1998 AMC 12 Solutions

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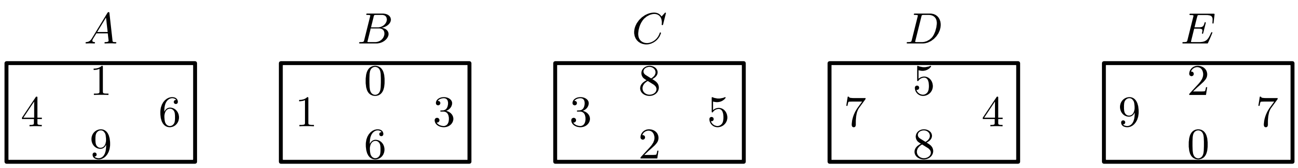
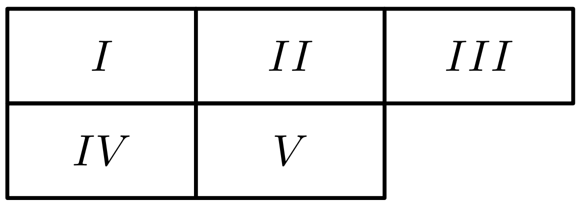
1. Each of the sides of five congruent rectangles is labeled with an integer, as shown. These five rectangles are placed, without rotating or reflecting, in positions I through V so that the labels on coincident sides are equal. Which of the rectangles is in position I?





- A A
- B B
- C C
- D D
- E E

Solution:

Position II needs both its left and right labels to occur as right and left labels, respectively, on two other rectangles. Only D, with side labels 7 and 4, permits both matches: E has right label 7, and A has left label 4. Hence position I contains E, so the correct answer is **E**.

2. Letters A, B, C, and D represent four different digits selected from $0, 1, 2, \ldots, 9$. If $\frac{A+B}{C+D}$ is an integer that is as large as possible, what is the value of $A + B$?

- A 13
- B 14
- C 15
- D 16
- E 17

Solution:

The least positive denominator is $0 + 1 = 1$, and the greatest numerator using two other digits is $8 + 9 = 17$. Their ratio is the integer 17, which is plainly maximal. Thus $A + B = 17$, and **E** is correct.

3. If $a, b,$ and c are digits for which

$$\begin{array}{r} 7a2 \\ -\ 48b \\ \hline c73 \end{array}$$

then $a+b+c=$

A 14

B 15

C 16

D 17

E 18

Solution:

The units column gives $12-b=3$, so $b=9$. In the tens column, $a-1$ must borrow, and $a-1+10-8=7$, giving $a=6$. The hundreds column then gives $7-1-4=c=2$. Therefore $a+b+c=6+9+2=17$, so **D** is correct.

4. Define $[a, b, c]$ to mean $\frac{a+b}{c}$, where $c \neq 0$. What is the value of

$$[[60, 30, 90], [2, 1, 3], [10, 5, 15]]?$$

- A 0
- B 0.5
- C 1
- D 1.5
- E 2

Solution:

Each inner value is $1: \frac{90}{90} = 1$, $\frac{3}{3} = 1$, and $\frac{15}{15} = 1$. Thus the outer expression is $[1, 1, 1] = \frac{1+1}{1} = 2$. The correct answer is **E**.

5. If $2^{1998} - 2^{1997} - 2^{1996} + 2^{1995} = k \cdot 2^{1995}$, what is the value of k?

- A 1
- B 2
- C 3
- D 4
- E 5

Solution:

Factoring gives $2^{1995}(8 - 4 - 2 + 1) = 3 \cdot 2^{1995}$. Hence $k = 3$, so **C** is correct.

6. If 1998 is written as a product of two positive integers whose difference is as small as possible, then the difference is

- A 8
- B 15
- C 17
- D 47
- E 93

Solution:

Since $1998 = 2 \cdot 3^3 \cdot 37,$ its factor pair nearest $\sqrt{1998}$ is $37 \cdot 54.$ Their difference is $54 - 37 = 17,$ so **C** is correct.

