1994 AMC 12 Solutions

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1. $4^4 \cdot 9^4 \cdot 4^9 \cdot 9^9 =$

- A 13^{13}
- B 13^{36}
- C 36^{13}
- D 36^{36}
- E 1296^{26}

Solution:

Combining like bases and then pairing them,

$$\begin{aligned} 4^4 \cdot 4^9 \cdot 9^4 \cdot 9^9 &= 4^{13} 9^{13} \\ &= (4 \cdot 9)^{13} \\ &= 36^{13}. \end{aligned}$$

Thus the correct answer is **C**.

2. A large rectangle is partitioned into four rectangles by two segments parallel to its sides. The areas of three of the resulting rectangles are shown. What is the area of the fourth rectangle?

6	14
?	35

A 10

B 15

C 20

D 21

E 25

Solution:

Let the column widths be u, v and the row heights be s, t. The displayed areas give $us = 6, vs = 14$, and $vt = 35$. Therefore the missing area is

$$ut = \frac{(us)(vt)}{vs} = \frac{6 \cdot 35}{14} = 15.$$

Thus the correct answer is **B**.

3. How many of the following are equal to $x^x + x^x$ for all $x > 0$?

$$\begin{aligned} \mathrm{I}: &\quad 2x^x \\ \mathrm{II}: &\quad x^{2x} \\ \mathrm{III}: &\quad (2x)^x \\ \mathrm{IV}: &\quad (2x)^{2x} \end{aligned}$$

- A 0
- **B** 1
- C 2
- D 3
- E 4

Solution:

The given sum is $2x^x$, so expression I is always equal to it. Expressions II, III, and IV change the exponent, the base, or both; for example, at $x = 1$ their values are $1, 2,$ and $4,$ respectively, while the target is $2.$ Hence only one expression works. Thus the correct answer is **B**.

4. In the xy-plane, the segment with endpoints $(-5, 0)$ and $(25, 0)$ is the diameter of a circle. If the point $(x, 15)$ is on the circle, then $x =$

A 10

B 12.5

C 15

D 17.5

E 20

Solution:

The center is $(10, 0)$ and the radius is 15. Thus

$$(x-10)^2 + 15^2 = 15^2,$$

so $x = 10$. Thus the correct answer is **A**.

5. Pat intended to multiply a number by 6 but instead divided by $6.$ Pat then meant to add 14 but instead subtracted $14.$ After these mistakes, the result was $16.$ If the correct operations had been used, the value produced would have been

- A less than 400
- B between 400 and 600
- C between 600 and 800
- D between 800 and 1000
- E greater than 1000

Solution:

If the starting number is $x,$ the mistaken calculation gives

$$\frac{x}{6} - 14 = 16,$$

so $x = 180.$ The intended calculation would produce $6(180) + 14 = 1094,$ which is greater than $1000.$ Thus the correct answer is **E**.

6. In the sequence

$$\ldots, a, b, c, d, 0, 1, 1, 2, 3, 5, 8, \ldots$$

each term is the sum of the two terms to its left. Find a.

A -3

B -1

C 0

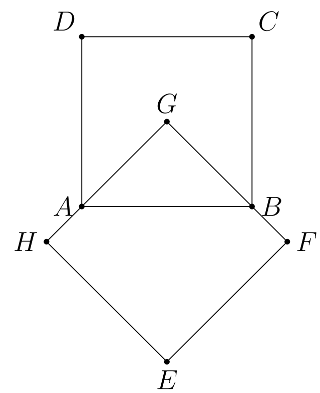
D 1

E 3

Solution:

Working backward, $d + 0 = 1$ gives $d = 1$, then $c + d = 0$ gives $c = -1$, and $b + c = d$ gives $b = 2$. Finally $a + b = c$, so $a = -1 - 2 = -3$. Thus the correct answer is **A**.

7. Squares $ABCD$ and $EFGH$ are congruent, $AB = 10$, and G is the center of square $ABCD$. The area of the region in the plane covered by these squares is



A 75

B 100

C 125

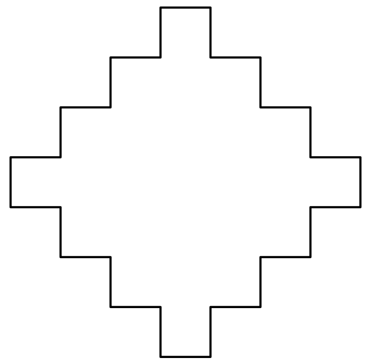
D 150

E 175

Solution:

Each square has area 100. Their overlap is $\triangle GAB$, whose base is $AB = 10$ and whose altitude from the center G to $\overline{AB}$ is 5. Its area is $\frac{1}{2}(10)(5) = 25$. Hence the union has area $100 + 100 - 25 = 175$. Thus the correct answer is **E**.

