# 1993 AMC 12 Solutions

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1. For integers $a, b$, and $c$, define $\boxed{a, b, c}$ to mean $a^b - b^c + c^a$. Then $\boxed{1, -1, 2}$ equals

A $-4$

B $-2$

C $0$

D $2$

E $4$

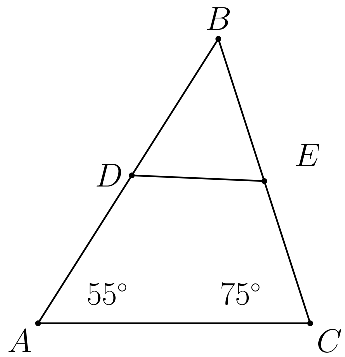
## Solution:

By the definition,

$$\begin{aligned} \boxed{1, -1, 2} &= 1^{-1} - (-1)^2 + 2^1 \\ &= 1 - 1 + 2 \\ &= 2. \end{aligned}$$

Thus the correct answer is **D**.

**2.** In $\triangle ABC, \angle A = 55^\circ, \angle C = 75^\circ, D$ is on side $\overline{AB}$ and $E$ is on side $\overline{BC}$. If $DB = BE$, then $\angle BED =$




- A $50^\circ$
- B $55^\circ$
- C $60^\circ$
- D $65^\circ$
- E $70^\circ$

## Solution:

We have $\angle ABC = 180^\circ - 55^\circ - 75^\circ = 50^\circ$. Because $D$ and $E$ lie on $\overline{BA}$ and $\overline{BC}$, respectively, $\angle DBE = 50^\circ$. Since $DB = BE$, the two base angles of $\triangle DBE$ are

$$\frac{180^\circ - 50^\circ}{2} = 65^\circ.$$

Thus the correct answer is **D**.

**3.**

$$\frac{15^{30}}{45^{15}} =$$

A $\left(\frac{1}{3}\right)^{15}$

B $\left(\frac{1}{3}\right)^{2}$

C $1$

D $3^{15}$

E $5^{15}$

## Solution:

Using a common exponent,

$$\frac{15^{30}}{45^{15}} = \left(\frac{15^2}{45}\right)^{15} = 5^{15}.$$

Thus the correct answer is **E**.

**4.** Define the operation "$\circ$" by $x \circ y = 4x - 3y + xy$, for all real numbers $x$ and $y$. For how many real numbers $y$ does $3 \circ y = 12$?

A $0$

B $1$

C $3$

D $4$

E more than $4$

## Solution:

For every real $y$,

$$3 \circ y = 4(3) - 3y + 3y = 12.$$

Hence every real number $y$ works, so there are more than $4$ such numbers. Thus the correct answer is **E**.

**5.** Last year a bicycle cost $\$160$ and a cycling helmet cost $\$40$. This year the cost of the bicycle increased by $5\%$, and the cost of the helmet increased by $10\%$. The percent increase in the combined cost of the bicycle and the helmet is

A $6\%$

B $7\%$

C $7.5\%$

D $8\%$

E $15\%$

## Solution:

The bicycle increases by $\$8$ and the helmet by $\$4$, for a total increase of $\$12$. Last year's combined cost was $\$200$, so the percent increase is

$$\frac{12}{200} \cdot 100\% = 6\%.$$

Thus the correct answer is **A**.

**6.**

$$\sqrt{\frac{8^{10}+4^{10}}{8^4+4^{11}}} =$$

A $\sqrt{2}$

B $16$

C $32$

D $12^{\frac{2}{3}}$

E $512.5$

## Solution:

Writing all terms as powers of $2$,

$$\begin{aligned} \frac{8^{10}+4^{10}}{8^4+4^{11}} &= \frac{2^{30}+2^{20}}{2^{12}+2^{22}} \\ &= \frac{2^{20}(2^{10}+1)}{2^{12}(1+2^{10})} \\ &= 2^8. \end{aligned}$$

Its positive square root is $2^4 = 16$. Thus the correct answer is **B**.

