 21B = 29C$

E $\frac{1}{A^2} + \frac{1}{B^2} = \frac{1}{C^2}$

Solution:

Since $20^2 + 21^2 = 841 = 29^2$, the triangle is right-angled, and its hypotenuse of length 29 is a diameter of the circle. Thus the largest region C is the semicircle on one side of that diameter.

The other semicircle consists of the triangle together with regions A and B. Since the two semicircles are congruent and the triangle has area $\frac{1}{2} \cdot 20 \cdot 21 = 210$, we get

$$A + B + 210 = C.$$

Thus, the correct answer is **B**.

22. The graphs of $y = -|x - a| + b$ and $y = |x - c| + d$ intersect at points $(2, 5)$ and $(8, 3)$. Find $a + c$.

- A 7
- B 8
- C 10
- D 13
- E 18

Solution:

The first graph is an inverted right angle with vertex (a, b), and the second is an upright right angle with vertex (c, d). Because each consists of two lines of slope ± 1, the four points $(a, b), (2, 5), (c, d), (8, 3)$ are the vertices of a rectangle in order.

The diagonals of a rectangle share a midpoint, so $\dfrac{a + c}{2} = \dfrac{2 + 8}{2} = 5$, giving $a + c = 10$.

Thus, the correct answer is **C**.

23. The equiangular convex hexagon $ABCDEF$ has $AB = 1, BC = 4, CD = 2,$ and $DE = 4.$ The area of the hexagon is

A $\frac{15}{2}\sqrt{3}$

B $9\sqrt{3}$

C 16

D $\frac{39}{4}\sqrt{3}$

E $\frac{43}{4}\sqrt{3}$

Solution:

Each interior angle is 120°, so extending sides FA and BC, BC and DE, and DE and FA cuts off three equilateral corner triangles and forms a large equilateral triangle.

The corner triangles built on $AB, CD,$ and EF are equilateral, and one finds the large triangle has side $1 + 4 + 2 = 7,$ while the removed triangles have sides $1, 2,$ and $1.$ The area is

$$\frac{\sqrt{3}}{4}\left(7^2 - 1^2 - 2^2 - 1^2\right) = \frac{43\sqrt{3}}{4}.$$

Thus, the correct answer is **E**.

24. Six points on a circle are given. Four of the chords joining pairs of the six points are selected at random. What is the probability that the four chords form a convex quadrilateral?

A $\frac{1}{15}$

B $\frac{1}{91}$

C $\frac{1}{273}$

D $\frac{1}{455}$

E $\frac{1}{1365}$

Solution:

There are $\binom{6}{2} = 15$ chords, so $\binom{15}{4} = 1365$ ways to select four of them. A convex quadrilateral arises exactly when the four chords are the sides of a quadrilateral on four of the six points, and each choice of 4 points gives exactly one such quadrilateral. Hence there are $\binom{6}{4} = 15$ favorable outcomes, and the probability is $\dfrac{15}{1365} = \dfrac{1}{91}$.

Thus, the correct answer is **B**.

25. There are unique integers $a_2, a_3, a_4, a_5, a_6, a_7$ such that

$$\frac{5}{7} = \frac{a_2}{2!} + \frac{a_3}{3!} + \frac{a_4}{4!} + \frac{a_5}{5!} + \frac{a_6}{6!} + \frac{a_7}{7!},$$

where $0 \le a_i < i$ for $i = 2, 3, \ldots, 7$. Find $a_2 + a_3 + a_4 + a_5 + a_6 + a_7$.

A 8

B 9

C 10

D 11

E 12

Solution:

Multiplying by $7! = 5040$ gives

$$3600 = 2520a_2 + 840a_3 + 210a_4 + 42a_5 + 7a_6 + a_7.$$

Reducing modulo 7, $a_7 = 2$.

Then $\dfrac{3600 - 2}{7} = 514 = 360a_2 + 120a_3 + 30a_4 + 6a_5 + a_6$. Reducing modulo 6 gives $a_6 = 4$, and continuing this way yields $a_5 = 0, a_4 = 1, a_3 = 1, a_2 = 1$.

The sum is $1 + 1 + 1 + 0 + 4 + 2 = 9$.

Thus, the correct answer is **B**.

26. Three non-overlapping regular plane polygons, at least two of which are congruent, all have sides of length 1. The polygons meet at a point A in such a way that the sum of the three interior angles at A is 360°. Thus the three polygons form a new polygon with A as an interior point. What is the largest possible perimeter that this polygon can have?

