1997 AMC 12 Solutions

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1. If a and b are digits for which

$$\begin{array}{r} 2a \\ \times\ b3 \\ \hline 69 \\ 92 \\ \hline 989 \end{array}$$

then $a + b =$

A 3

B 4

C 7

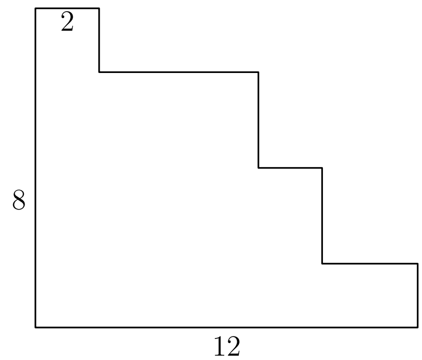
D 9

E 12

Solution:

The first partial product says $3(20 + a) = 69$, so $a = 3$. The second says $23b = 92$, so $b = 4$. Thus $a + b = 7$, and the correct answer is **C**.

2. The adjacent sides of the decagon shown meet at right angles. What is its perimeter?



A 22

B 32

C 34

D 44

E 50

Solution:

The total of the rightward horizontal sides is 12, so all horizontal sides total 24. The full height is $8+2=10$, so the vertical sides total 20. Therefore the perimeter is $24+20=44$, and the correct answer is **D**.

3. If $x, y,$ and z are real numbers such that

$$(x-3)^2+(y-4)^2 \\ +(z-5)^2=0,$$

then $x+y+z=$

A -12

B 0

C 8

D 12

E 50

Solution:

All three squares are nonnegative, so each must be 0. Hence $x=3,\ y=4,\ z=5,$ and $x+y+z=12.$ The correct answer is **D**.

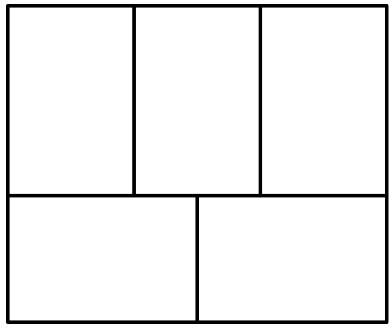
4. If a is 50% larger than c, and b is 25% larger than c, then a is what percent larger than b?

- A 20%
- B 25%
- C 50%
- D 100%
- E 200%

Solution:

We have $a = 1.5c$ and $b = 1.25c$. Thus $\frac{a}{b} = \frac{1.5}{1.25} = 1.2$, so a is 20% larger than b. The correct answer is **A**.

5. A rectangle with perimeter 176 is divided into five congruent rectangles as shown in the diagram. What is the perimeter of one of the five congruent rectangles?



A 35.2

B 76

C 80

D 84

E 86

Solution:

Let a small rectangle have sides x and y. The common width gives $3x = 2y$, while the large rectangle has dimensions $3x$ by $x + y$. Its perimeter is $2(4x + y) = 176$. Since $y = \frac{3x}{2}$, this is $11x = 176$, so $x = 16,\ y = 24$. One small perimeter is $2(16 + 24) = 80$, making **C** correct.

6. Consider the sequence

$$1, -2, 3, -4, 5, -6, \ldots,$$

whose nth term is $(-1)^{n+1}n$. What is the average of the first 200 terms of the sequence?

A -1

B -0.5

C 0

D 0.5

E 1

Solution:

Each pair $(2k-1) - 2k$ has sum -1. The 100 pairs therefore total -100, and their 200-term average is $-\frac{100}{200} = -0.5$. The correct answer is **B**.

7. The sum of seven integers is -1. What is the maximum number of the seven integers that can be larger than 13?

A 1

B 4

C 5

D 6

E 7

Solution:

If all seven integers exceeded $13,$ their sum would be at least $98.$ Six can exceed 13: take six copies of 14 and a seventh integer of $-85.$ Thus the maximum is $6,$ and the correct answer is **D**.