8. In the polygon shown, each side is perpendicular to its adjacent sides, and all 28 of the sides are congruent. The perimeter of the polygon is 56. The area of the region bounded by the polygon is



A 84

B 96

C 100

D 112

E 196

Solution:

Each side has length 2. Measured in side-length units, the seven horizontal strips have widths

$$1, 3, 5, 7, 5, 3, 1,$$

totaling 25 unit squares. Each such square has area $2^2 = 4$, so the polygon's area is $25 \cdot 4 = 100$. Thus the correct answer is **C**.

9. If $\angle A$ is four times $\angle B$, and the complement of $\angle B$ is four times the complement of $\angle A$, then $\angle B =$

A 10°

B 12°

C 15°

D 18°

E 22.5°

Solution:

Let $\angle B = x$, so $\angle A = 4x$. The complement condition gives

$$90 - x = 4(90 - 4x).$$

Hence $15x = 270$ and $x = 18^\circ$. Thus the correct answer is **D**.

10. For distinct real numbers x and y, let $M(x, y)$ be the larger of x and y and let $m(x, y)$ be the smaller of x and y. If

$$a < b < c < d < e,$$

then

$$M\big(M(a, m(b, c)),\\ m(d, m(a, e))\big) =$$

- A a
- B b
- C c
- D d
- E e

Solution:

From the ordering, $m(b, c) = b$, so $M(a, m(b, c)) = M(a, b) = b$. Also $m(a, e) = a$, hence $m(d, m(a, e)) = m(d, a) = a$. The outer expression is therefore $M(b, a) = b$. Thus the correct answer is **B**.

11. Three cubes of volume $1, 8$ and 27 are glued together at their faces. The smallest possible surface area of the resulting configuration is

A 36

B 56

C 70

D 72

E 74

Solution:

Before gluing, the total surface area is

$$6(1^2 + 2^2 + 3^2) = 84.$$

The side-2 cube can share area 4 with the side-3 cube, while the side-1 cube is placed at their common edge so that it shares area 1 with each larger cube. Thus the total contact area is $6.$ Each glued area removes two exposed copies, giving $84 - 2(6) = 72.$ No pair can share more than the smaller face, so this is maximal contact and minimal surface area. Thus the correct answer is **D**.

12. If $i^2 = -1$, then

$$\left(i - i^{-1}\right)^{-1} =$$

A 0

B $-2i$

C $2i$

D $-\frac{i}{2}$

E $\frac{i}{2}$

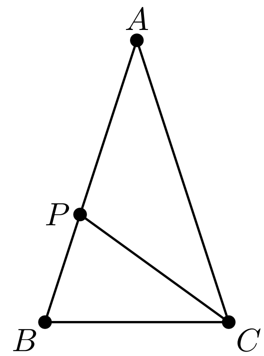
Solution:

Because $i^{-1} = -i$,

$$\begin{aligned}\left(i - i^{-1}\right)^{-1} &= (i+i)^{-1} \\ &= \frac{1}{2i} \\ &= -\frac{i}{2}.\end{aligned}$$

Thus the correct answer is **D**.

13. In triangle ABC, $AB = AC$. If there is a point P strictly between A and B such that $AP = PC = CB$, then $\angle A =$



A $30°$

B $36°$

C $48°$

D $60°$

E $72°$

Solution:

Let $\angle A = \theta$. Since $AP = PC$, triangle APC has $\angle PAC = \angle PCA = \theta$. Since $AB = AC$, each base angle of $\triangle ABC$ is $\frac{180° - \theta}{2}$. Also $PC = CB$, so the base angles of $\triangle PCB$ at P and B are equal; the one at B is $\frac{180° - \theta}{2}$. Hence $\angle PCB = \theta$. At C,

$$2\theta = \frac{180° - \theta}{2},$$

giving $5\theta = 180°$ and $\theta = 36°$. Thus the correct answer is **B**.

14. Find the sum of the arithmetic series

$$20 + 20\frac{1}{5} + 20\frac{2}{5} + \cdots + 40.$$

A 3000

B 3030

C 3150

D 4100

E 6000

Solution:

The number of terms is

$$\frac{40 - 20}{\frac{1}{5}} + 1 = 101.$$

Their average is $\frac{20+40}{2} = 30$, so the sum is $101 \cdot 30 = 3030$. Thus the correct answer is **B**.