**7.** The symbol $R_k$ stands for an integer whose base-ten representation is a sequence of $k$ ones. For example, $R_3 = 111, R_5 = 11111,$ etc. When $R_{24}$ is divided by $R_4,$ the quotient $Q = \frac{R_{24}}{R_4}$ is an integer whose base-ten representation is a sequence containing only ones and zeros. The number of zeros in $Q$ is

A $10$

B $11$

C $12$

D $13$

E $15$

## Solution:

Grouping the $24$ digits into six blocks of four gives

$$\begin{aligned} R_{24} = R_4 \left(1 + 10^4 + 10^8 \right. \\ \left. + 10^{12} + 10^{16} \right. \\ \left. + 10^{20}\right). \end{aligned}$$

Thus $Q$ has six ones, with three zeros between each consecutive pair. It therefore has $5 \cdot 3 = 15$ zeros. Thus the correct answer is **E**.

**8.** Let $C_1$ and $C_2$ be circles of radius $1$ that are in the same plane and tangent to each other. How many circles of radius $3$ are in this plane and tangent to both $C_1$ and $C_2$?

A $2$

B $4$

C $5$

D $6$

E $8$

## Solution:

Let the centers of the tangent unit circles be $2$ units apart. The new center must be $4$ units from a unit-circle center for external tangency and $2$ units away for internal tangency. The distance pair $(4,4)$ gives two centers, as does $(2,2)$. Each mixed pair $(4,2)$ and $(2,4)$ gives one center because the corresponding center-circles are internally tangent. Hence there are $2+2+1+1=6$ circles. Thus the correct answer is **D**.

**9.** Country $A$ has $c\%$ of the world's population and owns $d\%$ of the world's wealth. Country $B$ has $e\%$ of the world's population and $f\%$ of its wealth. Assume that the citizens of $A$ share the wealth of $A$ equally, and assume that those of $B$ share the wealth of $B$ equally. Find the ratio of the wealth of a citizen of $A$ to the wealth of a citizen of $B$.

A $\frac{cd}{ef}$

B $\frac{ce}{df}$

C $\frac{cf}{de}$

D $\frac{de}{cf}$

E $\frac{df}{ce}$

## Solution:

Country $A$'s per-capita share is proportional to $\frac{d}{c}$, while country $B$'s is proportional to $\frac{f}{e}$. Their ratio is

$$\frac{\frac{d}{c}}{\frac{f}{e}} = \frac{de}{cf}.$$

Thus the correct answer is **D**.

**10.** Let $r$ be the number that results when both the base and the exponent of $a^b$ are tripled, where $a, b > 0$. If $r$ equals the product of $a^b$ and $x^b$ where $x > 0$, then $x =$

A $3$

B $3a^2$

C $27a^2$

D $2a^{3b}$

E $3a^{2b}$

## Solution:

We have

$$r = (3a)^{3b} = \left(27a^3\right)^b.$$

Also $r = a^b x^b = (ax)^b$. Positivity allows us to equate the bases, so $ax = 27a^3$ and $x = 27a^2$. Thus the correct answer is **C**.

**11.** If $\log_2(\log_2(\log_2(x))) = 2$, then how many digits are in the base-ten representation for $x$?

A $5$

B $7$

C $9$

D $11$

E $13$

## Solution:

Undoing the logarithms gives

$$\begin{aligned} \log_2(\log_2 x) &= 4, \\ \log_2 x &= 16, \\ x &= 2^{16} = 65536. \end{aligned}$$

This number has $5$ decimal digits. Thus the correct answer is **A**.

**12.** If $f(2x) = \frac{2}{2+x}$ for all $x > 0$, then $2f(x) =$

A $\frac{2}{1+x}$

B $\frac{2}{2+x}$

C $\frac{4}{1+x}$

D $\frac{4}{2+x}$

E $\frac{8}{4+x}$

## Solution:

Replacing the given $x$ by $\frac{x}{2}$ yields

$$f(x) = \frac{2}{2 + \frac{x}{2}} = \frac{4}{4 + x}.$$

Therefore $2f(x) = \frac{8}{4+x}$. Thus the correct answer is **E**.