8. Mientka Publishing Company prices its best seller *Where's Walter?* as follows:

$$C(n) = \begin{cases} 12n, & 1 \le n \le 24, \\ 11n, & 25 \le n \le 48, \\ 10n, & 49 \le n, \end{cases}$$

where n is the number of books ordered, and $C(n)$ is the cost in dollars of n books. Notice that 25 books cost less than 24 books. For how many values of n is it cheaper to buy more than n books than to buy exactly n books?

A 3

B 4

C 5

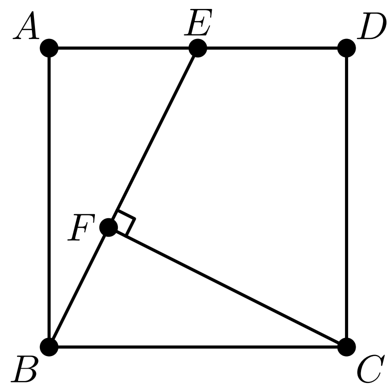
D 6

E 8

Solution:

At the first break, $C(25) = 275$ is below $C(23) = 276$ and $C(24) = 288$, giving $n = 23, 24$. At the second, $C(49) = 490$ is below $C(n) = 11n$ for $n = 45, 46, 47, 48$. No other n works because each piece increases. There are $2 + 4 = 6$, so **D** is correct.

9. In the figure, $ABCD$ is a 2 by 2 square, E is the midpoint of $\overline{AD}$, and F is on $\overline{BE}$. If $\overline{CF}$ is perpendicular to $\overline{BE}$, then the area of quadrilateral $CDEF$ is



A 2

B $3-\frac{\sqrt{3}}{2}$

C $\frac{11}{5}$

D $\sqrt{5}$

E $\frac{9}{4}$

Solution:

With the coordinates in the first hint, BE has equation $y = 2x$, and the perpendicular through C is $y = -\frac{(x-2)}{2}$. Their intersection is $F = \left(\frac{2}{5}, \frac{4}{5}\right)$. The square has area 4, while triangles AEF and BCF have areas $\frac{3}{5}$ and $\frac{4}{5}$, respectively. Hence $[CDEF] = 4 - \frac{3}{5} - \frac{4}{5} = \frac{11}{5}$, so **C** is correct.

10. Two six-sided dice are fair in the sense that each face is equally likely to turn up. However, one of the dice has the 4 replaced by 3 and the other die has the 3 replaced by $4.$ When these dice are rolled, what is the probability that the sum is odd?

- A $\frac{1}{3}$
- B $\frac{4}{9}$
- C $\frac{1}{2}$
- D $\frac{5}{9}$
- E $\frac{11}{18}$

Solution:

The first modified die has 4 odd faces and 2 even faces; the second has 2 odd and 4 even. Thus the probability is $(\frac{4}{6})(\frac{4}{6})+(\frac{2}{6})(\frac{2}{6})=\frac{20}{36}=\frac{5}{9}$. The correct answer is **D**.

11. In the sixth, seventh, eighth, and ninth basketball games of the season, a player scored $23, 14, 11,$ and 20 points, respectively. Her points-per-game average was higher after nine games than it was after the first five games. If her average after ten games was greater than $18,$ what is the least number of points she could have scored in the tenth game?

- A 26
- B 27
- C 28
- D 29
- E 30

Solution:

Games 6–9 total $68.$ The condition $\frac{S+68}{9} > \frac{S}{5}$ gives $S < 85,$ so the greatest integral S is $84.$ A ten-game average above 18 requires a total at least $181,$ hence the tenth score is at least $181 - (84 + 68) = 29.$ This is attainable, so **D** is correct.

12. If m and b are real numbers and $mb > 0$, then the line whose equation is $y = mx + b$ cannot contain the point

A $(0, 1997)$

B $(0, -1997)$

C $(19, 97)$

D $(19, -97)$

E $(1997, 0)$

Solution:

If $(1997, 0)$ were on the line, then $0 = 1997m + b$, so $b = -1997m$ and $mb = -1997m^2 < 0$, a contradiction. Each other point can occur for suitable same-sign m, b. Thus the correct answer is **E**.

13. How many two-digit positive integers N have the property that the sum of N and the number obtained by reversing the order of the digits of N is a perfect square?

