1995 AMC 12 Solutions

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1. Kim earned scores of $87, 83$ and 88 on her first three mathematics examinations. If Kim receives a score of 90 on the fourth exam, then her average will

A remain the same

B increase by 1

C increase by 2

D increase by 3

E increase by 4

Solution:

The first three scores total $87 + 83 + 88 = 258,$ so their average is $\frac{258}{3} = 86.$ With the fourth score, the total is $348,$ and the new average is $\frac{348}{4} = 87.$ It increases by $1,$ so the correct answer is **B**.

2. If $\sqrt{2+\sqrt{x}} = 3$, then $x =$

- A 1
- B $\sqrt{7}$
- C 7
- D 49
- E 121

Solution:

Squaring gives $2+\sqrt{x} = 9$, so $\sqrt{x} = 7$. Squaring again gives $x = 49$. Thus the correct answer is **D**.

3. The total in-store price for an appliance is $\$99.99$. A television commercial advertises the same product for three easy payments of $\$29.98$ and a one-time shipping and handling charge of $\$9.98$. How much is saved by buying the appliance from the television advertiser?

- A 6 cents
- B 7 cents
- C 8 cents
- D 9 cents
- E 10 cents

Solution:

The advertised total is $3(29.98) + 9.98 = 99.92$ dollars. The savings is $99.99 - 99.92 = 0.07$ dollars, or 7 cents. Thus the correct answer is **B**.

4. If M is 30% of Q, Q is 20% of P, and N is 50% of P, then $\frac{M}{N} =$

A $\frac{3}{250}$

B $\frac{3}{25}$

C 1

D $\frac{6}{5}$

E $\frac{4}{3}$

Solution:

We have $M = (0.30)(0.20)P = 0.06P$ and $N = 0.50P$. Hence $\frac{M}{N} = \frac{0.06}{0.50} = \frac{3}{25}$. Thus the correct answer is **B**.

5. A rectangular field is 300 feet wide and 400 feet long. Random sampling indicates that there are, on the average, three ants per square inch throughout the field. [12 inches = 1 foot.] Of the following, the number that most closely approximates the number of ants in the field is

A 500 thousand

B 5 million

C 50 million

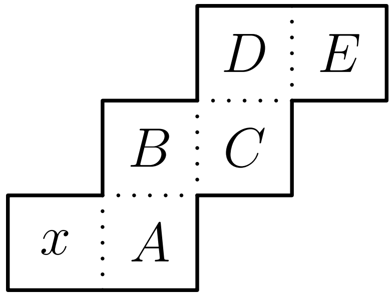
D 500 million

E 5 billion

Solution:

The area is $300 \cdot 400 = 120{,}000$ square feet, or $120{,}000 \cdot 144 = 17{,}280{,}000$ square inches. The estimate is $3(17{,}280{,}000) = 51{,}840{,}000$, closest to 50 million. Thus the correct answer is **C**.

6. The figure shown can be folded into the shape of a cube. In the resulting cube, which of the lettered faces is opposite the face marked x?



A A

B B

C C

D D

E E

Solution:

Hold A as the front face. Folding makes x the left face and B the top face. The face C, attached to the right of B, then folds to the right face, opposite x. Thus the correct answer is **C**.

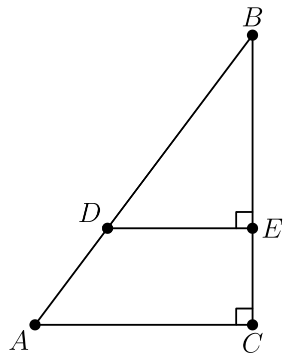
7. The radius of Earth at the equator is approximately 4000 miles. Suppose a jet flies once around Earth at a speed of 500 miles per hour relative to Earth. If the flight path is a negligible height above the equator, then, among the following choices, the best estimate of the number of hours of flight is

- A 8
- B 25
- C 50
- D 75
- E 100

Solution:

The equator is approximately $2\pi(4000) = 8000\pi$ miles long. The flight time is $\frac{8000\pi}{500} = 16\pi \approx 50$ hours. Thus the correct answer is **C**.

