1992 AMC 12 Solutions

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1. If $3(4x+5\pi)=P$, then $6(8x+10\pi)=$

A $2P$

B $4P$

C $6P$

D $8P$

E $18P$

Solution:

We have $8x+10\pi=2(4x+5\pi)$, so

$$\begin{aligned} &6(8x+10\pi)\\ &=12(4x+5\pi)\\ &=4\big[3(4x+5\pi)\big]\\ &=4P. \end{aligned}$$

Thus the correct answer is **B**.

2. An urn is filled with coins and beads, all of which are either silver or gold. Twenty percent of the objects in the urn are beads. Forty percent of the coins in the urn are silver. What percent of the objects in the urn are gold coins?

A 40%

B 48%

C 52%

D 60%

E 80%

Solution:

Coins make up 80% of the objects, and 60% of the coins are gold. Thus gold coins make up

$$0.80 \cdot 0.60 = 0.48 = 48\%$$

of all the objects.

Thus the correct answer is **B**.

3. If $m > 0$ and the points $(m, 3)$ and $(1, m)$ lie on a line with slope m, then $m =$

A 1

B $\sqrt{2}$

C $\sqrt{3}$

D 2

E $\sqrt{5}$

Solution:

The slope condition gives

$$\frac{m-3}{1-m} = m.$$

Therefore $m - 3 = m - m^2$, so $m^2 = 3$. Since $m > 0$, we obtain $m = \sqrt{3}$.

Thus the correct answer is **C**.

4. If $a, b,$ and c are positive integers and a and b are odd, then $3^a + (b-1)^2 c$ is

A odd for all choices of c

B even for all choices of c

C odd if c is even; even if c is odd

D odd if c is odd; even if c is even

E odd if c is not a multiple of 3; even if c is a multiple of 3

Solution:

The power 3^a is odd. Since $b-1$ is even, $(b-1)^2 c$ is even for every positive integer c. The sum of an odd integer and an even integer is always odd.

Thus the correct answer is **A**.

5. $6^6 + 6^6 + 6^6 + 6^6 + 6^6 + 6^6 =$

A 6^6

B 6^7

C 36^6

D 6^{36}

E 36^{36}

Solution:

Factoring gives $6 \cdot 6^6 = 6^{1+6} = 6^7$.

Thus the correct answer is **B**.

6. If $x > y > 0$, then

$$\frac{x^y y^x}{y^y x^x} =$$

A $(x-y)^{\frac{y}{x}}$

B $\left(\frac{x}{y}\right)^{x-y}$

C 1

D $\left(\frac{x}{y}\right)^{y-x}$

E $(x-y)^{\frac{x}{y}}$

Solution:

Combining like bases,

$$\frac{x^y y^x}{y^y x^x} = x^{y-x} y^{x-y} = \left(\frac{x}{y}\right)^{y-x}.$$

Thus the correct answer is **D**.

7. The ratio of w to x is $4 : 3$, of y to z is $3 : 2$, and of z to x is $1 : 6$. What is the ratio of w to y?

A $1 : 3$

B $16 : 3$

C $20 : 3$

D $27 : 4$

E $12 : 1$

Solution:

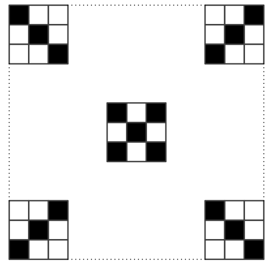
Multiplying the given ratios,

$$\begin{aligned} \frac{w}{y} &= \frac{w}{x} \cdot \frac{x}{z} \cdot \frac{z}{y} \\ &= \frac{4}{3} \cdot 6 \cdot \frac{2}{3} = \frac{16}{3}. \end{aligned}$$

Thus $w : y = 16 : 3.$

Thus the correct answer is **B**.

