1996 AMC 12 Solutions

Typeset by: LIVE by Po-Shen Loh
https://live.poshenloh.com/past-contests/amc12/1996/solutions

1. The addition below is incorrect. What is the largest digit that can be changed to make the addition correct?

$$\begin{array}{r} 641 \\ 852 \\ +\,973 \\ \hline 2456 \end{array}$$

A 4

B 5

C 6

D 7

E 8

Solution:

The three addends total $641 + 852 + 973 = 2466,$ which is 10 too large. Changing the tens digit 7 in 973 to 6 lowers the sum by 10 and gives $2456.$ No larger listed digit can make that change, so the correct answer is **D**.

2. Each day Walter gets $\$3$ for doing his chores or $\$5$ for doing them exceptionally well. After 10 days of doing his chores daily, Walter has received a total of $\$36.$ On how many days did Walter do them exceptionally well?

A 3

B 4

C 5

D 6

E 7

Solution:

Ten ordinary days would pay $10 \cdot 3 = 30$ dollars. Each exceptional day adds $5 - 3 = 2$ dollars, and the actual total is 6 dollars higher. Thus there were $\frac{6}{2} = 3$ exceptional days, so the correct answer is **A**.

3. $\frac{(3!)!}{3!} =$

A 1

B 2

C 6

D 40

E 120

Solution:

Since $3! = 6,$ the expression is $\frac{6!}{6} = 5! = 120.$ Thus the correct answer is **E**.

4. Six numbers from a list of nine integers are $7, 8, 3, 5, 9,$ and 5. The largest possible value of the median of all nine numbers in this list is

- A 5
- B 6
- C 7
- D 8
- E 9

Solution:

The six specified values in order are $3, 5, 5, 7, 8, 9$. Even if all three unspecified integers exceed 9, the fifth entry of the full sorted list is 8. This is attainable, so the correct answer is **D**.

5. Given that $0 < a < b < c < d$, which of the following is the largest?

- A $\frac{a+b}{c+d}$
- B $\frac{a+d}{b+c}$
- C $\frac{b+c}{a+d}$
- D $\frac{b+d}{a+c}$
- E $\frac{c+d}{a+b}$

Solution:

Among the displayed numerators, $c + d$ is largest; among the denominators, $a + b$ is smallest. Therefore $\frac{c+d}{a+b}$ exceeds every other positive fraction listed, so the correct answer is **E**.

6. If $f(x) = x^{x+1}(x+2)^{x+3}$, then $f(0) + f(-1) + f(-2) + f(-3) =$

A $-\frac{8}{9}$

B 0

C $\frac{8}{9}$

D 1

E $\frac{10}{9}$

Solution:

Direct substitution gives

$$\begin{aligned} f(0) &= 0, \\ f(-1) &= 1, \\ f(-2) &= 0, \\ f(-3) &= \frac{1}{9}. \end{aligned}$$

Their sum is $\frac{10}{9}$, so the correct answer is **E**.

7. A father takes his twins and a younger child out to dinner on the twins' birthday. The restaurant charges $\$4.95$ for the father and $\$0.45$ for each year of a child's age, where age is defined as the age at the most recent birthday. If the bill is $\$9.45$, which of the following could be the age of the youngest child?

A 1

B 2

C 3

D 4

E 5

Solution:

The children account for $9.45 - 4.95 = 4.50$ dollars, so their ages total $\frac{4.50}{0.45} = 10$. If each twin is t and the younger child is y, then $2t + y = 10$ with $y < t$. Of the choices, $y = 2$ gives $t = 4$, which works. Thus the correct answer is **B**.

8. If $3 = k \cdot 2^r$ and $15 = k \cdot 4^r$, then $r =$

A $-\log_2 5$

B $\log_5 2$

C $\log_{10} 5$

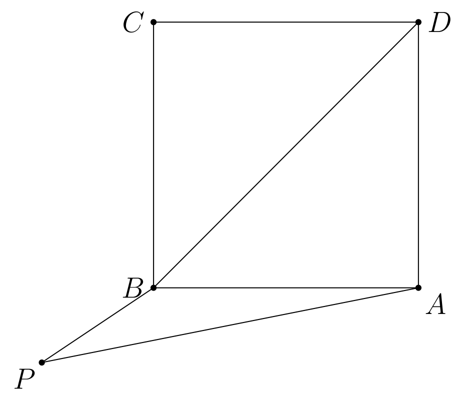
D $\log_2 5$

E $\frac{5}{2}$

Solution:

Dividing the equations gives $5 = (\frac{4}{2})^r = 2^r$. Hence $r = \log_2 5$, so the correct answer is **D**.