A 4

B 5

C 6

D 7

E 8

Solution:

The sum is $11(x + y)$. Since $1 \leq x + y \leq 18$, this is square only when $x + y = 11$, giving 121. The tens digit x can be any of $2, 3, \ldots, 9$, with $y = 11 - x$. There are 8 integers, so **E** is correct.

14. The number of geese in a flock increases so that the difference between the populations in year $n+2$ and year n is directly proportional to the population in year $n+1$. If the populations in the years $1994, 1995$, and 1997 were $39, 60$, and 123, respectively, then the population in 1996 was

A 81

B 84

C 87

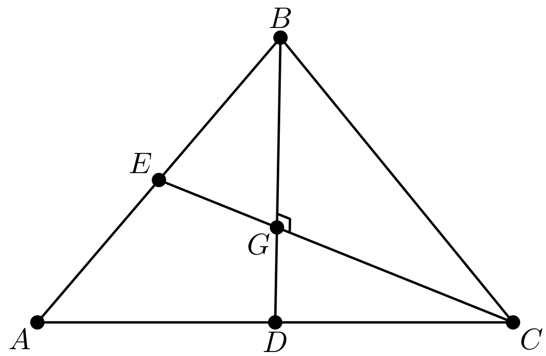
D 90

E 102

Solution:

The rule gives $x-39=60k$ and $123-60=kx$. Eliminating k yields $x(x-39)=3780$, or $(x-84)(x+45)=0$. The population is positive, so $x=84$, and **B** is correct.

15. Medians $\overline{BD}$ and $\overline{CE}$ of triangle ABC are perpendicular, $BD = 8,$ and $CE = 12.$ The area of triangle ABC is



A 24

B 32

C 48

D 64

E 96

Solution:

Quadrilateral $BCDE$ has perpendicular diagonals $BD = 8$ and $CE = 12,$ so its area is $\frac{1}{2}(8)(12) = 48.$ Since D, E are midpoints, triangle ADE has one fourth the area of $ABC,$ making $BCDE$ three fourths of it. Thus $[ABC] = \frac{48 \cdot 4}{3} = 64,$ so **D** is correct.

16. The three row sums and the three column sums of the array

$$\begin{bmatrix} 4 & 9 & 2 \\ 8 & 1 & 6 \\ 3 & 5 & 7 \end{bmatrix}$$

are the same. What is the least number of entries that must be altered to make all six sums different from one another?

A 1

B 2

C 3

D 4

E 5

Solution:

With at most three alterations, either two of the six lines are unchanged, or some altered entry is the only alteration in both its row and its column. In the first case those two sums remain equal; in the second, that entry changes its row sum and column sum by the same amount, so those two sums remain equal. Four suffice: replace $4, 1, 2, 6$ by $5, 3, 7, 9$, respectively. The resulting row sums are $21, 20, 15$ and column sums are $16, 17, 23$. Hence the minimum is 4, and **D** is correct.

17. A line $x = k$ intersects the graph of $y = \log_5 x$ and the graph of $y = \log_5(x+4)$. The distance between the points of intersection is 0.5. Given that $k = a + \sqrt{b}$, where a and b are integers, what is $a + b$?

A 6

B 7

C 8

D 9

E 10

Solution:

The vertical distance is

$$\begin{aligned} \log_5(k+4) - \log_5 k &= \frac{1}{2}, \\ \frac{k+4}{k} &= \sqrt{5}. \end{aligned}$$

Thus $\frac{k+4}{k} = \sqrt{5}$, so $k = \frac{4}{\sqrt{5}-1} = 1 + \sqrt{5}$. Therefore $a + b = 1 + 5 = 6$, and **A** is correct.

18. A list of integers has mode 32 and mean 22. The smallest number in the list is 10. The median m of the list is a member of the list. If the list member m were replaced by $m + 10$, the mean and median of the new list would be 24 and $m + 10$, respectively. If m were instead replaced by $m - 8$, the median of the new list would be $m - 4$. What is m?