8. A square floor is tiled with congruent square tiles. The tiles on the two diagonals of the floor are black. The rest of the tiles are white. If there are 101 black tiles, then the total number of tiles is



A 121

B 625

C 676

D 2500

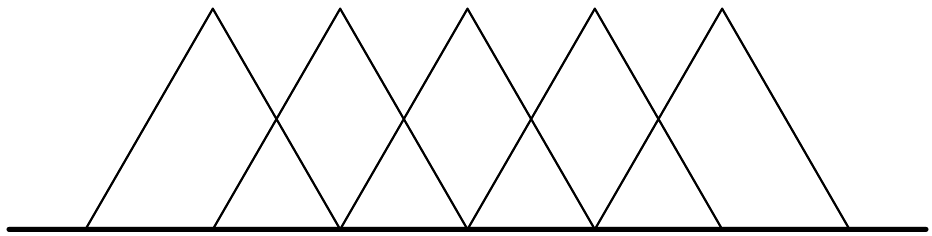
E 2601

Solution:

For an odd n, each diagonal contains n tiles and the two diagonals share only the center tile. Hence $2n - 1 = 101$, so $n = 51$. The floor therefore contains $51^2 = 2601$ tiles.

Thus the correct answer is **E**.

9. Five equilateral triangles, each with side $2\sqrt{3}$, are arranged so they are all on the same side of a line containing one side of each. Along this line, the midpoint of the base of one triangle is a vertex of the next. The area of the region of the plane that is covered by the union of the five triangular regions is



A 10

B 12

C 15

D $10\sqrt{3}$

E $12\sqrt{3}$

Solution:

Each large triangle has area

$$\frac{\sqrt{3}}{4}(2\sqrt{3})^2 = 3\sqrt{3}.$$

Each of the four overlaps is an equilateral triangle of side $\sqrt{3}$, hence area $\frac{3\sqrt{3}}{4}$. There are no triple overlaps, so the union has area

$$5(3\sqrt{3}) - 4\left(\frac{3\sqrt{3}}{4}\right) = 12\sqrt{3}.$$

Thus the correct answer is **E**.

10. The number of positive integers k for which the equation

$$kx - 12 = 3k$$

has an integer solution for x is

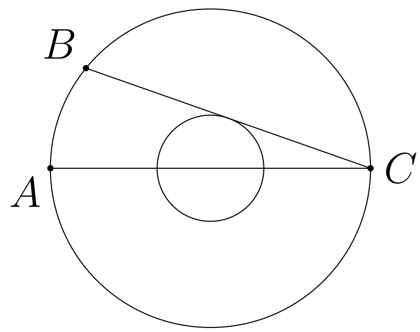
- A 3
- B 4
- C 5
- D 6
- E 7

Solution:

Solving gives $x = 3 + \frac{12}{k}$. Thus x is an integer precisely when k is a positive divisor of 12. The possibilities are $1, 2, 3, 4, 6,$ and 12, for a total of 6.

Thus the correct answer is **D**.

11. The ratio of the radii of two concentric circles is $1:3$. If $\overline{AC}$ is a diameter of the larger circle, $\overline{BC}$ is a chord of the larger circle that is tangent to the smaller circle, and $AB = 12$, then the radius of the larger circle is



A 13

B 18

C 21

D 24

E 26

Solution:

Let O be the common center and T the tangency point on $\overline{BC}$. Since $OT \perp BC$ and $\angle ABC = 90^\circ$, triangles OTC and ABC are similar. If the larger radius is R, then $OT = \frac{R}{3},\ OC = R$, and $AC = 2R$. Thus

$$\frac{OT}{AB} = \frac{OC}{AC} = \frac{1}{2}.$$

Hence $OT = 6$ and $R = 3OT = 18$.

Thus the correct answer is **B**.

12. Let $y = mx + b$ be the image when the line $x - 3y + 11 = 0$ is reflected across the x-axis. The value of $m + b$ is

- A -6
- B -5
- C -4
- D -3
- E -2

Solution:

A point (x, y) is on the reflected line exactly when $(x, -y)$ is on the original line. Hence $x + 3y + 11 = 0$, or

$$y = -\frac{1}{3}x - \frac{11}{3}.$$

Therefore $m + b = -\frac{1}{3} - \frac{11}{3} = -4.$

Thus the correct answer is **C**.

13. How many pairs of positive integers (a, b) with $a + b \leq 100$ satisfy the equation

$$\frac{a + b^{-1}}{a^{-1} + b} = 13?$$

A 1

B 5

C 7

D 9

E 13

Solution:

Multiplying the numerator and denominator by ab gives

$$\frac{a^2b + a}{b + ab^2} = \frac{a(ab + 1)}{b(ab + 1)} = \frac{a}{b}.$$

Thus $a = 13b$. The condition $a + b \leq 100$ becomes $14b \leq 100$, so b may be any integer from 1 through 7. There are 7 pairs.