9. Triangle PAB and square $ABCD$ are in perpendicular planes. Given that $PA = 3$, $PB = 4$, and $AB = 5$, what is PD?



- A 5
- B $\sqrt{34}$
- C $\sqrt{41}$
- D $2\sqrt{13}$
- E 8

Solution:

Since $PA^2 + PB^2 = 3^2 + 4^2 = 5^2 = AB^2$, triangle PAB is right at P. In square $ABCD$, $AD = 5$ and $AD \perp AB$. Because the two planes are perpendicular along AB, AD is perpendicular to the plane of PAB, and hence to AP. Thus $PD^2 = PA^2 + AD^2 = 3^2 + 5^2 = 34$. The correct answer is **B**.

10. How many line segments have both their endpoints located at the vertices of a given cube?

A 12

B 15

C 24

D 28

E 56

Solution:

Every unordered pair of the cube's 8 vertices determines one segment, including edges and diagonals. There are $\binom{8}{2} = 28$, so the correct answer is **D**.

11. Given a circle of radius 2, there are many line segments of length 2 that are tangent to the circle at their midpoints. Find the area of the region consisting of all such line segments.

- A $\frac{\pi}{4}$
- B $4-\pi$
- C $\frac{\pi}{2}$
- D π
- E 2π

Solution:

Each segment is centered at a tangency point 2 units from the circle's center and extends 1 unit along the tangent in both directions. As the tangency point rotates, the segments fill the annulus with inner radius 2 and outer radius $\sqrt{2^2+1^2}=\sqrt{5}$. Its area is $\pi(5-4)=\pi$, so the correct answer is **D**.

12. A function f from the integers to the integers is defined as follows:

$$f(n) = \begin{cases} n+3 & \text{if } n \text{ is odd,} \\ \frac{n}{2} & \text{if } n \text{ is even.} \end{cases}$$

Suppose k is odd and $f(f(f(k))) = 27$. What is the sum of the digits of k?

A 3

B 6

C 9

D 12

E 15

Solution:

Since k is odd, $f(k) = k+3$ is even, so $f(f(k)) = \frac{k+3}{2}$. If this value were odd, adding 3 could not produce 27 because that would require the value 24. Therefore it is even and is halved to 27, so $\frac{k+3}{2} = 54$. Hence $k = 105$, whose digits sum to 6. The correct answer is **B**.

13. Sunny runs at a steady rate, and Moonbeam runs m times as fast, where m is a number greater than 1. If Moonbeam gives Sunny a head start of h meters, how many meters must Moonbeam run to overtake Sunny?

A hm

B $\frac{h}{h+m}$

C $\frac{h}{m-1}$

D $\frac{hm}{m-1}$

E $\frac{h+m}{m-1}$

Solution:

Suppose Moonbeam runs x meters. The elapsed time is $\frac{x}{mv}$, so Sunny runs $\frac{v \cdot x}{mv} = \frac{x}{m}$ meters after the start. At the catch, $x = h + \frac{x}{m}$. Solving gives $x = \frac{hm}{m-1}$, so the correct answer is **D**.

14. Let $E(n)$ denote the sum of the even digits of n. For example, $E(5681) = 6 + 8 = 14$. Find $E(1) + E(2) + E(3) + \cdots + E(100)$.