15. For how many n in $\{1, 2, 3, \ldots, 100\}$ is the tens digit of n^2 odd?

- A 10
- B 20
- C 30
- D 40
- E 50

Solution:

Write $n = 10a + b$. Modulo 100,

$$n^2 \equiv 20ab + b^2.$$

The term $20ab$ changes the tens digit by an even amount, so only the tens digit of b^2 matters. Among $b = 0, 1, \ldots, 9$, it is odd only for $b = 4$ and $b = 6$. Each units digit occurs 10 times from 1 through 100, giving $2 \cdot 10 = 20$. Thus the correct answer is **B**.

16. Some marbles in a bag are red and the rest are blue. If one red marble is removed, then one-seventh of the remaining marbles are red. If two blue marbles are removed instead of one red, then one-fifth of the remaining marbles are red. How many marbles were in the bag originally?

A 8

B 22

C 36

D 57

E 71

Solution:

The two conditions give $\frac{R-1}{T-1} = \frac{1}{7}$ and $\frac{R}{T-2} = \frac{1}{5}$. Thus $T = 7R - 6$ and $T = 5R + 2$. Hence $R = 4$ and $T = 22$. Thus the correct answer is **B**.

17. An 8 by $2\sqrt{2}$ rectangle has the same center as a circle of radius 2. The area of the region common to both the rectangle and the circle is

A 2π

B $2\pi+2$

C $4\pi-4$

D $2\pi+4$

E $4\pi-2$

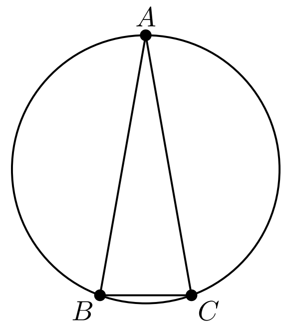
Solution:

The rectangle is wider than the circle, so the common region is the circle with the top and bottom caps beyond distance $\sqrt{2}$ from the center removed. One cap has area

$$2^2\left(\frac{\pi}{4}\right)-\sqrt{2}\sqrt{2^2-(\sqrt{2})^2}$$
$$=\pi-2.$$

Therefore the common area is $4\pi-2(\pi-2)=2\pi+4$. Thus the correct answer is **D**.

18. Triangle ABC is inscribed in a circle, and $\angle B = \angle C = 4\angle A.$ If B and C are adjacent vertices of a regular polygon of n sides inscribed in this circle, then $n =$



A 5

B 7

C 9

D 15

E 18

Solution:

The angle sum gives $9\angle A = 180^\circ$, so $\angle A = 20^\circ$. The central angle subtending chord $\overline{BC}$ is therefore 40°. Adjacent vertices of a regular n-gon subtend $\frac{360^\circ}{n}$, so

$$\frac{360^\circ}{n} = 40^\circ,$$

and $n = 9.$ Thus the correct answer is **C**.

19. Label one disk "1," two disks "2," three disks "3," $\ldots$, fifty disks "50." Put these $1 + 2 + 3 + \cdots + 50 = 1275$ labeled disks in a box. Disks are then drawn from the box at random without replacement. The minimum number of disks that must be drawn to guarantee drawing at least ten disks with the same label is

A 10

B 51

C 415

D 451

E 501

Solution:

To avoid ten equal labels, one may draw all

$$1 + 2 + \cdots + 9 = 45$$

disks with labels below 10, and at most 9 from each of the 41 labels 10 through 50. Thus $45 + 9(41) = 414$ disks can be drawn without forcing ten alike, and the next draw guarantees them. The minimum is 415. Thus the correct answer is **C**.

20. Suppose x, y, z is a geometric sequence with common ratio r and $x \neq y$. If $x, 2y, 3z$ is an arithmetic sequence, then r is

A $\frac{1}{4}$

B $\frac{1}{3}$

C $\frac{1}{2}$

D 2

E 4

Solution:

Because $y = xr$ and $z = xr^2$, the arithmetic-sequence condition gives

$$4xr = x + 3xr^2.$$

Here $x \neq 0$, so $3r^2 - 4r + 1 = 0$, yielding $r = 1$ or $r = \frac{1}{3}$. The condition $x \neq y$ excludes $r = 1$, leaving $r = \frac{1}{3}$. Thus the correct answer is **B**.

21. Find the number of counterexamples to the statement:

"If N is an odd positive integer the sum of whose digits is 4 and none of whose digits is 0, then N is prime."