A 16

B 17

C 18

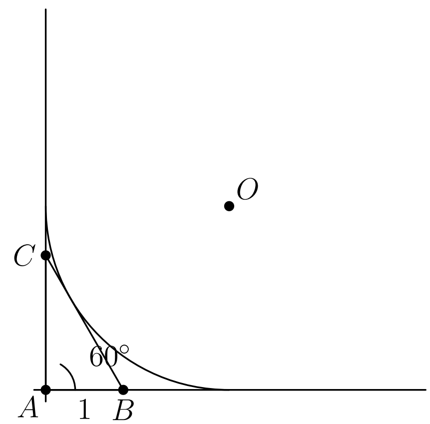
D 19

E 20

Solution:

The list has $\frac{10}{2} = 5$ entries. Write them $10 \leq a \leq m \leq b \leq c$. Replacing m by $m + 10$ makes that value the median, so $b, c \geq m + 10$; because the mode is 32, we must have $b = c = 32$. The original total is 110, giving $a + m = 36$. Replacing m by $m - 8$ makes the median $a = m - 4$, so $2m - 4 = 36$ and $m = 20$. Thus **E** is correct.

19. A circle with center O is tangent to the coordinate axes and to the hypotenuse of the $30°$-$60°$-$90°$ triangle ABC as shown, where $AB = 1$. To the nearest hundredth, what is the radius of the circle?



A 2.18

B 2.24

C 2.31

D 2.37

E 2.41

Solution:

Take $A = (0,0)$, $B = (1,0)$, and $C = (0,\sqrt{3})$. The hypotenuse is $\sqrt{3}x + y = \sqrt{3}$, and the circle center is $O = (r,r)$. Tangency gives

$$\frac{\left|(\sqrt{3}+1)r - \sqrt{3}\right|}{2} = r.$$

The pictured external circle has $(\sqrt{3}+1)r - \sqrt{3} = 2r$, so $r = \frac{\sqrt{3}}{\sqrt{3}-1} = \frac{3+\sqrt{3}}{2} \approx 2.37$. Thus **D** is correct.

20. Which one of the following integers can be expressed as the sum of 100 consecutive positive integers?

A $1{,}627{,}384{,}950$

B $2{,}345{,}678{,}910$

C $3{,}579{,}111{,}300$

D $4{,}692{,}581{,}470$

E $5{,}815{,}937{,}260$

Solution:

The sum is $100a + (1 + \cdots + 100) = 100a + 5050,$ so it ends in $50.$ Only $1,627,384,950$ has that property, and it gives the positive integer $a = \frac{1{,}627{,}384{,}950 - 5050}{100}$. Thus **A** is correct.

21. For any positive integer n, let

$$f(n) = \begin{cases} \log_8 n, & \text{if } \log_8 n \text{ is rational}, \\ 0, & \text{otherwise.} \end{cases}$$

What is $\sum_{n=1}^{1997} f(n)$?

A $\log_8 2047$

B 6

C $\frac{55}{3}$

D $\frac{58}{3}$

E 585

Solution:

For integer n, $\log_8 n$ is rational exactly when n is a power of 2. The relevant values are $n = 2^k$ for $0 \le k \le 10$, and $f(2^k) = \frac{k}{3}$. Therefore the sum is $\frac{0+1+\cdots+10}{3} = \frac{55}{3}$, so **C** is correct.

22. Ashley, Betty, Carlos, Dick, and Elgin went shopping. Each had a whole number of dollars to spend, and together they had $\$56$. The absolute difference between the amounts Ashley and Betty had to spend was $\$19$. The absolute difference between the amounts Betty and Carlos had was $\$7$, between Carlos and Dick was $\$5$, between Dick and Elgin was $\$4$, and between Elgin and Ashley was $\$11$. How much did Elgin have?