**13.** A square of perimeter $20$ is inscribed in a square of perimeter $28$. What is the greatest distance between a vertex of the inner square and a vertex of the outer square?

A $\sqrt{58}$

B $\frac{7\sqrt{5}}{2}$

C $8$

D $\sqrt{65}$

E $5\sqrt{3}$

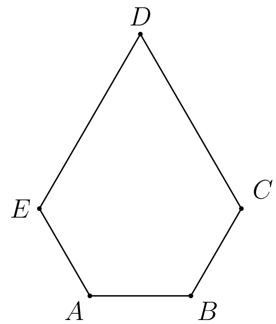
## Solution:

The outer and inner side lengths are $7$ and $5$. At any inner vertex, the two pieces of the outer side have lengths $x, y$ with

$$x + y = 7, \qquad x^2 + y^2 = 25.$$

Hence $2xy = 49 - 25 = 24$, so $\{x, y\} = \{3, 4\}$. From an inner vertex on one side, the farther opposite outer vertex has horizontal displacement $4$ and vertical displacement $7$. The greatest distance is therefore $\sqrt{4^2 + 7^2} = \sqrt{65}$. Thus the correct answer is **D**.

**14.** The convex pentagon $ABCDE$ has $\angle A = \angle B = 120^\circ$, $EA = AB = BC = 2$ and $CD = DE = 4$. What is the area of $ABCDE$?




- A $10$
- B $7\sqrt{3}$
- C $15$
- D $9\sqrt{3}$
- E $12\sqrt{5}$

## Solution:

Place $A = (0,0)$ and $B = (2,0)$. The $120^\circ$ angles and side lengths $EA = BC = 2$ give

$$E = (-1, \sqrt{3}), \qquad C = (3, \sqrt{3}).$$

Thus $ABCE$ is a trapezoid of height $\sqrt{3}$ and bases $2, 4$, so its area is $3\sqrt{3}$. Also $CE = 4 = CD = DE$, making $\triangle CDE$ equilateral with area $4\sqrt{3}$. The total area is $7\sqrt{3}$. Thus the correct answer is **B**.

**15.** For how many values of $n$ will an $n$-sided regular polygon have interior angles with integral degree measures?

A $16$

B $18$

C $20$

D $22$

E $24$

## Solution:

The interior angle is $180^\circ - \frac{360^\circ}{n}$, which is integral exactly when $n \mid 360$. Since $360 = 2^3 \cdot 3^2 \cdot 5$, it has

$$(3+1)(2+1)(1+1) = 24$$

positive divisors. Excluding $n = 1, 2$ leaves $22$ allowable values. Thus the correct answer is **D**.

**16.** Consider the non-decreasing sequence of positive integers

$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4,$$
$$5, 5, 5, 5, 5, \ldots$$

in which the $n$th positive integer appears $n$ times. The remainder when the $1993$rd term is divided by $5$ is

A $0$

B $1$

C $2$

D $3$

E $4$

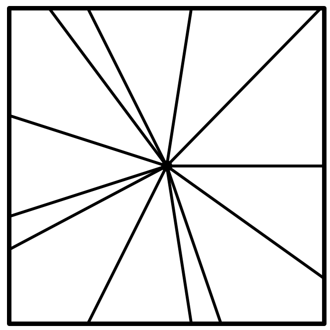
## Solution:

The last $62$ occurs in position

$$\frac{62 \cdot 63}{2} = 1953,$$

while the last $63$ occurs in position $\frac{63 \cdot 64}{2} = 2016$. Thus the $1993$rd term is $63$, whose remainder modulo $5$ is $3$. Thus the correct answer is **D**.

