1991 AMC 12 Solutions

Typeset by: LIVE by Po-Shen Loh
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1. If for any three distinct numbers $a, b,$ and c we define

$$\boxed{a, b, c} = \frac{c+a}{c-b},$$

then $\boxed{1, -2, -3} =$

A -2

B $-\frac{2}{5}$

C $-\frac{1}{4}$

D $\frac{2}{5}$

E 2

Solution:

Direct substitution gives

$$\begin{aligned}\boxed{1, -2, -3} &= \frac{-3+1}{-3-(-2)} \\ &= \frac{-2}{-1} = 2.\end{aligned}$$

Thus the correct answer is **E**.

2. $|3-\pi| =$

A $\frac{1}{7}$

B 0.14

C $3-\pi$

D $3+\pi$

E $\pi-3$

Solution:

Since $\pi > 3,$ the quantity $3-\pi$ is negative. Therefore $|3-\pi| = -(3-\pi) = \pi - 3.$

Thus the correct answer is **E**.

3. $\left(4^{-1}-3^{-1}\right)^{-1} =$

A -12

B -1

C $\frac{1}{12}$

D 1

E 12

Solution:

We have $4^{-1} - 3^{-1} = \frac{1}{4} - \frac{1}{3} = -\frac{1}{12}$. Raising this result to the power -1 takes its reciprocal, giving $-12.$

Thus the correct answer is **A**.

4. Which of the following triangles cannot exist?

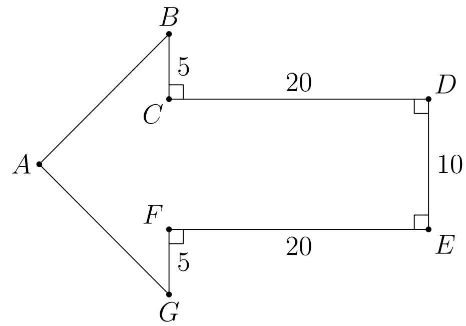
- A: An acute isosceles triangle
- B: An isosceles right triangle
- C: An obtuse right triangle
- D: A scalene right triangle
- E: A scalene obtuse triangle

Solution:

A right triangle has one 90° angle, leaving only 90° for its other two positive angles together. Neither remaining angle can therefore be obtuse. Hence a triangle cannot be both right and obtuse.

Thus the correct answer is **C**.

5. In the arrow-shaped polygon shown, the angles at vertices A, C, D, E, and F are right angles, $BC = FG = 5, CD = FE = 20, DE = 10$, and $AB = AG$. The area of the polygon is closest to



A 288

B 291

C 294

D 297

E 300

Solution:

The shaft $CDEF$ is a 20-by-10 rectangle, so its area is 200. Also $BG = BC + CF + FG = 5 + 10 + 5 = 20$. Because $\angle BAG = 90^\circ$ and $AB = AG$, triangle ABG is an isosceles right triangle with hypotenuse 20. Its legs are $10\sqrt{2}$, so its area is $\frac{1}{2}(10\sqrt{2})^2 = 100$. The total area is $200 + 100 = 300$.

Thus the correct answer is **E**.

6. If $x \geq 0$, then $\sqrt{x\sqrt{x\sqrt{x}}} =$

A $x\sqrt{x}$

B $x\sqrt[4]{x}$

C $\sqrt[8]{x}$

D $\sqrt[8]{x^3}$

E $\sqrt[8]{x^7}$

Solution:

Starting inside and using fractional exponents,

$$\begin{aligned}\sqrt{x\sqrt{x\sqrt{x}}} &= \sqrt{x \cdot x^{\frac{3}{4}}} \\ &= x^{\frac{7}{8}} = \sqrt[8]{x^7}.\end{aligned}$$

Thus the correct answer is **E**.

7. If $x = \frac{a}{b}$, $a \neq b$, and $b \neq 0$, then $\dfrac{a+b}{a-b} =$

A $\frac{x}{x+1}$

B $\frac{x+1}{x-1}$

C 1

D $x - \frac{1}{x}$

E $x + \frac{1}{x}$

Solution:

Dividing both numerator and denominator by the nonzero number b gives

$$\frac{a+b}{a-b} = \frac{\frac{a}{b}+1}{\frac{a}{b}-1} = \frac{x+1}{x-1}.$$

Thus the correct answer is **B**.