Thus the correct answer is **C**.

14. Which of the following equations have the same graph?

$$\text{I. } y = x - 2$$
$$\text{II. } y = \frac{x^2 - 4}{x + 2}$$
$$\text{III. } (x + 2)y = x^2 - 4$$

A I and II only

B I and III only

C II and III only

D I, II and III

E None. All the equations have different graphs

Solution:

Equation I is the whole line $y = x - 2$. Equation II simplifies to that line only when $x \neq -2$, so it omits $(-2, -4)$. In equation III, every point with $x = -2$ satisfies $0 = 0$; for other x it gives $y = x - 2$. Thus III is the line together with the entire vertical line $x = -2$. The three graphs are different.

Thus the correct answer is **E**.

15. Let $i = \sqrt{-1}$. Define a sequence of complex numbers by

$$\begin{aligned} z_1 &= 0, \\ z_{n+1} &= z_n^2 + i \quad \text{for } n \geq 1. \end{aligned}$$

In the complex plane, how far from the origin is z_{111}?

A 1

B $\sqrt{2}$

C $\sqrt{3}$

D $\sqrt{110}$

E $\sqrt{2^{55}}$

Solution:

The first terms are

$$\begin{aligned} &z_1 = 0, \quad z_2 = i, \quad z_3 = -1 + i, \\ &z_4 = -i, \quad z_5 = -1 + i. \end{aligned}$$

Hence the sequence alternates between $-1+i$ at odd indices and $-i$ at even indices from z_3 onward. Therefore $z_{111} = -1 + i$, whose distance from the origin is $\sqrt{(-1)^2 + 1^2} = \sqrt{2}$.

Thus the correct answer is **B**.

16. If

$$\frac{y}{x-z}=\frac{x+y}{z}=\frac{x}{y}$$

for three positive numbers $x, y,$ and z, all different, then $\frac{x}{y}=$

A $\frac{1}{2}$

B $\frac{3}{5}$

C $\frac{2}{3}$

D $\frac{5}{3}$

E 2

Solution:

Let $r=\frac{x}{y}>0.$ From $\frac{x+y}{z}=r,$ we get $\frac{z}{y}=\frac{r+1}{r}.$ Then

$$\frac{y}{x-z}=\frac{1}{r-\frac{r+1}{r}}=r.$$

Thus $r^2-r-1=1,$ or $r^2-r-2=0.$ Hence $r=2$ or $-1,$ and positivity gives $r=2.$

Thus the correct answer is **E**.

17. The two-digit integers from 19 to 92 are written consecutively to form the large integer

$$N = 19202122\ldots909192.$$

If 3^k is the highest power of 3 that is a factor of N, then $k =$

A 0

B 1

C 2

D 3

E more than 3

Solution:

Modulo 9, a number is congruent to its digit sum, so

$$\begin{aligned} N &\equiv 19 + 20 + \cdots + 92 \\ &= \frac{74(19+92)}{2} \\ &= 4107 \equiv 3 \pmod 9. \end{aligned}$$

Thus N is divisible by 3 but not by 9, so the highest power of 3 dividing it is 3^1.

Thus the correct answer is **B**.

18. The increasing sequence of positive integers $a_1, a_2, a_3, \ldots$ has the property that

$$a_{n+2} = a_n + a_{n+1} \quad \text{for all } n \geq 1.$$

If $a_7 = 120$, then a_8 is

A 128

B 168

C 193

D 194

E 210

Solution:

Let $a_1 = a$ and $a_2 = b$. Repeated use of the recurrence gives

$$\begin{aligned} a_7 &= 5a + 8b = 120, \\ a_8 &= 8a + 13b. \end{aligned}$$

Modulo 5, the first equation shows that b is a multiple of 5. The positive possibilities are $b = 5, 10,$ or 15. The conditions $b > a > 0$ leave only $b = 10,\ a = 8$. Hence $a_8 = 8(8) + 13(10) = 194.$

Thus the correct answer is **D**.