**17.** Amy painted a dart board over a square clock face using the “hour positions” as boundaries. [See figure.] If $t$ is the area of one of the eight triangular regions such as that between $12$ o’clock and $1$ o’clock, and $q$ is the area of one of the four corner quadrilaterals such as that between $1$ o’clock and $2$ o’clock, then $\frac{q}{t} =$



A $2\sqrt{3} - 2$

B $\frac{3}{2}$

C $\frac{\sqrt{5}+1}{2}$

D $\sqrt{3}$

E $2$

## Solution:

Let the square be $[-1, 1]^2$ and its center be the origin. The $1$ o’clock ray meets the top side at $(\frac{1}{\sqrt{3}}, 1)$, so

$$t = \frac{1}{2}\left(\frac{1}{\sqrt{3}}\right)(1) = \frac{1}{2\sqrt{3}}.$$

The corner quadrilateral has vertices $(0, 0)$, $(\frac{1}{\sqrt{3}}, 1)$, $(1, 1)$, $(1, \frac{1}{\sqrt{3}})$. Its area is $q = 1 - \frac{1}{\sqrt{3}}$. Hence

$$\frac{q}{t} = \frac{1 - \frac{1}{\sqrt{3}}}{\frac{1}{2\sqrt{3}}} = 2\sqrt{3} - 2.$$

Thus the correct answer is **A**.

**18.** Al and Barb start their new jobs on the same day. Al's schedule is $3$ work-days followed by $1$ rest-day. Barb's schedule is $7$ work-days followed by $3$ rest-days. On how many of their first $1000$ days do both have rest-days on the same day?

- A $48$
- B $50$
- C $72$
- D $75$
- E $100$

## Solution:

The combined pattern repeats every $20$ days. Al rests on days $4, 8, 12, 16, 20,$ while Barb rests on days $8, 9, 10, 18, 19, 20.$ Their common rest days are $8$ and $20,$ two per cycle. There are $\frac{1000}{20} = 50$ cycles, giving $2 \cdot 50 = 100$ common rest days. Thus the correct answer is **E**.

**19.** How many ordered pairs $(m, n)$ of positive integers are solutions to

$$\frac{4}{m} + \frac{2}{n} = 1?$$

A $1$

B $2$

C $3$

D $4$

E more than $4$

## Solution:

Clearing denominators and rearranging gives

$$\begin{aligned} mn - 4n - 2m = 0 \quad &\Longleftrightarrow \\ (m-4)(n-2) = 8. \end{aligned}$$

Both factors must be positive. The four ordered positive factor pairs of $8$ give

$$\begin{aligned} (m, n) = {} & (5, 10), (6, 6), \\ & (8, 4), (12, 3). \end{aligned}$$

Thus the correct answer is **D**.

**20.** Consider the equation $10z^2 - 3iz - k = 0$, where $z$ is a complex variable and $i^2 = -1$. Which of the following statements is true?

A For all positive real numbers $k$, both roots are pure imaginary.

B For all negative real numbers $k$, both roots are pure imaginary.

C For all pure imaginary numbers $k$, both roots are real and rational.

D For all pure imaginary numbers $k$, both roots are real and irrational.

E For all complex numbers $k$, neither root is real.

## Solution:

The roots are

$$z = \frac{3i \pm \sqrt{-9 + 40k}}{20}.$$

If $k$ is negative and real, then $-9 + 40k$ is negative real, so its square roots are pure imaginary. Both resulting values of $z$ are therefore pure imaginary. The other universal claims fail, for example by taking $k = 1,\ i$, or $0$ as appropriate. Thus the correct answer is **B**.

**21.** Let $a_1, a_2, \ldots, a_k$ be a finite arithmetic sequence with

$$a_4 + a_7 + a_{10} = 17$$

and

$$\begin{gathered} a_4 + a_5 + a_6 + \cdots + a_{12} \\ + a_{13} + a_{14} = 77. \end{gathered}$$