8. Liquid X does not mix with water. Unless obstructed, it spreads out on the surface of water to form a circular film 0.1 cm thick. A rectangular box measuring 6 cm by 3 cm by 12 cm is filled with liquid X. Its contents are poured onto a large body of water. What will be the radius, in centimeters, of the resulting circular film?

A $\frac{\sqrt{216}}{\pi}$

B $\sqrt{\frac{216}{\pi}}$

C $\sqrt{\frac{2160}{\pi}}$

D $\frac{216}{\pi}$

E $\frac{2160}{\pi}$

Solution:

The liquid volume is $6 \cdot 3 \cdot 12 = 216$ cubic centimeters. Thus

$$0.1\pi r^2 = 216,$$

so $r^2 = \frac{2160}{\pi}$ and $r = \sqrt{\frac{2160}{\pi}}$.

Thus the correct answer is **C**.

9. From time $t = 0$ to time $t = 1$ a population increased by $i\%$, and from time $t = 1$ to time $t = 2$ the population increased by $j\%$. Therefore, from time $t = 0$ to time $t = 2$ the population increased by

A $(i + j)\%$

B $ij\%$

C $(i + ij)\%$

D $\left(i + j + \frac{ij}{100}\right)\%$

E $\left(i + j + \frac{i+j}{100}\right)\%$

Solution:

The combined growth factor is

$$\left(1 + \frac{i}{100}\right)\left(1 + \frac{j}{100}\right) = 1 + \frac{i+j}{100} + \frac{ij}{10000}.$$

Hence the percent increase is $i + j + \frac{ij}{100}$.

Thus the correct answer is **D**.

10. Point P is 9 units from the center of a circle of radius $15.$ How many different chords of the circle contain P and have integer lengths?

A 11

B 12

C 13

D 14

E 29

Solution:

The longest chord through P is a diameter of length $30.$ The shortest is perpendicular to the radius through $P,$ and has length

$$2\sqrt{15^2 - 9^2} = 24.$$

As the chord rotates between these positions, its length varies continuously from 24 to $30.$ The endpoint lengths each occur once, while each of the five interior integer lengths $25, 26, 27, 28, 29$ occurs twice. Thus the count is $1 + 2(5) + 1 = 12.$

Thus the correct answer is **B**.

11. Jack and Jill run 10 kilometers. They start at the same point, run 5 kilometers up a hill, and return to the starting point by the same route. Jack has a 10-minute head start and runs at the rate of 15 km/hr uphill and 20 km/hr downhill. Jill runs 16 km/hr uphill and 22 km/hr downhill. How far from the top of the hill are they when they pass going in opposite directions?

A $\frac{5}{4}$ km

B $\frac{35}{27}$ km

C $\frac{27}{20}$ km

D $\frac{7}{3}$ km

E $\frac{28}{9}$ km

Solution:

Jack reaches the top in $\frac{5}{15} = \frac{1}{3}$ hour. Jill has then run for $\frac{1}{3} - \frac{1}{6} = \frac{1}{6}$ hour and is $\frac{16}{6} = \frac{8}{3}$ km uphill, so they are $5 - \frac{8}{3} = \frac{7}{3}$ km apart. Their closing speed is $20 + 16 = 36$ km/hr, so they meet after $\frac{\frac{7}{3}}{36} = \frac{7}{108}$ hour. Jack descends

$$20\left(\frac{7}{108}\right) = \frac{35}{27} \text{ km.}$$

Thus the correct answer is **B**.

12. The measures (in degrees) of the interior angles of a convex hexagon form an arithmetic sequence of positive integers. Let m° be the measure of the largest interior angle of the hexagon. The largest possible value of m° is

A 165°

B 167°

C 170°

D 175°

E 179°

Solution:

The angle sum is 720°, so

$$\begin{aligned} 6a + 15d &= 720, \\ 2a + 5d &= 240. \end{aligned}$$

Hence d is even. The largest angle is $a + 5d = 120 + \frac{5}{2}d$, which must be less than 180. Thus $d < 24$, and the largest allowable even value is $d = 22$. It gives $a = 65$ and $m = 65 + 5(22) = 175$.