7. If $N > 1$, then

$$\sqrt[3]{N\sqrt[3]{N\sqrt[3]{N}}} =$$

A $N^{\frac{1}{27}}$

B $N^{\frac{1}{9}}$

C $N^{\frac{1}{3}}$

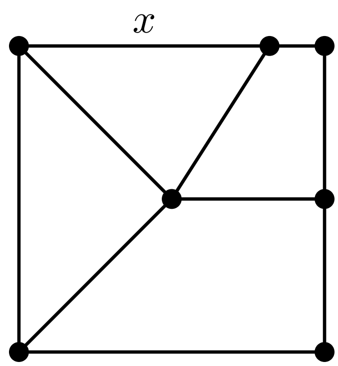
D $N^{\frac{13}{27}}$

E N

Solution:

The innermost radical is $N^{\frac{1}{3}}$. The next is $(N \cdot N^{\frac{1}{3}})^{\frac{1}{3}} = N^{\frac{4}{9}}$. The outermost is $(N \cdot N^{\frac{4}{9}})^{\frac{1}{3}} = N^{\frac{13}{27}}$. Thus **D** is correct.

8. A square with sides of length 1 is divided into two congruent trapezoids and a pentagon, which have equal areas, by joining the center of the square with points on three of the sides, as shown. Find x, the length of the longer parallel side of each trapezoid.



A $\frac{3}{5}$

B $\frac{2}{3}$

C $\frac{3}{4}$

D $\frac{5}{6}$

E $\frac{7}{8}$

Solution:

Each trapezoid has area $\frac{1}{3}$. Its height is $\frac{1}{2}$, and its parallel sides have lengths x and $\frac{1}{2}$. Therefore

$$\frac{1}{2} \cdot \frac{1}{2}\left(x + \frac{1}{2}\right) = \frac{1}{3},$$

which gives $x = \frac{5}{6}$. Thus **D** is correct.

9. A speaker talked for sixty minutes to a full auditorium. Twenty percent of the audience heard the entire talk and ten percent slept through the entire talk. Half of the remainder heard one third of the talk and the other half heard two thirds of the talk. What was the average number of minutes of the talk heard by members of the audience?

A 24

B 27

C 30

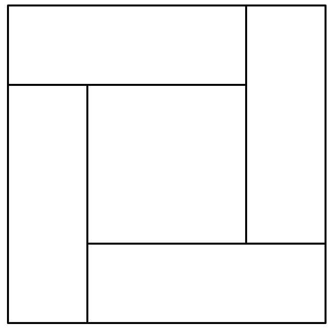
D 33

E 36

Solution:

The average is $0.20(60)+0.10(0)+0.35(20)+0.35(40)=12+7+14=33$ minutes. Thus **D** is correct.

10. A large square is divided into a small square surrounded by four congruent rectangles as shown. The perimeter of each of the congruent rectangles is 14. What is the area of the large square?



A 49

B 64

C 100

D 121

E 196

Solution:

If a rectangle has sides $x, y,$ then $2x + 2y = 14,$ so $x + y = 7.$ The large square has side $x + y,$ hence area $7^2 = 49.$ The correct answer is **A**.

11. Let R be a rectangle. How many circles in the plane of R have a diameter both of whose endpoints are vertices of R?

- A 1
- B 2
- C 4
- D 5
- E 6

Solution:

The four sides determine four distinct diameter circles. The two diagonals are equal and share a midpoint, so they determine the same fifth circle. Hence there are $5,$ and **D** is correct.

12. How many different prime numbers are factors of N if

$$\log_2(\log_3(\log_5(\log_7 N))) = 11?$$

A 1

B 2

C 3

D 4

E 7

Solution:

Successively exponentiating gives

$$N = 7^{5^{3^{2^{11}}}}.$$

Thus 7 is the only prime factor of N, so the correct answer is **A**.

13. Walter rolls four standard six-sided dice and finds that the product of the numbers on the upper faces is 144. Which of the following could not be the sum of the upper four faces?

A 14

B 15

C 16

D 17

E 18

Solution:

Valid rolls include $(4, 4, 3, 3)$ with sum 14, $(6, 4, 3, 2)$ with sum 15, $(6, 6, 2, 2)$ with sum 16, and $(6, 6, 4, 1)$ with sum 17. If two 6s occur, the other two faces have product 4 and sum at most 5, giving total at most 17; the cases with fewer 6s also cannot total 18 while retaining product 144. Thus 18 is impossible, and **E** is correct.

14. A parabola has vertex at $(4, -5)$ and has two x-intercepts, one positive and one negative. If this parabola is the graph of $y = ax^2 + bx + c$, which of $a, b,$ and c must be positive?

A only a

B only b

C only c

D a and b only

E none

Solution:

The parabola opens upward, so $a > 0$. Its roots have opposite signs, so $\frac{c}{a} < 0$ and hence $c < 0$. Also $-\frac{b}{2a} = 4 > 0$, so $b < 0$. Therefore only a must be positive, and **A** is correct.