A $\$6$

B $\$7$

C $\$8$

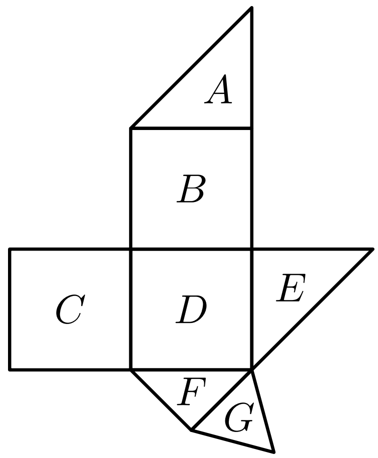
D $\$9$

E $\$10$

Solution:

The signed differences around the cycle have magnitudes $19, 7, 5, 4, 11$ and sum 0. Thus one side of the sign split must total half of 46, namely 23. The only split is $19 + 4 = 7 + 5 + 11$. In one orientation, the amounts are $A = E + 11, B = E - 8, C = E - 1$, and $D = E + 4$. Their total is $5E + 6$, so $5E + 6 = 56$ and $E = 10$. Reversing every sign would give $5E - 6 = 56$, not an integral E. Thus Elgin had $\$10$ and **E** is correct.

23. In the figure, polygons $A, E,$ and F are isosceles right triangles; $B, C,$ and D are squares with sides of length 1; and G is an equilateral triangle. The figure can be folded along its edges to form a polyhedron having the polygons as faces. The volume of this polyhedron is



A $\frac{1}{2}$

B $\frac{2}{3}$

C $\frac{3}{4}$

D $\frac{5}{6}$

E $\frac{4}{3}$

Solution:

The net forms a unit cube with one corner cut off. The removed corner is a triangular pyramid with three mutually perpendicular unit edges, so its volume is $\frac{1}{3} \cdot \frac{1}{2} \cdot 1 = \frac{1}{6}$. The remaining polyhedron has volume $1 - \frac{1}{6} = \frac{5}{6}$, and the correct answer is **D**.

24. A *rising number*, such as $34689,$ is a positive integer each digit of which is larger than each of the digits to its left. There are $\binom{9}{5}=126$ five-digit rising numbers. When these numbers are arranged from smallest to largest, the 97th number in the list does not contain the digit

A 4

B 5

C 6

D 7

E 8

Solution:

There are $\binom{8}{4}=70$ beginning with $1.$ Among those beginning with $2,$ the first $\binom{6}{3}=20$ begin with $23,$ occupying positions 71–$90.$ Thus the 97th overall is the seventh beginning with $24:$

$$\begin{gathered}2456,2457,2458,2459,\\2467,2468,2469.\end{gathered}$$

The number 2469 omits $5,$ so **B** is correct.

25. Let $ABCD$ be a parallelogram and let $\overrightarrow{AA'}, \overrightarrow{BB'}, \overrightarrow{CC'}$, and $\overrightarrow{DD'}$ be parallel rays in space on the same side of the plane determined by $ABCD$. If $AA' = 10, BB' = 8$, $CC' = 18, DD' = 22$, and M and N are the midpoints of $A'C'$ and $B'D'$, respectively, then $MN =$

A 0

B 1

C 2

D 3

E 4

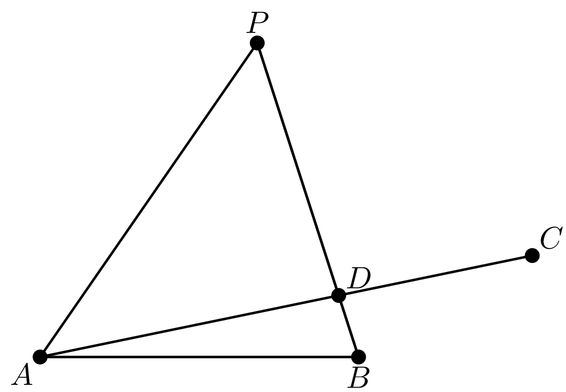
Solution:

Choose the common ray direction as a unit vector u. Then

$$M = \frac{A+C}{2} + 14u,$$
$$N = \frac{B+D}{2} + 15u.$$

Since $A + C = B + D$ for a parallelogram, $N - M = u$, so $MN = 1$. The correct answer is **B**.