If $a_k = 13$, then $k =$

A $16$

B $18$

C $20$

D $22$

E $24$

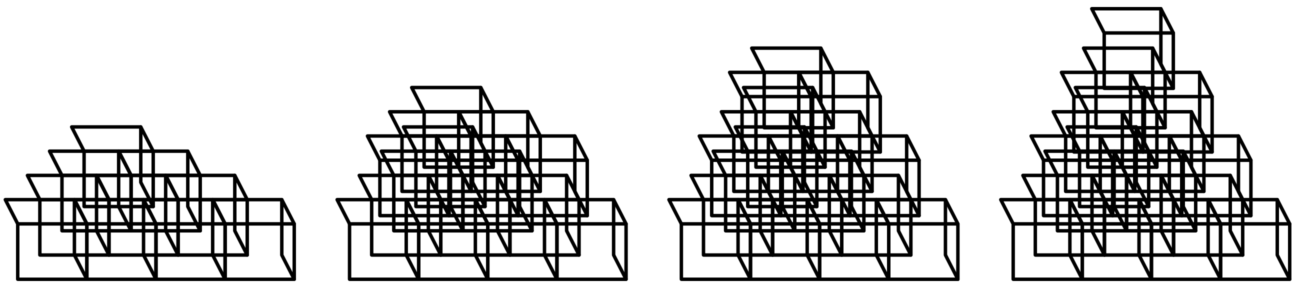
## Solution:

Because symmetric terms of an arithmetic sequence average to the middle term,

$$3a_7 = 17, \qquad 11a_9 = 77.$$

Thus $a_7 = \frac{17}{3}$ and $a_9 = 7$, so the common difference is $d = \frac{7-\frac{17}{3}}{2} = \frac{2}{3}$. Since $13 - 7 = 6 = 9d$, we have $a_k = a_{9+9} = a_{18}$. Thus the correct answer is **B**.

**22.** Twenty cubical blocks are arranged as shown. First, $10$ are arranged in a triangular pattern; then a layer of $6$, arranged in a triangular pattern, is centered on the $10$; then a layer of $3$, arranged in a triangular pattern, is centered on the $6$; and finally one block is centered on top of the third layer. The blocks in the bottom layer are numbered $1$ through $10$ in some order. Each block in layers $2, 3$ and $4$ is assigned the number which is the sum of the numbers assigned to the three blocks on which it rests. Find the smallest possible number which could be assigned to the top block.



A $55$

B $83$

C $114$

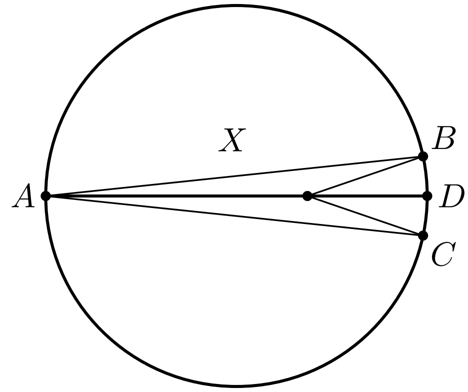
D $137$

E $144$

## Solution:

Expanding the sums layer by layer, the top value is $6c + 3(e_1 + e_2 + e_3 + e_4 + e_5 + e_6) + (v_1 + v_2 + v_3)$, where $c$ is the center bottom block, the $e_i$ are the six non-corner boundary blocks, and the $v_i$ are the three corner blocks. To minimize this weighted sum, assign $1$ to the coefficient-$6$ position, $2, \ldots, 7$ to the coefficient-$3$ positions, and $8, 9, 10$ to the coefficient-$1$ positions. The minimum is $6 + 3(2 + 3 + 4 + 5 + 6 + 7) + (8 + 9 + 10) = 114.$ Thus the correct answer is **C**.