8. In triangle ABC, $\angle C = 90°$, $AC = 6$ and $BC = 8$. Points D and E are on $\overline{AB}$ and $\overline{BC}$, respectively, and $\angle BED = 90°$. If $DE = 4$, then $BD =$



A 5

B $\frac{16}{3}$

C $\frac{20}{3}$

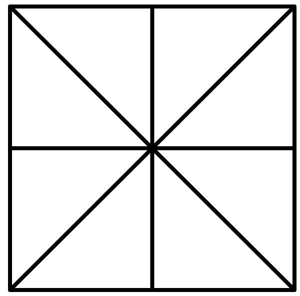
D $\frac{15}{2}$

E 8

Solution:

The large right triangle has $AB = \sqrt{6^2 + 8^2} = 10$. Since $\overline{DE} \parallel \overline{AC}$, triangles BDE and BAC are similar, with scale factor $\frac{DE}{AC} = \frac{4}{6} = \frac{2}{3}$. Therefore $BD = (\frac{2}{3})(10) = \frac{20}{3}$. Thus the correct answer is **C**.

9. Consider the figure consisting of a square, its diagonals, and the segments joining the midpoints of opposite sides. The total number of triangles of any size in the figure is



A 10

B 12

C 14

D 16

E 18

Solution:

There are 8 smallest triangles, each bounded by a half-side, a half-diagonal, and a half-midline. There are 4 triangles whose base is a full side and vertex is the center, and 4 half-square triangles cut off by a diagonal. The total is $8+4+4=16.$ Thus the correct answer is **D**.

10. The area of the triangle bounded by the lines $y = x, y = -x$ and $y = 6$ is

A 12

B $12\sqrt{2}$

C 24

D $24\sqrt{2}$

E 36

Solution:

The vertices are $(0, 0)$, $(6, 6)$, and $(-6, 6)$. The horizontal base has length 12, and the height is 6, so the area is $\frac{1}{2}(12)(6) = 36$. Thus the correct answer is **E**.

11. How many base 10 four-digit numbers, $N = \underline{a\,b\,c\,d}$, satisfy all three of the following conditions?

(i) $4{,}000 \leq N < 6{,}000;$
(ii) N is a multiple of $5;$
(iii) $3 \leq b < c \leq 6.$

A 10

B 18

C 24

D 36

E 48

Solution:

The thousands digit a is 4 or 5, and divisibility by 5 makes d either 0 or 5. The pair (b, c) is obtained by choosing two of $3, 4, 5, 6$ in increasing order, giving $\binom{4}{2} = 6$ pairs. Thus the count is $2 \cdot 2 \cdot 6 = 24$, and the correct answer is **C**.

12. Let f be a linear function with the properties that $f(1) \leq f(2), f(3) \geq f(4),$ and $f(5) = 5.$ Which of the following statements is true?

A $f(0) < 0$

B $f(0) = 0$

C $f(1) < f(0) < f(-1)$

D $f(0) = 5$

E $f(0) > 5$

Solution:

From $f(1) \leq f(2),$ the slope is nonnegative. From $f(3) \geq f(4),$ it is nonpositive. Thus the slope is $0,$ so f is constant. Since $f(5) = 5,$ we have $f(0) = 5.$ Thus the correct answer is **D**.

13. The addition below is incorrect. The display can be made correct by changing one digit d, wherever it occurs, to another digit e. Find the sum of d and e.

$$\begin{array}{ccccccc} & 7 & 4 & 2 & 5 & 8 & 6 \\ + & 8 & 2 & 9 & 4 & 3 & 0 \\ \hline 1 & 2 & 1 & 2 & 0 & 1 & 6 \end{array}$$

A 4

B 6

C 8

D 10

E more than 10

Solution:

The units, tens, and hundreds columns work with carries $0, 1, 1$. In the thousands column, replacing every 2 by 6 gives $6 + 9 + 1 = 16$, so the result digit is also 6. The next columns then give $4 + 6 + 1 = 11$ and $7 + 8 + 1 = 16$, producing the corrected equation $746586 + 869430 = 1616016$. Thus $d + e = 2 + 6 = 8$, and the correct answer is **C**.