15. A regular hexagon and an equilateral triangle have equal areas. What is the ratio of the length of a side of the triangle to the length of a side of the hexagon?

A $\sqrt{3}$

B 2

C $\sqrt{6}$

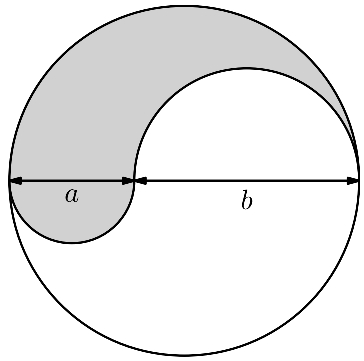
D 3

E 6

Solution:

A regular hexagon of side s is six equilateral triangles of side s. An equal-area equilateral triangle of side t therefore satisfies $t^2 = 6s^2$, so $\frac{t}{s} = \sqrt{6}$. The correct answer is **C**.

16. The figure shown is the union of a circle and two semicircles of diameters a and b, all of whose centers are collinear. The ratio of the area of the shaded region to that of the unshaded region is



A $\sqrt{\frac{a}{b}}$

B $\frac{a}{b}$

C $\frac{a^2}{b^2}$

D $\frac{a+b}{2b}$

E $\frac{a^2+2ab}{b^2+2ab}$

Solution:

The shaded area is

$$\frac{\pi}{2}\left(\frac{a+b}{2}\right)^2+\frac{\pi}{2}\left(\frac{a}{2}\right)^2 - \frac{\pi}{2}\left(\frac{b}{2}\right)^2=\frac{\pi a(a+b)}{4}.$$

Similarly, the unshaded area is $\frac{\pi b(a+b)}{4}$. Their ratio is $\frac{a}{b}$, so **B** is correct.

17. Let $f(x)$ be a function with the two properties:

(a) for any two real numbers x and y, $f(x+y)=x+f(y)$, and

(b) $f(0)=2$.

What is the value of $f(1998)$?

A 0

B 2

C 1996

D 1998

E 2000

Solution:

Taking $y=0$ gives $f(x)=x+f(0)=x+2$. Hence $f(1998)=2000$, so **E** is correct.

18. A right circular cone of volume A, a right circular cylinder of volume M, and a sphere of volume C all have the same radius, and the common height of the cone and the cylinder is equal to the diameter of the sphere. Then

A $A - M + C = 0$

B $A + M = C$

C $2A = M + C$

D $A^2 - M^2 + C^2 = 0$

E $2A + 2M = 3C$

Solution:

The volumes are $A = \frac{2}{3}\pi r^3$, $M = 2\pi r^3$, and $C = \frac{4}{3}\pi r^3$. Therefore $A - M + C = (\frac{2}{3} - 2 + \frac{4}{3})\pi r^3 = 0$, so **A** is correct.

19. How many triangles have area 10 and vertices at $(-5, 0)$, $(5, 0)$, and $(5\cos\theta, 5\sin\theta)$ for some angle θ?

A 0

B 2

C 4

D 6

E 8

Solution:

The area is $\frac{1}{2}(10)|5\sin\theta| = 25|\sin\theta|$. Thus $|\sin\theta| = \frac{2}{5}$. There are four corresponding points on the circle, and each gives a distinct triangle. Hence the answer is 4, making **C** correct.

20. Three cards, each with a positive integer written on it, are lying face-down on a table. Casey, Stacy, and Tracy are told that

(a) the numbers are all different,

(b) they sum to 13, and

(c) they are in increasing order, left to right.

First, Casey looks at the number on the leftmost card and says, "I don't have enough information to determine the other two numbers." Then Tracy looks at the number on the rightmost card and says, "I don't have enough information to determine the other two numbers." Finally, Stacy looks at the number on the middle card and says, "I don't have enough information to determine the other two numbers." Assume that each person knows that the other two reason perfectly and hears their comments. What number is on the middle card?

A 2

B 3

C 4

D 5

E There is not enough information to determine the number.

Solution:

The eight triples are

$$(1,2,10),(1,3,9),$$
$$(1,4,8),(1,5,7),$$
$$(2,3,8),(2,4,7),$$
$$(2,5,6),(3,4,6).$$

Casey's statement excludes $(3,4,6)$. With that known, Tracy's statement excludes rightmost values $10, 9$, and 6. The remaining triples are $(1,4,8), (2,3,8)$ and $(1,5,7), (2,4,7)$. A middle value 3 or 5 would now identify the triple, so Stacy's statement forces the middle value 4. Thus **C** is correct.