A 0

B 1

C 2

D 3

E 4

Solution:

The odd possibilities are $13, 31, 121, 211,$ and $1111.$ The numbers $13, 31,$ and 211 are prime, while $121 = 11^2$ and $1111 = 11 \cdot 101.$ Thus there are 2 counterexamples. Thus the correct answer is **C**.

22. Nine chairs in a row are to be occupied by six students and Professors Alpha, Beta and Gamma. These three professors arrive before the six students and decide to choose their chairs so that each professor will be between two students. In how many ways can Professors Alpha, Beta and Gamma choose their chairs?

A 12

B 36

C 60

D 84

E 630

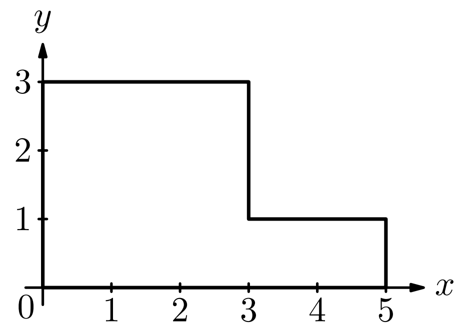
Solution:

The professors must occupy three nonconsecutive positions among chairs $2, 3, \ldots, 8$. The number of such 3-subsets of 7 consecutive positions is

$$\binom{7-3+1}{3} = \binom{5}{3} = 10.$$

The three distinct professors can be assigned to the selected chairs in $3! = 6$ ways, for $10 \cdot 6 = 60$. Thus the correct answer is **C**.

23. In the xy-plane, consider the L-shaped region bounded by horizontal and vertical segments with vertices at $(0,0)$, $(0,3)$, $(3,3)$, $(3,1)$, $(5,1)$ and $(5,0)$. The slope of the line through the origin that divides the area of this region exactly in half is



A $\frac{2}{7}$

B $\frac{1}{3}$

C $\frac{2}{3}$

D $\frac{3}{4}$

E $\frac{7}{9}$

Solution:

The region has area $3 \cdot 3 + 2 \cdot 1 = 11$, so each half has area $\frac{11}{2}$. If the line is $y = mx$, the part below it consists of the entire 2-unit rectangle from $x = 3$ to $x = 5$, together with a triangle of base 3 and height $3m$. Therefore

$$2 + \frac{1}{2}(3)(3m) = \frac{11}{2}.$$

This gives $m = \frac{7}{9}$, which indeed lies in the assumed range. Thus the correct answer is **E**.

24. A sample consisting of five observations has an arithmetic mean of 10 and a median of 12. The smallest value that the range (largest observation minus smallest) can assume for such a sample is

- A 2
- B 3
- C 5
- D 7
- E 10

Solution:

The sample $7, 7, 12, 12, 12$ has sum 50, median 12, and range 5, so range 5 is possible. Suppose the range were at most 4. If the largest value were at most 12, then the top three values would all be 12, leaving the bottom two with sum 14; yet each would be at least 8, a contradiction. If the largest value exceeded 12, then the smallest would exceed 8, making the total exceed $8 + 8 + 12 + 12 + 12 = 52$. Thus no smaller range works. The correct answer is **C**.

25. If x and y are non-zero real numbers such that

$$|x| + y = 3$$

and

$$|x|y + x^3 = 0,$$

then the integer nearest to $x - y$ is

A -3

B -1

C 2

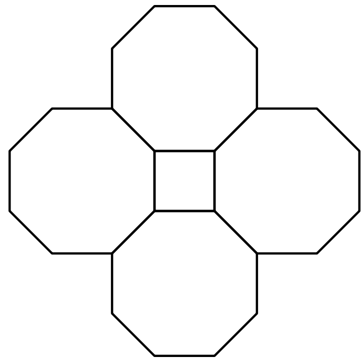
D 3

E 5

Solution:

If $x > 0$, the second equation gives $y = -x^2$, while the first gives $y = 3 - x$; these would require $x^2 - x + 3 = 0$, which has no real root. Hence $x < 0$. Put $t = -x > 0$. Then the equations become $t + y = 3$ and $ty - t^3 = 0$, so $y = t^2$. Therefore $x - y = -t - t^2 = -3$, already an integer. Thus the correct answer is **A**.

26. A regular polygon of m sides is exactly enclosed (no overlaps, no gaps) by m regular polygons of n sides each. (Shown here for $m=4, n=8$.) If $m=10$, what is the value of n?