A 12

B 14

C 18

D 21

E 24

Solution:

Let two congruent a-gons and one b-gon meet at A. Their interior angles satisfy

$$2 \cdot 180\left(1-\tfrac{2}{a}\right)+180\left(1-\tfrac{2}{b}\right)=360,$$

which reduces to $(a-4)(b-2)=8$.

The solutions (a, b) are $(5,10)$, $(6,6)$, $(8,4)$, and $(12,3)$. The new polygon's perimeter is $2a+b-6$, giving $14, 12, 14,$ and 21. The largest is 21.

Thus, the correct answer is **D**.

27. In triangle $ABC, 3 \sin A + 4 \cos B = 6$ and $4 \sin B + 3 \cos A = 1$. Then $\angle C$ in degrees is

A 30

B 60

C 90

D 120

E 150

Solution:

Squaring both equations and adding gives

$$9 + 16 + 24(\sin A \cos B + \cos A \sin B) = 37,$$

so $24 \sin(A + B) = 12$ and $\sin(A + B) = \frac{1}{2}$.

Then $\sin C = \sin(A + B) = \frac{1}{2}$, so $\angle C = 30^\circ$ or 150°. If $\angle C = 150^\circ$, then $A < 30^\circ$, making $3 \sin A + 4 \cos B < 6$, a contradiction. Hence $\angle C = 30^\circ$.

Thus, the correct answer is **A**.

28. Let $x_1, x_2, \ldots, x_n$ be a sequence of integers such that

(i) $-1 \leq x_i \leq 2$ for $i = 1, 2, 3, \ldots, n$;

(ii) $x_1 + x_2 + \cdots + x_n = 19$; and

(iii) $x_1^2 + x_2^2 + \cdots + x_n^2 = 99$.

Let m and M be the minimal and maximal possible values of $x_1^3 + x_2^3 + \cdots + x_n^3$, respectively. Then $\dfrac{M}{m} = ?$

A 3

B 4

C 5

D 6

E 7

Solution:

Let a, b, c be the numbers of -1s, 1s, and 2s. Then $-a + b + 2c = 19$ and $a + b + 4c = 99$, giving $a = 40 - c$ and $b = 59 - 3c$ with $0 \leq c \leq 19$.

The sum of cubes is $-a + b + 8c = 19 + 6c$. The minimum is at $c = 0$ (value 19) and the maximum at $c = 19$ (value 133), so $\dfrac{M}{m} = \dfrac{133}{19} = 7$.

Thus, the correct answer is **E**.

29. A tetrahedron with four equilateral triangular faces has a sphere inscribed within it and a sphere circumscribed about it. For each of the four faces, there is a sphere tangent externally to the face at its center and to the circumscribed sphere. A point P is selected at random inside the circumscribed sphere. The probability that P lies inside one of the five small spheres is closest to

- A 0
- B 0.1
- C 0.2
- D 0.3
- E 0.4

Solution:

Let O be the common center of the inscribed and circumscribed spheres. Splitting the tetrahedron into four congruent pieces from O shows the circumradius is 3 times the inradius, so the circumscribed sphere has 27 times the inscribed sphere's volume V.

Each externally tangent sphere fits between a face and the circumscribed sphere and is congruent to the inscribed sphere, so the five small spheres have total volume $5V$. The probability is $\frac{5V}{27V} = \frac{5}{27} \approx 0.185$, closest to 0.2.

Thus, the correct answer is **C**.

30. The number of ordered pairs of integers (m, n) for which $mn \geq 0$ and

$$m^3 + n^3 + 99mn = 33^3$$

is equal to

- A 2
- B 3
- C 33
- D 35
- E 99

Solution:

Writing $z = -33$, the equation becomes $m^3 + n^3 + z^3 - 3mnz = 0$, which factors as

$$(m + n - 33)\left(m^2 + n^2 + 33^2 - mn + 33m + 33n\right) = 0.$$

The second factor equals $\frac{1}{2}\left[(m-n)^2 + (m+33)^2 + (n+33)^2\right]$, which is 0 only at $(m, n) = (-33, -33)$; this satisfies $mn \geq 0$.

Otherwise $m + n = 33$. With $mn \geq 0$ both are nonnegative, giving $(0, 33), (1, 32), \ldots, (33, 0)$, which is 34 pairs. Together there are 35 solutions.

Thus, the correct answer is **D**.

Problems: https://live.poshenloh.com/past-contests/amc12/1999