21. In an h-meter race, Sunny is exactly d meters ahead of Windy when Sunny finishes the race. The next time they race, Sunny sportingly starts d meters behind Windy, who is at the starting line. Both runners run at the same constant speed as they did in the first race. How many meters ahead is Sunny when Sunny finishes the second race?

A $\frac{d}{h}$

B 0

C $\frac{d^2}{h}$

D $\frac{h^2}{d}$

E $\frac{d^2}{h-d}$

Solution:

If Sunny's speed is r, Windy's is $\frac{r(h-d)}{h}$. Sunny needs time $\frac{h+d}{r}$ in the second race, during which Windy runs

$$\frac{r(h-d)}{h} \cdot \frac{h+d}{r} = h - \frac{d^2}{h}.$$

Sunny finishes at position h, so the lead is $\frac{d^2}{h}$. Thus **C** is correct.

22. What is the value of the expression

$$\frac{1}{\log_2 100!} + \frac{1}{\log_3 100!} + \frac{1}{\log_4 100!} + \cdots + \frac{1}{\log_{100} 100!}?$$

A 0.01

B 0.1

C 1

D 2

E 10

Solution:

Each term equals $\log_{100!} b$. Therefore the sum is

$$\log_{100!}(2 \cdot 3 \cdots 100) = \log_{100!}(100!) = 1.$$

Thus **C** is correct.

23. The graphs of $x^2+y^2=4+12x+6y$ and $x^2+y^2=k+4x+12y$ intersect when k satisfies $a \le k \le b$, and for no other values of k. Find $b-a$.

A 5

B 68

C 104

D 140

E 144

Solution:

The circles are

$$(x-6)^2+(y-3)^2=49,$$
$$(x-2)^2+(y-6)^2=k+40.$$

Their centers are 5 units apart. With second radius $r=\sqrt{k+40}$, intersection requires $|7-r| \le 5 \le 7+r$, or $2 \le r \le 12$. Thus $-36 \le k \le 104$, and $b-a=140$. The correct answer is **D**.

24. Call a 7-digit telephone number $d_1d_2d_3$-$d_4d_5d_6d_7$ *memorable* if the prefix sequence $d_1d_2d_3$ is exactly the same as either of the sequences $d_4d_5d_6$ or $d_5d_6d_7$ (possibly both). Assuming that each d_i can be any of the ten decimal digits $0, 1, 2, \ldots, 9$, the number of different memorable telephone numbers is

A $19{,}810$

B $19{,}910$

C $19{,}990$

D $20{,}000$

E $20{,}100$

Solution:

Each matching condition gives $10^4 = 10,000$ numbers: choose the three prefix digits and the one unconstrained remaining digit. If both hold, then $d_1d_2d_3 = d_4d_5d_6 = d_5d_6d_7$, forcing all digits equal, so there are 10 overlaps. Inclusion-exclusion gives $10,000 + 10,000 - 10 = 19,990$, so **C** is correct.

25. A piece of graph paper is folded once so that $(0,2)$ is matched with $(4,0)$, and $(7,3)$ is matched with (m,n). Find $m+n$.

A 6.7

B 6.8

C 6.9

D 7.0

E 8.0

Solution:

The crease passes through $(2,1)$ with slope 2, so it is $y=2x-3$. The perpendicular through $(7,3)$ has slope $-\frac{1}{2}$ and meets the crease at $(\frac{19}{5}, \frac{23}{5})$. This intersection is the midpoint of $(7,3)$ and (m,n), giving $m=\frac{3}{5},\ n=\frac{31}{5}$. Hence $m+n=\frac{34}{5}=6.8$, so **B** is correct.

26. In quadrilateral $ABCD$, it is given that $\angle A = 120^\circ$, angles B and D are right angles, $AB = 13$, and $AD = 46$. Then $AC =$

A 60

B 62

C 64

D 65

E 72

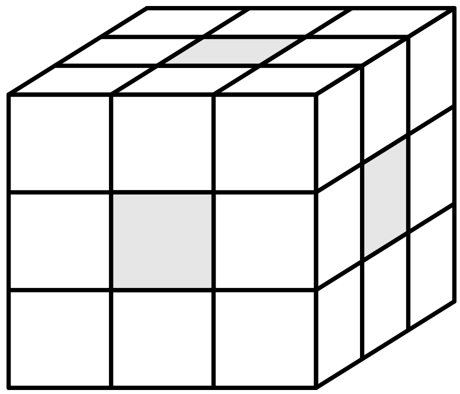
Solution:

Because $\angle B + \angle D = 180^\circ$, the quadrilateral is cyclic, and AC is its diameter. In triangle ABD,

$$\begin{aligned} BD^2 &= 13^2 + 46^2 \\ &\quad - 2(13)(46)\cos 120^\circ \\ &= 2883 = (31\sqrt{3})^2. \end{aligned}$$

By the extended law of sines, $AC = \frac{BD}{\sin 120^\circ} = \frac{31\sqrt{3}}{\frac{\sqrt{3}}{2}} = 62$. Thus **B** is correct.