A 5

B 6

C 14

D 20

E 26

Solution:

At a vertex of the inner polygon, its interior angle and two angles from the surrounding polygons total 360°. Thus

$$180^\circ\left(1-\frac{2}{m}\right) + 2\cdot 180^\circ\left(1-\frac{2}{n}\right) = 360^\circ.$$

This simplifies to $\frac{1}{m}+\frac{2}{n}=\frac{1}{2}$. With $m=10$, we get $\frac{2}{n}=\frac{2}{5}$, so $n=5$. Thus the correct answer is **A**.

27. A bag of popping corn contains $\frac{2}{3}$ white kernels and $\frac{1}{3}$ yellow kernels. Only $\frac{1}{2}$ of the white kernels will pop, whereas $\frac{2}{3}$ of the yellow ones will pop. A kernel is selected at random from the bag, and pops when placed in the popper. What is the probability that the kernel selected was white?

- A $\frac{1}{2}$
- B $\frac{5}{9}$
- C $\frac{4}{7}$
- D $\frac{3}{5}$
- E $\frac{2}{3}$

Solution:

The probabilities of selecting a kernel that pops are

$$P(W \text{ and pop}) = \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3},$$
$$P(Y \text{ and pop}) = \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}.$$

Hence $P(\text{pop}) = \frac{5}{9}$, and

$$P(W \mid \text{pop}) = \frac{\frac{1}{3}}{\frac{5}{9}} = \frac{3}{5}.$$

Thus the correct answer is **D**.

28. In the xy-plane, how many lines whose x-intercept is a positive prime number and whose y-intercept is a positive integer pass through the point $(4, 3)$?

A 0

B 1

C 2

D 3

E 4

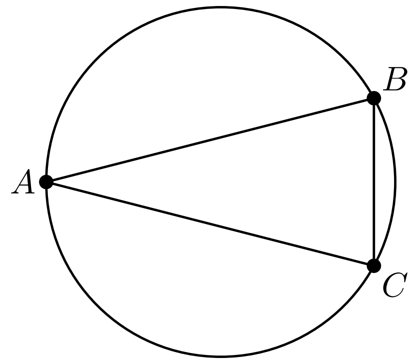
Solution:

Let the x-intercept be the prime p and the y-intercept be the positive integer q. Intercept form gives

$$\frac{4}{p}+\frac{3}{q}=1,$$
$$q=\frac{3p}{p-4}=3+\frac{12}{p-4}.$$

Thus $p-4$ is a positive divisor of 12. Testing $1, 2, 3, 4, 6, 12$ gives prime p only for $p=5$ and $p=7$. Therefore there are 2 lines. Thus the correct answer is **C**.

29. Points A, B and C on a circle of radius r are situated so that $AB = AC, AB > r$, and the length of minor arc $\overset{\frown}{BC}$ is r. If angles are measured in radians, then $\frac{AB}{BC} =$



A $\frac{1}{2}\csc\frac{1}{4}$

B $2\cos\frac{1}{2}$

C $4\sin\frac{1}{2}$

D $\csc\frac{1}{2}$

E $2\sec\frac{1}{2}$

Solution:

The central angle subtending minor arc $\overset{\frown}{BC}$ is $\frac{r}{r} = 1$, so $BC = 2r\sin\frac{1}{2}$. Since $AB = AC$ and $AB > r$, point A is the midpoint of the major arc BC. The minor central angle from A to B is $\pi - \frac{1}{2}$, so $AB = 2r\text{sin}\left(\frac{\pi-\frac{1}{2}}{2}\right) = 2r\cos\frac{1}{4}$. Therefore $\frac{AB}{BC} = \frac{\cos(\frac{1}{4})}{\sin(\frac{1}{2})} = \frac{1}{2\sin(\frac{1}{4})} = \frac{1}{2}\csc\frac{1}{4}$. Thus the correct answer is **A**.

30. When n standard 6-sided dice are rolled, the probability of obtaining a sum of 1994 is greater than zero and is the same as the probability of obtaining a sum of S. The smallest possible value of S is

- A 333
- B 335
- C 337
- D 339
- E 341

Solution:

The smallest feasible number of dice is

$$n = \left\lceil \frac{1994}{6} \right\rceil = 333.$$

Replacing every die result d by $7-d$ is a bijection between outcomes of sum T and outcomes of sum $7n-T$. Thus for $n=333$, the sum paired with 1994 is

$$S = 7(333) - 1994 = 337.$$

Dice-sum counts increase strictly from the minimum sum up to the mean, so for any feasible n no sum below 337 can match the count at 1994. Thus the smallest value is 337, and the correct answer is **C**.

Problems: https://live.poshenloh.com/past-contests/amc12/1994