A 200

B 360

C 400

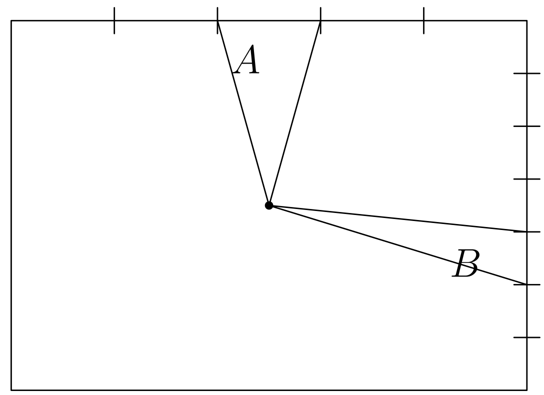
D 900

E 2250

Solution:

From 00 through 99, each digit occurs 10 times in each of the two positions. The positive even digits sum to $2 + 4 + 6 + 8 = 20$, so the total contribution is $2 \cdot 10 \cdot 20 = 400$. The number 100 contributes no even positive digit, so the correct answer is **C**.

15. Two opposite sides of a rectangle are each divided into n congruent segments, and the endpoints of one segment are joined to the center to form triangle A. The other sides are each divided into m congruent segments, and the endpoints of one of these segments are joined to the center to form triangle B. [See figure for $n = 5, m = 7$.] What is the ratio of the area of triangle A to the area of triangle B?



A 1

B $\frac{m}{n}$

C $\frac{n}{m}$

D $\frac{2m}{n}$

E $\frac{2n}{m}$

Solution:

Triangle A has base $\frac{w}{n}$ and altitude $\frac{h}{2}$, so its area is $\frac{wh}{4n}$. Triangle B has base $\frac{h}{m}$ and altitude $\frac{w}{2}$, so its area is $\frac{wh}{4m}$. Their ratio is $\frac{m}{n}$, and the correct answer is **B**.

16. A fair standard six-sided dice is tossed three times. Given that the sum of the first two tosses equals the third, what is the probability that at least one 2 is tossed?

A $\frac{1}{6}$

B $\frac{91}{216}$

C $\frac{1}{2}$

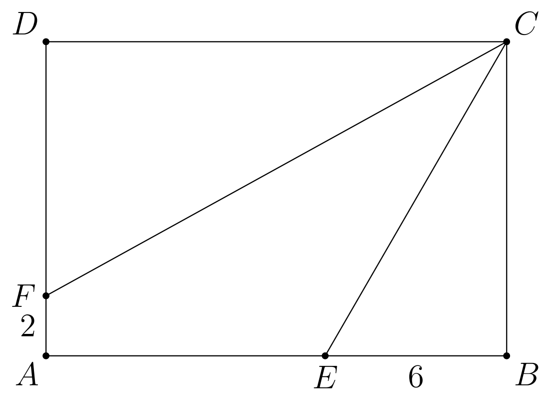
D $\frac{8}{15}$

E $\frac{7}{12}$

Solution:

For third tosses $2, 3, 4, 5, 6,$ the numbers of ordered first-two-toss pairs are $1, 2, 3, 4, 5,$ for 15 equally likely conditional outcomes. A third toss of 2 contributes $(1, 1, 2)$. A 2 in the first position gives 4 outcomes and a 2 in the second gives $4,$ with $(2, 2, 4)$ counted twice. Thus there are $1 + 4 + 4 - 1 = 8$ favorable outcomes, and the correct answer is **D**.

17. In rectangle $ABCD$, angle C is trisected by $\overline{CF}$ and $\overline{CE}$, where E is on $\overline{AB}$, F is on $\overline{AD}$, $BE = 6$, and $AF = 2$. Which of the following is closest to the area of the rectangle $ABCD$?



A 110

B 120

C 130

D 140

E 150

Solution:

Let the rectangle have width w and height h. In right triangle CBE, $\tan 30^\circ = \frac{BE}{CB} = \frac{6}{h}$, so $h = 6\sqrt{3}$. In the triangle using C, F, D, the horizontal run is w and the vertical drop is $h - 2$, so $\tan 60^\circ = \frac{w}{h-2}$. Hence

$$w = \sqrt{3}(6\sqrt{3} - 2) = 18 - 2\sqrt{3}.$$

The area is $6\sqrt{3}(18 - 2\sqrt{3}) = 108\sqrt{3} - 36 \approx 151.1$, closest to 150. Thus the correct answer is **E**.

18. A circle of radius 2 has center at $(2, 0)$. A circle of radius 1 has center at $(5, 0)$. A line is tangent to the two circles at points in the first quadrant. Which of the following is closest to the y-intercept of the line?