**23.** Points $A, B, C$ and $D$ are on a circle of diameter $1$, and $X$ is on diameter $\overline{AD}$. If $BX = CX$ and $3\angle BAC = \angle BXC = 36^\circ$, then $AX =$




A $\cos 6^\circ \cos 12^\circ \sec 18^\circ$

B $\cos 6^\circ \sin 12^\circ \csc 18^\circ$

C $\cos 6^\circ \sin 12^\circ \sec 18^\circ$

D $\sin 6^\circ \sin 12^\circ \csc 18^\circ$

E $\sin 6^\circ \sin 12^\circ \sec 18^\circ$

## Solution:

Since $BX = CX$, the line $AD$ bisects $\angle BXC$, and symmetry also gives $\angle BAD = 6^\circ$. Because $AD = 1$ is a diameter, $\angle ABD = 90^\circ$, so $AB = \cos 6^\circ$. Also

$$\angle AXB = 180^\circ - 18^\circ = 162^\circ,$$
$$\angle ABX = 180^\circ - 162^\circ$$
$$- 6^\circ = 12^\circ.$$

The Law of Sines in $\triangle ABX$ gives

$$AX = \frac{AB \sin 12^\circ}{\sin 162^\circ}$$
$$= \cos 6^\circ \sin 12^\circ$$
$$\csc 18^\circ.$$

Thus the correct answer is **B**.

**24.** A box contains $3$ shiny pennies and $4$ dull pennies. One by one, pennies are drawn at random from the box and not replaced. If the probability is $\frac{a}{b}$ that it will take more than four draws until the third shiny penny appears and $\frac{a}{b}$ is in lowest terms, then $a+b=$

A $11$

B $20$

C $35$

D $58$

E $66$

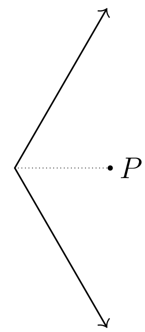
## Solution:

The shiny pennies occupy a uniformly chosen $3$-element subset of the $7$ draw positions. There are $\binom{7}{3}=35$ possibilities. The third shiny penny appears by draw $4$ exactly when all three shiny positions lie among the first four, which occurs in $\binom{4}{3}=4$ cases. The requested probability is

$$1-\frac{4}{35}=\frac{31}{35}.$$

Therefore $a+b=31+35=66.$ Thus the correct answer is **E**.

**25.** Let $S$ be the set of points on the rays forming the sides of a $120^\circ$ angle, and let $P$ be a fixed point inside the angle on the angle bisector. Consider all distinct equilateral triangles $PQR$ with $Q$ and $R$ in $S$. (Points $Q$ and $R$ may be on the same ray, and switching the names of $Q$ and $R$ does not create a distinct triangle.) There are




A exactly $2$ such triangles.

B exactly $3$ such triangles.

C exactly $7$ such triangles.

D exactly $15$ such triangles.

E more than $15$ such triangles.

## Solution:

Let $O$ be the vertex. Choose $A$ on one ray so that $OA = OP$; because $OP$ bisects the $120^\circ$ angle, $\triangle AOP$ is equilateral. For any $Q$ on $\overline{AO}$, choose $R$ on the other ray with $OR = AQ$. Then $\triangle POR \cong \triangle PAQ$ by SAS. Consequently $PQ = PR$ and $\angle QPR = \angle APO = 60^\circ$, so $\triangle PQR$ is equilateral. Continuously many choices of $Q$ give distinct triangles, hence certainly more than $15$. Thus the correct answer is **E**.

**26.** Find the largest positive value attained by the function

$$f(x) = \sqrt{8x - x^2} \\ - \sqrt{14x - x^2 - 48},$$

$x$ a real number.

A $\sqrt{7} - 1$

B $3$

C $2\sqrt{3}$

D $4$

E $\sqrt{55} - \sqrt{5}$

## Solution:

Both radicals are real exactly when $6 \leq x \leq 8$. Factoring and rationalizing,

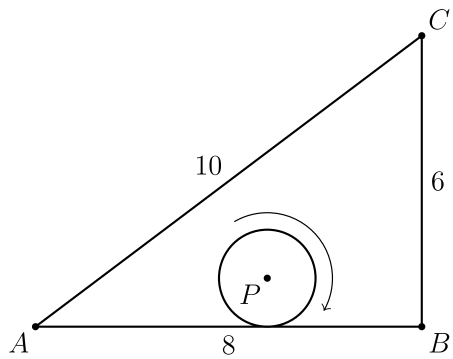
$$\begin{aligned} f(x) &= \sqrt{8-x}\left(\sqrt{x} - \sqrt{x-6}\right) \\ &= \frac{6\sqrt{8-x}}{\sqrt{x} + \sqrt{x-6}}. \end{aligned}$$

On this interval the numerator decreases while the denominator increases, so the maximum occurs at $x = 6$. Its value is

$$f(6) = \sqrt{12} = 2\sqrt{3}.$$

Thus the correct answer is **C**.