19. For each vertex of a solid cube, consider the tetrahedron determined by the vertex and the midpoints of the three edges that meet at that vertex. The portion of the cube that remains when these eight tetrahedra are cut away is called a *cuboctahedron*. The ratio of the volume of the cuboctahedron to the volume of the original cube is closest to which of these?

A 75%

B 78%

C 81%

D 84%

E 87%

Solution:

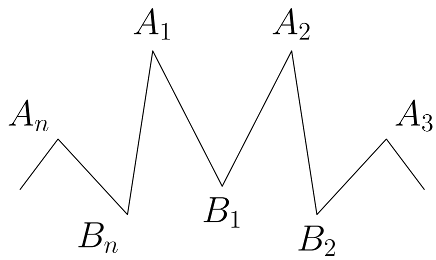
Take the cube's side length to be 2. Its volume is 8. Each corner tetrahedron has three perpendicular edges of length 1, so its volume is $\frac{1}{6}$. The eight removed tetrahedra have total volume $\frac{4}{3}$, leaving $\frac{20}{3}$. The ratio is

$$\frac{\frac{20}{3}}{8} = \frac{5}{6} = 83\tfrac{1}{3}\%,$$

which is closest to 84%.

Thus the correct answer is **D**.

20. Part of an "n-pointed regular star" is shown. It is a simple closed polygon in which all $2n$ edges are congruent, angles $A_1, A_2, \ldots, A_n$ are congruent, and angles $B_1, B_2, \ldots, B_n$ are congruent. If the acute angle at A_1 is 10° less than the acute angle at B_1, then $n =$



A 12

B 18

C 24

D 36

E 60

Solution:

Let the acute angles at the tips and notches be α and β, respectively, so $\beta - \alpha = 10^\circ$. The $2n$-gon has n interior angles α and n reflex interior angles $360^\circ - \beta$. Therefore

$$n\alpha + n(360^\circ - \beta) \\ = (2n - 2)180^\circ.$$

Substituting $\beta - \alpha = 10^\circ$ gives $350n = 360n - 360$, so $n = 36$.

Thus the correct answer is **D**.

21. For a finite sequence $A = (a_1, a_2, \ldots, a_n)$ of numbers, the *Cesàro sum* of A is defined to be

$$\frac{S_1 + S_2 + \cdots + S_n}{n},$$

where

$$S_k = a_1 + a_2 + a_3 + \cdots + a_k$$
$$(1 \leq k \leq n).$$

If the Cesàro sum of the 99-term sequence $(a_1, a_2, \ldots, a_{99})$ is 1000, what is the Cesàro sum of the 100-term sequence $(1, a_1, a_2, \ldots, a_{99})$?

A 991

B 999

C 1000

D 1001

E 1009

Solution:

The given condition says $S_1 + \cdots + S_{99} = 99{,}000$. For the new sequence, the partial sums are $1, 1 + S_1, \ldots, 1 + S_{99}$. Their sum is

$$100 + (S_1 + \cdots + S_{99}) = 99{,}100.$$

Dividing by 100 gives the new Cesàro sum 991.

Thus the correct answer is **A**.

22. Ten points are selected on the positive x-axis, X^{+}, and five points are selected on the positive y-axis, Y^{+}. The fifty segments connecting the ten selected points on X^{+} to the five selected points on Y^{+} are drawn. What is the maximum possible number of points of intersection of these fifty segments that could lie in the interior of the first quadrant?

A 250

B 450

C 500

D 1250

E 2500

Solution:

Choose two of the 10 points on the x-axis and two of the 5 points on the y-axis. Among the four connecting segments, the two with reversed endpoint order cross exactly once. The points can be chosen so no three segments meet at one interior point, so the maximum is

$$\binom{10}{2}\binom{5}{2} = 45 \cdot 10 = 450.$$

Thus the correct answer is **B**.

23. What is the size of the largest subset S of $\{1, 2, 3, \ldots, 50\}$ such that no pair of distinct elements of S has a sum divisible by 7?

A 6

B 7

C 14

D 22

E 23

Solution:

The residue classes $1, 2, \ldots, 6, 0$ contain $8, 7, 7, 7, 7, 7, 7$ numbers, respectively. We may take numbers from at most one class in each complementary pair $(1, 6), (2, 5), (3, 4)$, and at most one multiple of 7. Thus

$$|S| \leq \max(8, 7) + \max(7, 7) \\ + \max(7, 7) + 1 = 23.$$

Taking every number congruent to $1, 2,$ or $3 \pmod 7$, together with one multiple of 7, attains 23.