Thus the correct answer is **D**.

13. Horses X, Y, and Z are entered in a three-horse race in which ties are not possible. If the odds against X winning are 3-to-1 and the odds against Y winning are 2-to-3, what are the odds against Z winning? (By "odds against H winning are p-to-q" we mean that the probability of H winning the race is $\frac{q}{p+q}$.)

A 3-to-20

B 5-to-6

C 8-to-5

D 17-to-3

E 20-to-3

Solution:

The given probabilities are $P(X) = \frac{1}{4}$ and $P(Y) = \frac{3}{5}$. Therefore

$$P(Z) = 1 - \frac{1}{4} - \frac{3}{5} = \frac{3}{20}.$$

The probability that Z loses is $\frac{17}{20}$, so the odds against Z are 17-to-3.

Thus the correct answer is **D**.

14. If x is the cube of a positive integer and d is the number of positive integers that are divisors of x, then d could be

- A 200
- B 201
- C 202
- D 203
- E 204

Solution:

For any prime p, take $x = p^{201} = (p^{67})^3$. This is a cube, and its positive divisors are $1, p, p^2, \ldots, p^{201}$, a total of 202. Thus 202 can occur.

Thus the correct answer is **C**.

15. A circular table has exactly 60 chairs around it. There are N people seated at this table in such a way that the next person to be seated must sit next to someone. The smallest possible value of N is

- A 15
- B 20
- C 30
- D 40
- E 58

Solution:

The condition says that every empty chair has an occupied neighbor, so no three consecutive chairs may all be empty. Therefore each occupied chair can account for at most itself and the two empty chairs following it, giving $60 \leq 3N$ and $N \geq 20.$ Repeating the pattern occupied-empty-empty around the table attains $N = 20.$

Thus the correct answer is **B**.

16. One hundred students at Century High School participated in the AHSME last year, and their mean score was 100. The number of non-seniors taking the AHSME was 50% more than the number of seniors, and the mean score of the seniors was 50% higher than that of the non-seniors. What was the mean score of the seniors?

A 100

B 112.5

C 120

D 125

E 150

Solution:

If there are s seniors, then there are $1.5s$ non-seniors, so $2.5s = 100$ and $s = 40$. Let the non-senior mean be x; the senior mean is $1.5x$. The total score equation is

$$60x + 40(1.5x) = 100(100),$$

so $120x = 10000$, and the senior mean is $1.5x = 125$.

Thus the correct answer is **D**.

17. A positive integer N is a palindrome if the integer obtained by reversing the sequence of digits of N is equal to N. The year 1991 is the only year in the current century with the following two properties:

(a) It is a palindrome.

(b) It factors as a product of a 2-digit prime palindrome and a 3-digit prime palindrome.

How many years in the millenium between 1000 and 2000 (including the year 1991) have properties (a) and (b)?

A 1

B 2

C 3

D 4

E 5

Solution:

The palindromes in the interval are $1dd1$, where d is a digit. Algebraically,

$$\begin{aligned} 1dd1 &= 1001 + 110d \\ &= 11(91 + 10d). \end{aligned}$$

The two-digit prime palindrome must therefore be 11, and $91 + 10d$ must be a three-digit prime palindrome. For $d = 0, 1, \ldots, 9$, the prime values occur for $d = 1, 4, 6, 9$, giving $101, 131, 151, 181$. Thus there are 4 years.

Thus the correct answer is **D**.

18. If S is the set of points z in the complex plane such that $(3+4i)z$ is a real number, then S is a

A right triangle

B circle

C hyperbola

D line

E parabola

Solution:

Writing $z = x + yi$,

$$(3+4i)z = (3x-4y) \\ + (4x+3y)i.$$

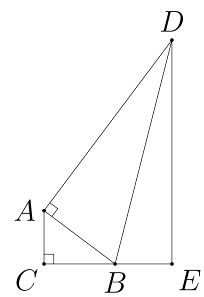
This is real exactly when $4x+3y=0$, which is the equation of a line through the origin.

Thus the correct answer is **D**.