**27.** The sides of $\triangle ABC$ have lengths $6, 8$ and $10$. A circle with center $P$ and radius $1$ rolls around the inside of $\triangle ABC$, always remaining tangent to at least one side of the triangle. When $P$ first returns to its original position, through what distance has $P$ traveled?




A $10$

B $12$

C $14$

D $15$

E $17$

## Solution:

The center $P$ remains one unit from the side it touches, so its path is the triangle formed by the three inward parallel lines at distance $1$. This triangle is similar to $\triangle ABC$. The original right triangle has area $24$ and semiperimeter $12$, hence inradius $\frac{24}{12} = 2$. The path triangle has inradius $2 - 1 = 1$, so its linear scale is $\frac{1}{2}$. Its perimeter, and therefore the distance traveled, is

$$\frac{1}{2}(6 + 8 + 10) = 12.$$

Thus the correct answer is **B**.

**28.** How many triangles with positive area are there whose vertices are points in the $xy$-plane whose coordinates are integers $(x, y)$ satisfying $1 \leq x \leq 4$ and $1 \leq y \leq 4$?

A $496$

B $500$

C $512$

D $516$

E $560$

## Solution:

There are $\binom{16}{3} = 560$ triples of grid points. The four rows and four columns contribute

$$8\binom{4}{3} = 32$$

collinear triples. For slope $1$, the diagonal lengths capable of containing three points are $3, 4, 3,$ contributing $\binom{3}{3} + \binom{4}{3} + \binom{3}{3} = 6$; slope $-1$ contributes another $6.$ No other slope fits three lattice points inside a $4$-by-$4$ point array. Thus the number of positive-area triangles is $560 - 32 - 6 - 6 = 516.$ Thus the correct answer is **D**.

**29.** Which of the following sets could NOT be the lengths of the external diagonals of a right rectangular prism [a "box"]? (An external diagonal is a diagonal of one of the rectangular faces of the box.)

A $\{4, 5, 6\}$

B $\{4, 5, 7\}$

C $\{4, 6, 7\}$

D $\{5, 6, 7\}$

E $\{5, 7, 8\}$

## Solution:

If the edge lengths are $a, b, c$, then the three face-diagonal squares are $a^2 + b^2,\ a^2 + c^2,\ b^2 + c^2$. For sorted diagonals $p \le q \le r$,

$$a^2 = \frac{p^2 + q^2 - r^2}{2},$$

so necessarily $r^2 < p^2 + q^2$; the analogous formulas show this is also sufficient. Only $\{4, 5, 7\}$ fails, since $7^2 = 49 > 16 + 25 = 41$. Thus the correct answer is **B**.

**30.** Given $0 \le x_0 < 1$, let

$$x_n = \begin{cases} 2x_{n-1}, & \text{if } 2x_{n-1} < 1, \\ 2x_{n-1} - 1, & \text{if } 2x_{n-1} \ge 1 \end{cases}$$

for all integers $n > 0$. For how many $x_0$ is it true that $x_0 = x_5$?

A $0$

B $1$

C $5$

D $31$

E infinitely many

## Solution:

The recurrence is $x_n = \{2x_{n-1}\}$, so $x_5 = \{32x_0\}$. The equality $x_5 = x_0$ requires

$$32x_0 - x_0 = \lfloor 32x_0 \rfloor,$$

hence $31x_0$ is an integer. The values $x_0 = \frac{k}{31}$ for $k = 0, 1, \ldots, 30$ all satisfy the recurrence and lie in $[0, 1)$. There are $31$ values. Thus the correct answer is **D**.

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