A $\frac{\sqrt{2}}{4}$

B $\frac{\sqrt{8}}{3}$

C $1 + \sqrt{3}$

D $2\sqrt{2}$

E 3

Solution:

Write the upper common tangent as $mx - y + b = 0$ and let $s = \sqrt{m^2 + 1}$. The two center-to-line distances give

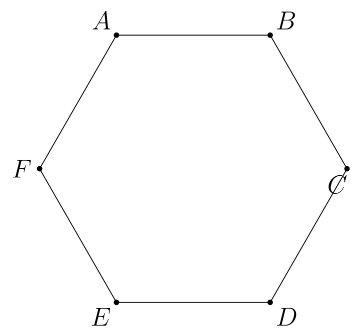
$$2m + b = 2s, \qquad 5m + b = s.$$

Thus $3m = -s$, so $9m^2 = m^2 + 1$ and $m = -\frac{1}{\sqrt{8}}$. From the second equation,

$$\begin{aligned} b &= s - 5m \\ &= \frac{3}{\sqrt{8}} + \frac{5}{\sqrt{8}} = 2\sqrt{2}. \end{aligned}$$

Therefore the correct answer is **D**.

19. The midpoints of the sides of a regular hexagon $ABCDEF$ are joined to form a smaller hexagon. What fraction of the area of $ABCDEF$ is enclosed by the smaller hexagon?



A $\frac{1}{2}$

B $\frac{\sqrt{3}}{3}$

C $\frac{2}{3}$

D $\frac{3}{4}$

E $\frac{\sqrt{3}}{2}$

Solution:

Two adjacent midpoints and their shared vertex form a triangle with two sides $\frac{s}{2}$ and included angle $120°$. Its opposite side has squared length

$$\frac{s^2}{4} + \frac{s^2}{4} - 2\left(\frac{s}{2}\right)^2 \cos 120^\circ$$
$$= \frac{3s^2}{4}.$$

Thus the smaller hexagon has scale factor $\frac{\sqrt{3}}{2}$, and its area ratio is $\frac{3}{4}$. The correct answer is **D**.

20. In the xy-plane, what is the length of the shortest path from $(0,0)$ to $(12,16)$ that does not go inside the circle $(x-6)^2+(y-8)^2=25$?

A $10\sqrt{3}$

B $10\sqrt{5}$

C $10\sqrt{3}+\frac{5\pi}{3}$

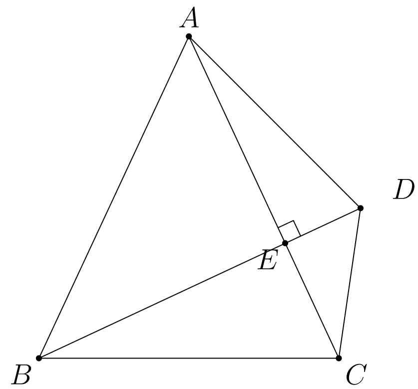
D $\frac{40\sqrt{3}}{3}$

E $10+5\pi$

Solution:

Each endpoint is 10 units from the circle's center, so each tangent segment has length $\sqrt{10^2-5^2}=5\sqrt{3}$. In the right triangle at a tangency point, the angle at the center is 60°. Since the endpoint rays are opposite, the intervening minor arc subtends $180^\circ-2(60^\circ)=60^\circ$, so its length is $\frac{5\pi}{3}$. The total is $10\sqrt{3}+\frac{5\pi}{3}$, and the correct answer is **C**.

21. Triangles ABC and ABD are isosceles with $AB = AC = BD$, and $\overline{BD}$ intersects $\overline{AC}$ at E. If $\overline{BD} \perp \overline{AC}$, then $\angle C + \angle D$ is



A 115°

B 120°

C 130°

D 135°

E not uniquely determined

Solution:

Let $\angle BAC = \alpha$. Since $AB = AC$,

$$\angle C = \frac{180^\circ - \alpha}{2} = 90^\circ - \frac{\alpha}{2}.$$

Because $BD \perp AC$, the angle between BA and BD is $90^\circ - \alpha$. In isosceles triangle ABD, $AB = BD$, so its two base angles are

$$\angle D = \frac{180^\circ - (90^\circ - \alpha)}{2} = 45^\circ + \frac{\alpha}{2}.$$

Their sum is 135°, so the correct answer is **D**.