26. Triangle ABC and point P in the same plane are given. Point P is equidistant from A and B, angle APB is twice angle ACB, and $\overline{AC}$ intersects $\overline{BP}$ at point D. If $PB = 3$ and $PD = 2$, then $AD \cdot CD =$



A 5

B 6

C 7

D 8

E 9

Solution:

Because $PA = PB$, draw their circle with center P. The condition $\angle APB = 2\angle ACB$ is the central-inscribed angle relation, so C lies on the same circle. Along line PB, the other circle intersection is E with $PE = 3$. Power of D gives

$$\begin{aligned} AD \cdot CD &= DE \cdot DB \\ &= (3-2)(3+2) = 5. \end{aligned}$$

Thus **A** is correct.

27. Consider those functions f that satisfy $f(x+4)+f(x-4)=f(x)$ for all real x. Any such function is periodic, and there is a least common positive period p for all of them. Find p.

- A 8
- B 12
- C 16
- D 24
- E 32

Solution:

For $u_n = f(x+4n)$, the equation is $u_{n+1} = u_n - u_{n-1}$. Starting from u_0, u_1, the sequence is

$$u_0, u_1, u_1 - u_0, -u_0,$$
$$-u_1, u_0 - u_1, u_0, u_1, \ldots,$$

so every such function has period $6 \cdot 4 = 24$. This is least because $f(x) = \sin(\frac{\pi x}{12})$ satisfies the equation and has fundamental period 24. Hence **D** is correct.

28. How many ordered triples of integers (a, b, c) satisfy

$$\begin{aligned} |a+b|+c &= 19, \\ ab+|c| &= 97? \end{aligned}$$

A 0

B 4

C 6

D 10

E 12

Solution:

If $c \geq 0$, substitution and factoring lead to no pair consistent with the required sign of $a+b$. Thus $c<0$, so $s>19$ and $ab+s=116$. If $a+b=s$, then $(a+1)(b+1)=117$, producing the unordered pairs $\{0,116\}, \{2,38\}, \{8,12\}$. If $a+b=-s$, then $(a-1)(b-1)=117$, producing $\{-116,0\}, \{-38,-2\}, \{-12,-8\}$. Each unordered pair has two orders, giving $6 \cdot 2 = 12$ triples. The correct answer is **E**.

29. Call a positive real number *special* if it has a decimal representation that consists entirely of digits 0 and 7. For example, $\frac{700}{99} = 7.\overline{07} = 7.070707\ldots$ and 77.007 are special numbers. What is the smallest n such that 1 can be written as a sum of n special numbers?

A 7

B 8

C 9

D 10

E 1 cannot be represented as a sum of finitely many special numbers

Solution:

Suppose 1 is a sum of n special numbers, and let a_k count summands having a 7 in the kth decimal place. Dividing by 7 gives

$$\begin{aligned} \frac{1}{7} &= \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots \\ &= 0.\overline{142857}. \end{aligned}$$

For $n \leq 9$, each a_k is a digit, so $a_1, a_2, \ldots = 1, 4, 2, 8, 5, 7, \ldots$; hence $n \geq 8$. Eight suffice because the repeating special decimals represented by

$$\begin{aligned} 700700 &+ 2(070707) \\ &+ 2(077777) \\ &+ 3(000777) = 999999. \end{aligned}$$

Their sum is 1. Therefore the minimum is 8, and **B** is correct.

30. For positive integers n, denote by $D(n)$ the number of pairs of different adjacent digits in the binary (base two) representation of n. For example, $D(3) = D(11_2) = 0$, $D(21) = D(10101_2) = 4$, and $D(97) = D(1100001_2) = 2$. For how many positive integers n less than or equal to 97 does $D(n) = 2$?

- A 16
- B 20
- C 26
- D 30
- E 35

Solution:

A d-bit numeral with exactly two changes has form $1^a 0^b 1^c$ with positive a, b, c, giving $\binom{d-1}{2}$ choices. For $d = 3, 4, 5, 6$, the total is $1 + 3 + 6 + 10 = 20$. Among seven-bit numbers at most $1100001_2 = 97$, the five forms beginning with one 1 all work, and the only form beginning with at least two 1s is 1100001_2 itself. Thus there are $20 + 6 = 26$, and **C** is correct.

Problems: https://live.poshenloh.com/past-contests/amc12/1997