14. If $f(x) = ax^4 - bx^2 + x + 5$ and $f(-3) = 2$, then $f(3) =$

A -5

B -2

C 1

D 3

E 8

Solution:

The even-power terms and constant are the same at 3 and -3, while the linear term changes from -3 to 3. Hence $f(3) - f(-3) = 6$, so $f(3) = 2 + 6 = 8$. Thus the correct answer is **E**.

15. Five points on a circle are numbered $1, 2, 3, 4,$ and 5 in clockwise order. A bug jumps in a clockwise direction from one point to another around the circle; if it is on an odd-numbered point, it moves one point, and if it is on an even-numbered point, it moves two points. If the bug begins on point 5, after 1995 jumps it will be on point

A 1

B 2

C 3

D 4

E 5

Solution:

Starting at 5, the landing points are $1, 2, 4, 1, 2, 4, \ldots$ Thus the cycle $1, 2, 4$ has length 3. Since 1995 is divisible by 3, the 1995th landing point is 4. Thus the correct answer is **D**.

16. Anita attends a baseball game in Atlanta and estimates that there are $50{,}000$ fans in attendance. Bob attends a baseball game in Boston and estimates that there are $60{,}000$ fans in attendance. A league official who knows the actual numbers attending the two games notes that:

i. The actual attendance in Atlanta is within 10% of Anita's estimate.

ii. Bob's estimate is within 10% of the actual attendance in Boston.

To the nearest $1{,}000$, the largest possible difference between the numbers attending the two games is

A $10{,}000$

B $11{,}000$

C $20{,}000$

D $21{,}000$

E $22{,}000$

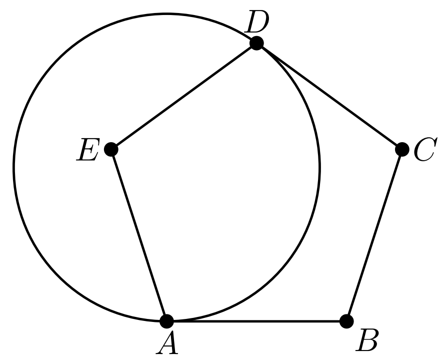
Solution:

Atlanta's actual attendance A satisfies $45{,}000 \le A \le 55{,}000$. For Boston's actual attendance B, the condition is $|60{,}000 - B| \le 0.1B$, so $\frac{60{,}000}{1.1} \le B \le \frac{60{,}000}{0.9}$. The largest difference is therefore

$$\frac{60{,}000}{0.9} - 45{,}000 = 21{,}666.\overline{6},$$

which rounds to $22{,}000$. Thus the correct answer is **E**.

17. Given regular pentagon $ABCDE,$ a circle can be drawn that is tangent to $\overline{DC}$ at D and to $\overline{AB}$ at $A.$ The number of degrees in minor arc $\overset{\frown}{AD}$ is



A 72

B 108

C 120

D 135

E 144

Solution:

Let O be the circle's center. The direction changes by 72° at each vertex of the pentagon, so the acute angle between lines AB and DC is $36^\circ.$ The radii to their tangency points are perpendicular to these lines; for the circle shown, the minor central angle $\angle AOD$ is the supplementary angle $180^\circ - 36^\circ = 144^\circ.$ Therefore minor arc $\overset{\frown}{AD}$ measures $144^\circ,$ and the correct answer is **E**.