22. Four distinct points, $A, B, C,$ and $D,$ are to be selected from 1996 points evenly spaced around a circle. All quadruples are equally likely to be chosen. What is the probability that the chord $\overline{AB}$ intersects the chord $\overline{CD}$?

- A $\frac{1}{4}$
- B $\frac{1}{3}$
- C $\frac{1}{2}$
- D $\frac{2}{3}$
- E $\frac{3}{4}$

Solution:

For any fixed four points, there are three ways to partition their labels into two unordered chord pairs. Exactly one pairing joins alternating points around the circle and therefore crosses. The named pair $\overline{AB}, \overline{CD}$ is equally likely to be any of these three pairings, so the probability is $\frac{1}{3}$. The correct answer is **B**.

23. The sum of the lengths of the twelve edges of a rectangular box is 140, and the distance from one corner of the box to the farthest corner is 21. The total surface area of the box is

A 776

B 784

C 798

D 800

E 812

Solution:

The edge sum gives $4(a+b+c)=140$, so $a+b+c=35$. The space diagonal gives $a^2+b^2+c^2=21^2=441$. Therefore the surface area is

$$\begin{aligned}&2(ab+bc+ca)\\&=(a+b+c)^2\\&-(a^2+b^2+c^2)\\&=35^2-441=784.\end{aligned}$$

Thus the correct answer is **B**.

24. The sequence

$$1, 2, 1, 2, 2, 1, 2, 2, 2, 1, 2,$$
$$2, 2, 2, 1, 2, 2, 2, 2, 2, 1, 2, \ldots$$

consists of 1's separated by blocks of 2's with n 2's in the nth block. The sum of the first 1234 terms of this sequence is

A 1996

B 2419

C 2429

D 2439

E 2449

Solution:

Through block k there are k ones and $\frac{k(k+1)}{2}$ twos, hence $\frac{k(k+3)}{2}$ terms. For $k = 48$ this is 1224. Their sum is

$$48 + 2\left(\frac{48 \cdot 49}{2}\right) = 2400.$$

The next 10 terms are one 1 and nine 2's, with sum 19. The requested sum is 2419, so the correct answer is **B**.

25. Given that $x^2 + y^2 = 14x + 6y + 6,$ what is the largest possible value that $3x + 4y$ can have?

A 72

B 73

C 74

D 75

E 76

Solution:

Completing the square gives

$$(x-7)^2 + (y-3)^2 = 64.$$

At the center, $3x + 4y = 3(7) + 4(3) = 33.$ Moving 8 units in the direction of the vector $(3, 4)$ increases the expression by $8\sqrt{3^2 + 4^2} = 40.$ The maximum is $73,$ so the correct answer is **B**.

26. An urn contains marbles of four colors: red, white, blue, and green. When four marbles are drawn without replacement, the following events are equally likely:

(a) the selection of four red marbles;

(b) the selection of one white and three red marbles;

(c) the selection of one white, one blue, and two red marbles; and

(d) the selection of one marble of each color.

What is the smallest number of marbles satisfying the given condition?

A 19

B 21

C 46

D 69

E more than 69

Solution:

Equal probabilities have the same common denominator, so their favorable selection counts satisfy $\binom{r}{4} = w\binom{r}{3} = wb\binom{r}{2} = wbgr$. Successive ratios give $w = \frac{r-3}{4}$, $b = \frac{r-2}{3}$, and $g = \frac{r-1}{2}$. The least $r \geq 4$ making all three positive integers is $r = 11$. Then $(w, b, g) = (2, 3, 5)$, for $11 + 2 + 3 + 5 = 21$ marbles. Thus the correct answer is **B**.

27. Consider two solid spherical balls, one centered at $(0, 0, \frac{21}{2})$ with radius 6, and the other centered at $(0, 0, 1)$ with radius $\frac{9}{2}$. How many points (x, y, z) with only integer coordinates (lattice points) are there in the intersection of the balls?