19. Triangle ABC has a right angle at C, $AC = 3$, and $BC = 4$. Triangle ABD has a right angle at A and $AD = 12$. Points C and D are on opposite sides of AB. The line through D parallel to AC meets CB extended at E. If

$$\frac{DE}{DB} = \frac{m}{n},$$

where m and n are relatively prime positive integers, then $m + n =$



A 25

B 128

C 153

D 243

E 256

Solution:

Place $C = (0,0)$, $A = (0,3)$, and $B = (4,0)$. Since $\overrightarrow{AB} = (4,-3)$, the length-12 vector perpendicular to $\overrightarrow{AB}$ and directed away from C is $(\frac{36}{5}, \frac{48}{5})$. Hence $D = (\frac{36}{5}, \frac{63}{5})$. Since $DE \parallel AC$, we have $E = (\frac{36}{5}, 0)$, so $DE = \frac{63}{5}$ and $DB = 13$. Therefore

$$\frac{DE}{DB} = \frac{\frac{63}{5}}{13} = \frac{63}{65}.$$

Thus $m + n = 63 + 65 = 128$.

Thus the correct answer is **B**.

20. The sum of all real x such that

$$(2^x - 4)^3 + (4^x - 2)^3 \\ = (4^x + 2^x - 6)^3$$

is

A $\frac{3}{2}$

B 2

C $\frac{5}{2}$

D 3

E $\frac{7}{2}$

Solution:

Let $a = 2^x - 4$ and $b = 4^x - 2$. The right side is $(a+b)^3$, so

$$a^3 + b^3 = (a+b)^3$$

is equivalent to $ab(a+b) = 0$. The cases give $2^x = 4,\ 4^x = 2$, or $4^x + 2^x = 6$, whose real solutions are $x = 2, \frac{1}{2}$, and 1, respectively. Their sum is $2 + \frac{1}{2} + 1 = \frac{7}{2}$.

Thus the correct answer is **E**.

21. If

$$f\left(\frac{x}{x-1}\right)=\frac{1}{x}$$
$$\text{for all } x \neq 0,1$$

and $0<\theta<\frac{\pi}{2}$, then $f(\sec^2\theta)=$

A $\sin^2\theta$

B $\cos^2\theta$

C $\tan^2\theta$

D $\cot^2\theta$

E $\csc^2\theta$

Solution:

Since

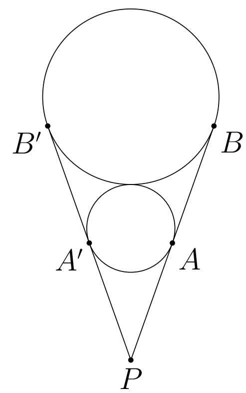
$$\sec^2\theta=\frac{1}{\cos^2\theta}=\frac{\csc^2\theta}{\csc^2\theta-1},$$

choose $x=\csc^2\theta$ in the definition. Then

$$f(\sec^2\theta)=\frac{1}{\csc^2\theta}=\sin^2\theta.$$

Thus the correct answer is **A**.

22. Two circles are externally tangent. Lines PAB and $PA'B'$ are common tangents with A and A' on the smaller circle and B and B' on the larger circle. If $PA = AB = 4$, then the area of the smaller circle is



A 1.44π

B 2π

C 2.56π

D $\sqrt{8}\pi$

E 4π

Solution:

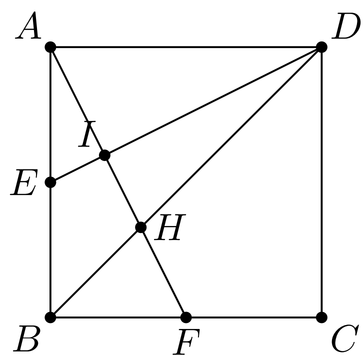
Since $PA = 4$ and $AB = 4$, we have $PB = 8$. The homothety centered at P taking the smaller circle to the larger therefore has ratio 2. If the smaller radius is r, its center is $3r$ from P : the centers are on the same ray, their distances from P have ratio 2, and their difference is $r + 2r = 3r$. Applying the right triangle formed by P, the small center, and A,

$$PA^2 = (3r)^2 - r^2 = 8r^2.$$

Thus $16 = 8r^2$, so $r^2 = 2$ and the area is 2π.