Thus the correct answer is **E**.

24. Let $ABCD$ be a parallelogram of area 10 with $AB = 3$ and $BC = 5$. Locate E, F, and G on segments $\overline{AB}, \overline{BC}$, and $\overline{AD}$, respectively, with $AE = BF = AG = 2$. Let the line through G parallel to $\overline{EF}$ intersect $\overline{CD}$ at H. The area of the quadrilateral $EFHG$ is

A 4

B 4.5

C 5

D 5.5

E 6

Solution:

Use AB and AD as basis vectors. Then

$$E = \left(\frac{2}{3}, 0\right),$$
$$F = \left(1, \frac{2}{5}\right),$$
$$G = \left(0, \frac{2}{5}\right).$$

A line from G parallel to EF meets CD at $H = (\frac{1}{2}, 1)$. The shoelace formula gives the coordinate area of $EFHG$ as $\frac{1}{2}$. Affine coordinates scale all areas by the area 10 of $ABCD$, so $[EFHG] = \frac{1}{2} \cdot 10 = 5$.

Thus the correct answer is **C**.

25. In triangle $ABC, \angle ABC = 120^\circ, AB = 3,$ and $BC = 4.$ If perpendiculars constructed to $\overline{AB}$ at A and to $\overline{BC}$ at C meet at $D,$ then $CD =$

A 3

B $\frac{8}{\sqrt{3}}$

C 5

D $\frac{11}{2}$

E $\frac{10}{\sqrt{3}}$

Solution:

Set $B = (0, 0)$ and $A = (3, 0).$ Then

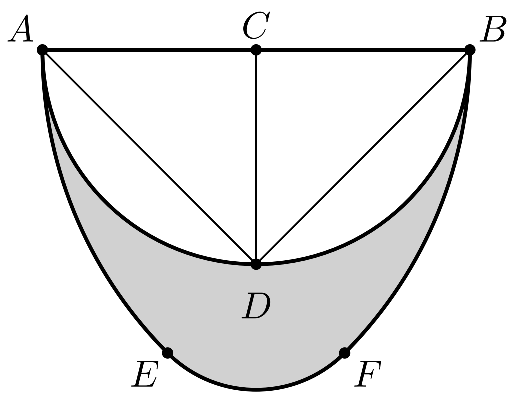
$$\begin{aligned} C &= 4(\cos 120^\circ, \sin 120^\circ) \\ &= (-2, 2\sqrt{3}). \end{aligned}$$

The perpendicular to AB at A is $x = 3.$ A vector perpendicular to $BC = (-2, 2\sqrt{3})$ is $(\sqrt{3}, 1),$ so the other perpendicular is $C + t(\sqrt{3}, 1).$ Its x-coordinate is 3 when $t = \frac{5}{\sqrt{3}}.$ Since $|(\sqrt{3}, 1)| = 2,$

$$CD = 2t = \frac{10}{\sqrt{3}}.$$

Thus the correct answer is **E**.

26. Semicircle $\overset{\frown}{AB}$ has center C and radius 1. Point D is on $\overset{\frown}{AB}$ and $CD \perp AB$. Extend $\overline{BD}$ and $\overline{AD}$ to E and F, respectively, so that circular arcs $\overset{\frown}{AE}$ and $\overset{\frown}{BF}$ have B and A as their respective centers. Circular arc $\overset{\frown}{EF}$ has center D. The area of the shaded "smile," $AEFBDA$, is



A $(2-\sqrt{2})\pi$

B $2\pi - \pi\sqrt{2} - 1$

C $\left(1-\frac{\sqrt{2}}{2}\right)\pi$

D $\frac{5\pi}{2} - \pi\sqrt{2} - 1$

E $(3-2\sqrt{2})\pi$

Solution:

Triangles ACD and BCD are isosceles right triangles, so $AD = BD = \sqrt{2}$, $\angle ADB = 90^\circ$, and $DE = DF = 2 - \sqrt{2}$. The smile is the 90° sector EDF, plus the congruent 45° sectors ABE and BAF, minus triangle ABD and the original semicircle. Its area is

$$\frac{1}{4}\pi(2-\sqrt{2})^2 + 2\left(\frac{1}{8}\pi \cdot 2^2\right) - \frac{1}{2}(2)(1) - \frac{1}{2}\pi = 2\pi - \pi\sqrt{2} - 1.$$

Thus the correct answer is **B**.