18. Two rays with common endpoint O form a 30° angle. Point A lies on one ray, point B on the other ray, and $AB = 1$. The maximum possible length of OB is

- A 1
- B $\frac{1+\sqrt{3}}{\sqrt{2}}$
- C $\sqrt{3}$
- D 2
- E $\frac{4}{\sqrt{3}}$

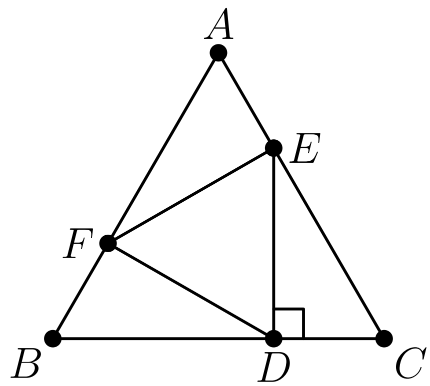
Solution:

Let $OA = x$ and $OB = y$. The Law of Cosines gives

$$1 = x^2 + y^2 - \sqrt{3}xy.$$

As a quadratic in x, this has discriminant $3y^2 - 4(y^2 - 1) = 4 - y^2$. A real x requires $y \leq 2$, and equality is attained when $x = \sqrt{3}$. Hence the maximum is 2, and the correct answer is **D**.

19. Equilateral triangle DEF is inscribed in equilateral triangle ABC as shown with $\overline{DE} \perp \overline{BC}$. The ratio of the area of $\triangle DEF$ to the area of $\triangle ABC$ is



A $\frac{1}{6}$

B $\frac{1}{4}$

C $\frac{1}{3}$

D $\frac{2}{5}$

E $\frac{1}{2}$

Solution:

Let ABC have side 1, with $B = (0,0)$ and $C = (1,0)$. If the inner side is s, write $D = (d,0)$ and $E = (d,s)$. Since E lies on AC, $s = \sqrt{3}(1-d)$. The third inner vertex is $F = (d - \frac{\sqrt{3}}{2}s, \frac{1}{2}s)$, and F lies on AB, so

$$\frac{s}{2} = \sqrt{3}\left(d - \frac{\sqrt{3}}{2}s\right),$$

giving $d = \frac{2s}{\sqrt{3}}$. Substitution yields $s = \frac{1}{\sqrt{3}}$. Areas of equilateral triangles scale as the square of their sides, so the ratio is $s^2 = \frac{1}{3}$. Thus the correct answer is **C**.

20. If a, b and c are three (not necessarily different) numbers chosen randomly and with replacement from the set $\{1, 2, 3, 4, 5\}$, the probability that $ab + c$ is even is

A $\frac{2}{5}$

B $\frac{59}{125}$

C $\frac{1}{2}$

D $\frac{64}{125}$

E $\frac{3}{5}$

Solution:

There are 3 odd and 2 even choices. The product ab is odd with probability $\frac{9}{25}$ and even with probability $\frac{16}{25}$. Therefore

$$\begin{aligned} P(ab + c \text{ even}) &= \frac{9}{25} \cdot \frac{3}{5} \\ &\quad + \frac{16}{25} \cdot \frac{2}{5} \\ &= \frac{59}{125}. \end{aligned}$$

Thus the correct answer is **B**.

21. Two nonadjacent vertices of a rectangle are $(4, 3)$ and $(-4, -3)$, and the coordinates of the other two vertices are integers. The number of such rectangles is

A 1

B 2

C 3

D 4

E 5

Solution:

The given diagonal has midpoint $(0, 0)$ and half-length 5. The other diagonal must therefore have endpoints (u, v) and $(-u, -v)$ with $u^2 + v^2 = 25$. Conversely, two equal diagonals with the same midpoint form a rectangle. There are 12 integer points on this circle:

$$(\pm 5, 0), (0, \pm 5),$$
$$(\pm 3, \pm 4), (\pm 4, \pm 3).$$

Antipodal points determine the same diagonal, giving 6 possibilities. One is the given diagonal itself, which is degenerate, so $6 - 1 = 5$ rectangles remain. Thus the correct answer is **E**.