Thus the correct answer is **B**.

23. If $ABCD$ is a 2×2 square, E is the midpoint of AB, F is the midpoint of BC, AF and DE intersect at I, and BD and AF intersect at H, then the area of quadrilateral $BEIH$ is



A $\frac{1}{3}$

B $\frac{2}{5}$

C $\frac{7}{15}$

D $\frac{8}{15}$

E $\frac{3}{5}$

Solution:

Set $B = (0, 0), C = (2, 0), D = (2, 2)$, and $A = (0, 2)$. Then $E = (0, 1),\ F = (1, 0)$, and the line intersections are

$$I = AF \cap DE = \left(\frac{2}{5}, \frac{6}{5}\right),$$
$$H = AF \cap BD = \left(\frac{2}{3}, \frac{2}{3}\right).$$

The shoelace formula gives

$$[BEIH] = \frac{1}{2}\left|\frac{4}{15} - \frac{2}{5} - \frac{4}{5}\right|$$
$$= \frac{7}{15}.$$

Thus the correct answer is **C**.

24. The graph, G, of $y = \log_{10} x$ is rotated 90° counter-clockwise about the origin to obtain a new graph G'. Which of the following is an equation for G'?

A $y = \log_{10}\left(\frac{x+90}{9}\right)$

B $y = \log_x 10$

C $y = \frac{1}{x+1}$

D $y = 10^{-x}$

E $y = 10^x$

Solution:

A point $(u, \log_{10} u)$ rotates to

$$(x, y) = (-\log_{10} u, u).$$

Hence $x = -\log_{10} y$, so $\log_{10} y = -x$ and $y = 10^{-x}$.

Thus the correct answer is **D**.

25. If $T_n = 1 + 2 + 3 + \cdots + n$ and

$$P_n = \frac{T_2}{T_2 - 1} \cdot \frac{T_3}{T_3 - 1} \cdot \frac{T_4}{T_4 - 1} \cdots \frac{T_n}{T_n - 1}$$

for $n = 2, 3, 4, \ldots$, then P_{1991} is closest to which of the following numbers?

A 2.0

B 2.3

C 2.6

D 2.9

E 3.2

Solution:

Since $T_k = \frac{k(k+1)}{2}$ and $T_k - 1 = \frac{(k-1)(k+2)}{2}$,

$$\frac{T_k}{T_k - 1} = \frac{k}{k-1} \cdot \frac{k+1}{k+2}.$$

The product telescopes:

$$P_n = \left(\prod_{k=2}^{n} \frac{k}{k-1}\right)\left(\prod_{k=2}^{n} \frac{k+1}{k+2}\right) = n \cdot \frac{3}{n+2} = \frac{3n}{n+2}.$$

For $n = 1991$, this is just under 3 and is closest to 2.9.

Thus the correct answer is **D**.

26. An n-digit positive integer is *cute* if its n digits are an arrangement of the set $\{1, 2, \ldots, n\}$ and its first k digits form an integer that is divisible by k, for $k = 1, 2, \ldots, n$. For example, 321 is a cute 3-digit integer because 1 divides 3, 2 divides 32, and 3 divides 321. How many cute 6-digit integers are there?

A 0

B 1

C 2

D 3

E 4

Solution:

Divisibility by 5 forces the fifth digit to be 5. Divisibility by 2 and 6 forces the second and sixth digits to be even. Now apply the divisibility tests successively: the first three digits must have sum divisible by 3, and the two-digit number formed by the third and fourth digits must be divisible by 4. Checking the remaining choices from $\{1, 2, 3, 4, 6\}$ leaves 123654 and 321654. Each number directly satisfies all six prefix divisibility conditions, so there are 2.

Thus the correct answer is **C**.