A 7

B 9

C 11

D 13

E 15

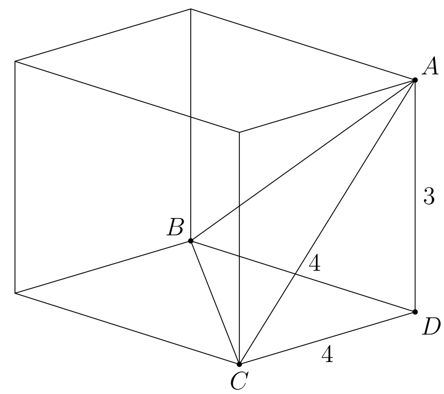
Solution:

The first ball permits integer heights 5 through 16, while the second permits -3 through 5, so an intersection lattice point must have $z = 5$. At that height the two bounds are

$$\begin{aligned} x^2 + y^2 &\le 36 - \left(5 - \frac{21}{2}\right)^2 \\ &= \frac{23}{4}, \\ x^2 + y^2 &\le \frac{81}{4} - (5-1)^2 \\ &= \frac{17}{4}. \end{aligned}$$

Thus $x^2 + y^2$ can be $0, 1, 2,$ or 4. These give $1, 4, 4,$ and 4 ordered integer pairs, respectively, for 13 points. The correct answer is **D**.

28. On a $4 \times 4 \times 3$ rectangular parallelepiped, vertices $A, B,$ and C are adjacent to vertex D. The perpendicular distance from D to the plane containing $A, B,$ and C is closest to



A 1.6

B 1.9

C 2.1

D 2.7

E 2.9

Solution:

Put $D = (0,0,0)$ and take the adjacent vertices as $(4,0,0), (0,4,0), (0,0,3)$. Their plane is

$$\frac{x}{4} + \frac{y}{4} + \frac{z}{3} = 1.$$

Its distance from the origin is

$$\frac{1}{\sqrt{(\frac{1}{4})^2 + (\frac{1}{4})^2 + (\frac{1}{3})^2}} = \frac{12}{\sqrt{34}} \approx 2.06,$$

closest to 2.1. Thus the correct answer is **C**.

29. If n is a positive integer such that $2n$ has 28 positive divisors and $3n$ has 30 positive divisors, then how many positive divisors does $6n$ have?

A 32

B 34

C 35

D 36

E 38

Solution:

Write $n = 2^a 3^b q$ with $\gcd(q, 6) = 1$, and let $t = \tau(q)$. Then

$$(a+2)(b+1)t = 28,$$
$$(a+1)(b+2)t = 30.$$

Hence t is 1 or 2. Checking the factor pairs of $14, 15$ gives no solution when $t = 2$. Checking those of $28, 30$ when $t = 1$ gives the unique solution $a = 5, b = 3$. Therefore

$$\tau(6n) = (a+2)(b+2)t$$
$$= 7 \cdot 5 = 35,$$

so the correct answer is **C**.

30. A hexagon inscribed in a circle has three consecutive sides each of length 3 and three consecutive sides each of length 5. The chord of the circle that divides the hexagon into two trapezoids, one with three sides each of length 3 and the other with three sides each of length 5, has length equal to $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

A 309

B 349

C 369

D 389

E 409

Solution:

Let the circle have radius R, and let α, β be the half-central angles for the sides $3, 5$. Then $3 = 2R\sin\alpha, 5 = 2R\sin\beta$, and $\alpha+\beta = 60^\circ$. Thus $\frac{5}{3} = \frac{\sin(60^\circ - \alpha)}{\sin\alpha} = \frac{\sqrt{3}}{2}\cot\alpha - \frac{1}{2}$, so $\tan\alpha = \frac{3\sqrt{3}}{13}$ and $\sin^2\alpha = \frac{27}{196}$. The dividing chord spans the three consecutive sides of length 3, so $\frac{L}{3} = \frac{\sin 3\alpha}{\sin\alpha} = 3 - 4\sin^2\alpha = 3 - \frac{27}{49} = \frac{120}{49}$. Therefore $L = \frac{360}{49}$, so $m+n = 409$. The correct answer is **E**.

Problems: https://live.poshenloh.com/past-contests/amc12/1996