22. A pentagon is formed by cutting a triangular corner from a rectangular piece of paper. The five sides of the pentagon have lengths $13, 19, 20, 25$ and $31,$ although this is not necessarily their order around the pentagon. The area of the pentagon is

- A 459
- B 600
- C 680
- D 720
- E 745

Solution:

The full rectangle sides must be longer than the two remnants on those same sides. The only assignment whose two differences and remaining cut side form a right triangle is the rectangle 25 by $31,$ with remnants 20 and $19.$ The removed corner then has legs $25 - 20 = 5$ and $31 - 19 = 12,$ whose hypotenuse is the listed side $13.$ Thus the pentagon's area is

$$\begin{aligned} 25 \cdot 31 - \frac{1}{2}(5)(12) &= 775 - 30 \\ &= 745. \end{aligned}$$

Hence the correct answer is **E**.

23. The sides of a triangle have lengths $11, 15$, and k, where k is an integer. For how many values of k is the triangle obtuse?

A 5

B 7

C 12

D 13

E 14

Solution:

The triangle inequality gives $5 \leq k \leq 25$. For $k \leq 15$, the longest side is 15, and the triangle is obtuse when

$$15^2 > 11^2 + k^2,$$

which gives $5 \leq k \leq 10$, or 6 values. For $k > 15$, it is obtuse when $k^2 > 11^2 + 15^2 = 346$, giving $19 \leq k \leq 25$, or 7 values. The total is $6 + 7 = 13$, so the correct answer is **D**.

24. There exist positive integers A, B, and C, with no common factor greater than 1, such that

$$A \log_{200} 5 + B \log_{200} 2 = C.$$

What is $A + B + C$?

A 6

B 7

C 8

D 9

E 10

Solution:

Combining logarithms gives

$$\log_{200}(5^A 2^B) = C,$$

so $5^A 2^B = 200^C = 5^{2C} 2^{3C}$. Hence $A = 2C$ and $B = 3C$. The relatively prime positive triple is $(A, B, C) = (2, 3, 1)$, whose sum is 6. Thus the correct answer is **A**.

25. A list of five positive integers has mean 12 and range 18. The mode and median are both 8. How many different values are possible for the second largest element of the list?

- A 4
- B 6
- C 8
- D 10
- E 12

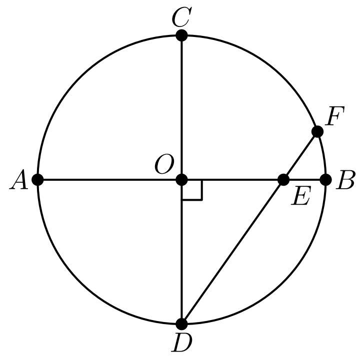
Solution:

Write the sorted list as $m, 8, 8, d, m + 18$. (If the fourth entry were 8, the sum and ordering conditions would be impossible.) Since the total is $5(12) = 60$,

$$m + 8 + 8 + d + (m + 18) = 60,$$

so $d = 26 - 2m$. The conditions $8 \leq d \leq m + 18$ and $m \leq 8$ give $3 \leq m \leq 8$. These six values yield $d = 20, 18, 16, 14, 12, 10,$ all valid. Thus there are 6 possibilities, and the correct answer is **B**.

26. In the figure, $\overline{AB}$ and $\overline{CD}$ are diameters of the circle with center O, $\overline{AB} \perp \overline{CD}$, and chord $\overline{DF}$ intersects $\overline{AB}$ at E. If $DE = 6$ and $EF = 2$, then the area of the circle is



A 23π

B $\frac{47}{2}\pi$

C 24π

D $\frac{49}{2}\pi$

E 25π

Solution:

Let the radius be r and $OE = x$. Intersecting chords gives

$$(r + x)(r - x) = 6 \cdot 2,$$

so $r^2 - x^2 = 12$. Put $O = (0, 0)$, $D = (0, -r)$, and $E = (x, 0)$. Since $DE : EF = 3 : 1$,

$$\begin{aligned} F &= D + \frac{4}{3}(E - D) \\ &= \left(\frac{4x}{3}, \frac{r}{3}\right). \end{aligned}$$

Because F lies on the circle, $\frac{16x^2}{9} + \frac{r^2}{9} = r^2$, so $r^2 = 2x^2$. Therefore $x^2 = 12$ and $r^2 = 24$. The area is 24π, so the correct answer is **C**.