27. A circle of radius r has chords $\overline{AB}$ of length 10 and $\overline{CD}$ of length 7. When $\overline{AB}$ and $\overline{CD}$ are extended through B and C, respectively, they intersect at P, which is outside the circle. If $\angle APD = 60^\circ$ and $BP = 8$, then $r^2 =$

- A 70
- B 71
- C 72
- D 73
- E 74

Solution:

Power of P gives

$$PA \cdot PB = PC \cdot PD.$$

Here $PA = 18,\ PB = 8$, and $PD = PC + 7$, so $PC(PC + 7) = 144$. Thus $PC = 9$ and $PD = 16$. Since $PA = 2PC$ and $\angle APC = 60^\circ$, triangle APC is right at C, with $AC = 9\sqrt{3}$. Therefore AD is a diameter, and

$$\begin{aligned}(2r)^2 &= AD^2 = AC^2 + CD^2\\ &= 243 + 49 = 292.\end{aligned}$$

Hence $r^2 = 73.$

Thus the correct answer is **D**.

28. Let $i = \sqrt{-1}$. The product of the real parts of the roots of $z^2 - z = 5 - 5i$ is

A -25

B -6

C -5

D $\frac{1}{4}$

E 25

Solution:

The quadratic formula gives

$$z = \frac{1 \pm \sqrt{21 - 20i}}{2}.$$

Since $(5 - 2i)^2 = 21 - 20i$, the roots are

$$\frac{1 + (5 - 2i)}{2} = 3 - i,$$
$$\frac{1 - (5 - 2i)}{2} = -2 + i.$$

Their real parts have product $3(-2) = -6$.

Thus the correct answer is **B**.

29. An "unfair" coin has a $\frac{2}{3}$ probability of turning up heads. If this coin is tossed 50 times, what is the probability that the total number of heads is even?

A $2^5\left(\frac{2}{3}\right)^{50}$

B $\frac{1}{2}\left(1-\frac{1}{3^{50}}\right)$

C $\frac{1}{2}$

D $\frac{1}{2}\left(1+\frac{1}{3^{50}}\right)$

E $\frac{2}{3}$

Solution:

Let E_n and O_n be the probabilities of an even and odd number of heads after n tosses. Then $E_n + O_n = 1,$ while

$$E_{n+1} - O_{n+1} = \left(\frac{1}{3} - \frac{2}{3}\right) \cdot (E_n - O_n).$$

Since $E_0 - O_0 = 1,$ we have $E_{50} - O_{50} = (-\frac{1}{3})^{50} = 3^{-50}$. Solving the sum and difference equations gives

$$E_{50} = \frac{1}{2}\left(1 + \frac{1}{3^{50}}\right).$$

Thus the correct answer is **D**.

30. Let $ABCD$ be an isosceles trapezoid with bases $AB = 92$ and $CD = 19$. Suppose $AD = BC = x$ and a circle with center on $\overline{AB}$ is tangent to segments $\overline{AD}$ and $\overline{BC}$. If m is the smallest possible value of x, then $m^2 =$

A 1369

B 1679

C 1748

D 2109

E 8825

Solution:

Place $A = (-46, 0), B = (46, 0), D = (-\frac{19}{2}, h)$, and $C = (\frac{19}{2}, h)$. Symmetry forces the circle's center to be $O = (0, 0)$. The horizontal offset along each leg is $\frac{73}{2}$, so

$$x^2 = h^2 + \left(\frac{73}{2}\right)^2.$$

The perpendicular from O to a leg first has its foot on the segment when the foot reaches the upper endpoint; this boundary condition is $OD \perp AD$. Thus

$$\begin{aligned}(-\frac{19}{2}, h) \cdot (\frac{73}{2}, h) &= 0,\\ h^2 &= \frac{1387}{4}.\end{aligned}$$

Therefore

$$m^2 = \frac{1387 + 5329}{4} = 1679.$$

Thus the correct answer is **B**.

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