27. If

$$x+\sqrt{x^2-1}+\frac{1}{x-\sqrt{x^2-1}} = 20,$$

then

$$x^2+\sqrt{x^4-1}+\frac{1}{x^2+\sqrt{x^4-1}} =$$

- A 5.05
- B 20
- C 51.005
- D 61.25
- E 400

Solution:

Because

$$\frac{1}{x-\sqrt{x^2-1}} = x+\sqrt{x^2-1},$$

the given equation implies $x+\sqrt{x^2-1}=10$. Its reciprocal is $x-\sqrt{x^2-1}=\frac{1}{10}$, so adding yields $2x=10.1$ and $x=5.05$. Also

$$\frac{1}{x^2+\sqrt{x^4-1}} = x^2-\sqrt{x^4-1}.$$

Therefore the requested expression is $2x^2=2(5.05)^2=51.005$.

Thus the correct answer is **C**.

28. Initially an urn contains 100 black marbles and 100 white marbles. Repeatedly, three marbles are removed from the urn and replaced from a pile outside the urn as follows:

Marbles removed	**Replaced with**
3 black	1 black
2 black, 1 white	1 black, 1 white
1 black, 2 white	2 white
3 white	1 black, 1 white

Which of the following sets of marbles could be the contents of the urn after repeated applications of this procedure?

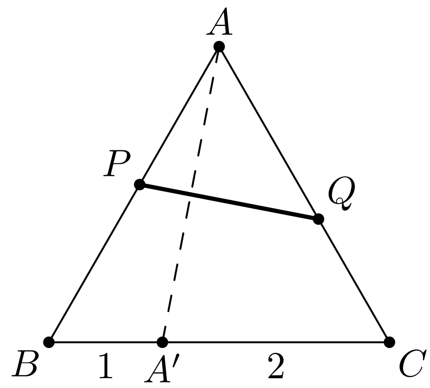
- A) 2 black marbles
- B) 2 white marbles
- C) 1 black marble
- D) 1 black and 1 white marble
- E) 1 white marble

Solution:

The parity of the number of white marbles never changes, so choices with one white marble are impossible. Also a final operation starting with three marbles can leave 1 black, 1 black and 1 white, or 2 white, but never 2 black. The state with 2 white is attainable: repeatedly replace 3 black by 1 black until 2 black remain; twice use the 1-black-2-white rule, leaving 100 white; then alternate the 3-white rule with the 1-black-2-white rule, reducing the number of white marbles by 2 per pair until 2 remain.

Thus the correct answer is **B**.

29. Equilateral triangle ABC has been creased and folded so that vertex A now rests at A' on BC as shown. If $BA' = 1$ and $A'C = 2$, then the length of crease PQ is



A $\frac{8}{5}$

B $\frac{7}{20}\sqrt{21}$

C $\frac{1+\sqrt{5}}{2}$

D $\frac{13}{8}$

E $\sqrt{3}$

Solution:

With the coordinates in the hint, the perpendicular-bisector condition for $X = (x, y)$ is $|X - A| = |X - A'|$, which simplifies to

$$x + 3\sqrt{3}\,y = 8.$$

Intersecting this line with $AB,\ y = \sqrt{3}x$, gives $P = (\frac{4}{5}, \frac{4\sqrt{3}}{5})$. Intersecting it with $AC,\ y = \sqrt{3}(3 - x)$, gives $Q = (\frac{19}{8}, \frac{5\sqrt{3}}{8})$. Hence

$$PQ = \sqrt{\left(\frac{63}{40}\right)^2 + \left(\frac{7\sqrt{3}}{40}\right)^2} = \frac{7}{20}\sqrt{21}.$$

Thus the correct answer is **B**.

30. For any set S, let $|S|$ denote the number of elements in S, and let $n(S)$ be the number of subsets of S, including the empty set and the set S itself. If A, B, and C are sets for which

$$n(A)+n(B)+n(C)$$
$$=n(A\cup B\cup C),$$
$$|A|=|B|=100,$$

then what is the minimum possible value of $|A\cap B\cap C|$?

A 96

B 97

C 98

D 99

E 100

Solution:

Let $c=|C|$ and $u=|A\cup B\cup C|$. Since $n(S)=2^{|S|}$, the equation becomes $2^{101}+2^c=2^u$. The two summands must be equal, so $c=101$ and $u=102$. Within the union, A, B, and C omit 2, 2, and 1 elements. Thus at most five distinct elements are absent from the triple intersection, giving $|A\cap B\cap C|\geq 97$. Equality is attained by making those omissions distinct.

Thus the correct answer is **B**.

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