27. Consider the triangular array of numbers with $0, 1, 2, 3, \ldots$ along the sides and interior numbers obtained by adding the two adjacent numbers in the previous row. Rows 1 through 6 are shown.

$$\begin{array}{ccccccccccc}
 & & & & & 0 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 2 & & 2 & & 2 & & & \\
 & & 3 & & 4 & & 4 & & 3 & & \\
 & 4 & & 7 & & 8 & & 7 & & 4 & \\
5 & & 11 & & 15 & & 15 & & 11 & & 5
\end{array}$$

Let $f(n)$ denote the sum of the numbers in row n. What is the remainder when $f(100)$ is divided by 100?

A 12

B 30

C 50

D 62

E 74

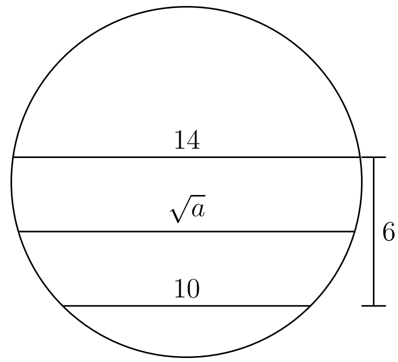
Solution:

Every previous entry contributes to two entries of the next row, and the two new boundary entries contribute an additional 2. Thus $f(n) = 2f(n-1) + 2$, with $f(1) = 0$. Hence

$$f(n) = 2^n - 2.$$

Since $2^{20} \equiv 76 \pmod{100}$ and $76^2 \equiv 76 \pmod{100}$, we have $2^{100} \equiv 76 \pmod{100}$. Therefore $f(100) \equiv 74 \pmod{100}$, and the correct answer is **E**.

28. Two parallel chords in a circle have lengths 10 and 14, and the distance between them is 6. The chord parallel to these chords and midway between them is of length $\sqrt{a}$, where a is



A 144

B 156

C 168

D 176

E 184

Solution:

Let the signed distances from the center to the 14- and 10-chords be u and v, with $u - v = 6$. Then

$$r^2 - u^2 = 7^2, \qquad r^2 - v^2 = 5^2.$$

Hence $v^2 - u^2 = 24$, so $(v - u)(v + u) = 24$. Since $v - u = -6$, we get $u + v = -4$, and therefore $u = 1, v = -5$. The midway chord is at signed distance -2, while $r^2 = 49 + 1 = 50$. Its squared length is $4(50 - 4) = 184$, so $a = 184$. Thus the correct answer is **E**.

29. For how many three-element sets of positive integers $\{a, b, c\}$ is it true that $a \cdot b \cdot c = 2310$?

A 32

B 36

C 40

D 43

E 45

Solution:

Since $2310 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$, each prime belongs to exactly one of the three factors. If all factors exceed 1, their unordered prime groups form a partition of five objects into three nonempty blocks, counted by

$$S(5,3) = \frac{3^5 - 3 \cdot 2^5 + 3}{6} = 25.$$

If one factor is 1, the primes are partitioned into two nonempty blocks, giving $S(5,2) = 2^4 - 1 = 15$. The factors are distinct because their disjoint prime sets differ. Thus the total is $25 + 15 = 40$, and the correct answer is **C**.

30. A large cube is formed by stacking 27 unit cubes. A plane is perpendicular to one of the internal diagonals of the large cube and bisects that diagonal. The number of unit cubes that the plane intersects is

A 16

B 17

C 18

D 19

E 20

Solution:

Take the large cube as $[0,3]^3$. The plane is $x+y+z=\frac{9}{2}$. A unit cube with lower corner (i,j,k), where $i,j,k \in \{0,1,2\}$, contains values of $x+y+z$ from $s=i+j+k$ to $s+3$. It intersects the plane exactly when $s=2,3,$ or 4. The coefficients of x^2, x^3, x^4 in

$$(1+x+x^2)^3$$

are $6,7,6$. Thus the plane intersects $6+7+6=19$ unit cubes, and the correct answer is **D**.

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