27. A $9 \times 9 \times 9$ cube is composed of twenty-seven $3 \times 3 \times 3$ cubes. The big cube is "tunneled" as follows: First, the six $3 \times 3 \times 3$ cubes which make up the center of each face as well as the center $3 \times 3 \times 3$ cube are removed as shown. Second, each of the twenty remaining $3 \times 3 \times 3$ cubes is diminished in the same way. That is, the center facial unit cubes as well as each center cube are removed. The surface area of the final figure is



A 384

B 729

C 864

D 1024

E 1056

Solution:

After the first stage, 8 corner subcubes contribute 27 exposed units each, and 12 edge subcubes contribute 36 each. Tunneling a corner subcube removes 3 exposed unit squares and adds 24 tunnel-wall squares; tunneling an edge subcube removes 4 and also adds 24. Hence the final area is

$$\begin{aligned} &8(27-3+24) \\ &\qquad + 12(36-4+24) \\ &= 384 + 672 = 1056. \end{aligned}$$

Thus **E** is correct.

28. In triangle ABC, angle C is a right angle and $CB > CA$. Point D is located on $\overline{BC}$ so that angle CAD is twice angle DAB. If $\frac{AC}{AD} = \frac{2}{3}$, then $\frac{CD}{BD} = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

A 10

B 14

C 18

D 22

E 26

Solution:

Let $\angle DAB = \alpha$. Since $\frac{AC}{AD} = \cos 2\alpha = \frac{2}{3}$, the identity for $\cos 2\alpha$ gives $\tan\alpha = \frac{1}{\sqrt{5}}$. Taking $AC = 1$,

$$CD = \tan 2\alpha = \frac{\sqrt{5}}{2},$$
$$CB = \tan 3\alpha = \frac{7\sqrt{5}}{5}.$$

Thus $BD = CB - CD = \frac{9\sqrt{5}}{10}$, and $\frac{CD}{BD} = \frac{5}{9}$. Therefore $m + n = 14$, so **B** is correct.

29. A point (x, y) in the plane is called a *lattice point* if both x and y are integers. The area of the largest square that contains exactly three lattice points in its interior is closest to

A 4.0

B 4.2

C 4.5

D 5.0

E 5.6

Solution:

It suffices to enclose the three noncollinear lattice points $(0,0), (0,1), (1,0)$. In a maximal placement, two opposite sides are pinned by neighboring lattice points; the greatest possible separation is the distance $\sqrt{5}$ between parallel lines through $(1,1)$ and $(0,-1)$. Hence the area is at most 5. This is attained by the square bounded by

$$\begin{aligned} y &= 2x+2, \\ y &= 2x-3, \\ 2y+x &= 3, \\ 2y+x &= -2. \end{aligned}$$

Its area is 5 and exactly the three stated lattice points lie inside. Thus **D** is correct.

30. For each positive integer n, let

$$a_n = \frac{(n+9)!}{(n-1)!}.$$

Let k denote the smallest positive integer for which the rightmost nonzero digit of a_k is odd. The rightmost nonzero digit of a_k is

- A 1
- B 3
- C 5
- D 7
- E 9

Solution:

The five even terms in any ten consecutive integers contribute at least 2^8. Thus the rightmost nonzero digit can be odd only when the block contains at least eight factors of 5, first possible when it contains $5^7 = 78125$. For $n = 5^7 - 9$, the block has $v_2 = 9$ and $v_5 = 8$, so the digit remains even. For $n = 5^7 - 8 = 78117$, both valuations are 8. Cancelling $2^8 5^8$ and multiplying the remaining odd unit digits gives

$$\begin{aligned} &7 \cdot 9 \cdot 1 \cdot 3 \cdot 9 \cdot 3 \\ &\quad \cdot 1 \cdot 1 \cdot 3 \equiv 9 \pmod{10}. \end{aligned}$$

Hence the first odd rightmost nonzero digit is 9, and **E** is correct.

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