1990 AMC 12 Solutions

Typeset by: LIVE by Po-Shen Loh
https://live.poshenloh.com/past-contests/amc12/1990/solutions

1. If $\dfrac{\frac{x}{4}}{2} = \dfrac{4}{\frac{x}{2}}$, then $x =$

A $\pm\frac{1}{2}$

B ± 1

C ± 2

D ± 4

E ± 8

Solution:

The equation simplifies to $\frac{x}{8} = \frac{8}{x}$. Since $x \neq 0$, multiplying by $8x$ gives $x^2 = 64$, so $x = \pm 8$.

Thus the correct answer is **E**.

2. $\left(\frac{1}{4}\right)^{-\frac{1}{4}} =$

A -16

B $-\sqrt{2}$

C $-\frac{1}{16}$

D $\frac{1}{256}$

E $\sqrt{2}$

Solution:

Taking the reciprocal gives $4^{\frac{1}{4}} = (2^2)^{\frac{1}{4}} = 2^{\frac{1}{2}} = \sqrt{2}.$

Thus the correct answer is **E**.

3. The consecutive angles of a trapezoid form an arithmetic sequence. If the smallest angle is 75°, then the largest angle is

A 95°

B 100°

C 105°

D 110°

E 115°

Solution:

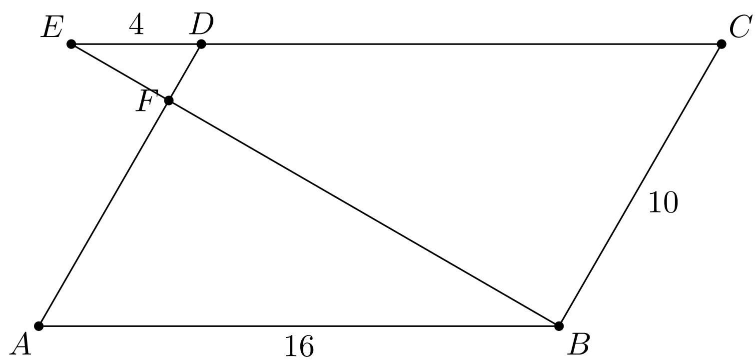
The four angles sum to 360°, so

$$\begin{aligned}75 &+ (75+d)\\ &+ (75+2d)\\ &+ (75+3d) = 360.\end{aligned}$$

Thus $d = 10$, and the largest angle is $75 + 3(10) = 105^\circ$.

Thus the correct answer is **C**.

4. Let $ABCD$ be a parallelogram with $\angle ABC = 120°$, $AB = 16$, and $BC = 10$. Extend $\overline{CD}$ through D to E so that $DE = 4$. If $\overline{BE}$ intersects $\overline{AD}$ at F, then FD is closest to



A 1

B 2

C 3

D 4

E 5

Solution:

Because $AB \parallel ED$, triangles ABF and DEF are similar. Hence

$$\frac{AF}{FD} = \frac{AB}{DE} = \frac{16}{4} = 4.$$

Since $AF + FD = AD = BC = 10$, we get $5FD = 10$, so $FD = 2$.

Thus the correct answer is **B**.

5. Which of these numbers is largest?

A $\sqrt{\sqrt[3]{5 \cdot 6}}$

B $\sqrt{6\sqrt[3]{5}}$

C $\sqrt{5\sqrt[3]{6}}$

D $\sqrt[3]{5\sqrt{6}}$

E $\sqrt[3]{6\sqrt{5}}$

Solution:

Raising the five positive choices to the sixth power preserves their order and gives, respectively,

$$\begin{gathered} 30, \qquad 6^3 \cdot 5 = 1080, \\ 5^3 \cdot 6 = 750, \\ 5^2 \cdot 6 = 150, \qquad 6^2 \cdot 5 = 180. \end{gathered}$$

The largest is the second.

Thus the correct answer is **B**.

6. Points A and B are 5 units apart. How many lines in a given plane containing A and B are 2 units from A and 3 units from B?

- A 0
- B 1
- C 2
- D 3
- E more than 3

Solution:

A qualifying line is a common tangent to the circle centered at A with radius 2 and the circle centered at B with radius $3.$ Their center distance equals the sum of their radii, so they are externally tangent. They have two external common tangents and one common tangent at their point of contact, for a total of $3.$

Thus the correct answer is **D**.

7. A triangle with integral sides has perimeter 8. The area of the triangle is

A $2\sqrt{2}$

B $\frac{16}{9}\sqrt{3}$

C $2\sqrt{3}$

D 4

E $4\sqrt{2}$

Solution:

The only unordered positive integral side lengths summing to 8 and satisfying the triangle inequality are $2, 3, 3.$ Their semiperimeter is $4,$ so Heron's formula gives

$$\begin{aligned} K &= \sqrt{4(4-2)(4-3)(4-3)} \\ &= 2\sqrt{2}. \end{aligned}$$

Thus the correct answer is **A**.

8. The number of real solutions of the equation

$$|x-2|+|x-3|=1$$

is

A 0

B 1

C 2

D 3

E more than 3

Solution:

For every $x \in [2,3]$,

$$\begin{aligned} &|x-2|+|x-3| \\ &\quad=(x-2)+(3-x) \\ &=1. \end{aligned}$$

Thus the entire interval $[2,3]$ consists of solutions, so there are more than 3.

Thus the correct answer is **E**.

9. Each edge of a cube is colored either red or black. Every face of the cube has at least one black edge. The smallest possible number of black edges is

- A 2
- B 3
- C 4
- D 5
- E 6

Solution:

Each black edge can cover only its two incident faces, so covering all 6 faces requires at least 3 black edges. Choose edges incident to the face-pairs top/front, bottom/left, and back/right. These three edges cover all six faces, so the minimum is $3.$

Thus the correct answer is **B**.

10. An $11 \times 11 \times 11$ wooden cube is formed by gluing together 11^3 unit cubes. What is the greatest number of unit cubes that can be seen from a single point?

A 328

B 329

C 330

D 331

E 332

Solution:

From a suitable point one can see three mutually adjacent faces. They contain

$$\begin{aligned} &3(11^2) - 3(11) + 1 \\ &= 363 - 33 + 1 \\ &= 331 \end{aligned}$$

distinct unit cubes: subtract the three shared edges and restore the corner cube.

Thus the correct answer is **D**.

11. How many positive integers less than 50 have an odd number of positive integer divisors?

A 3

B 5

C 7

D 9

E 11

Solution:

A positive integer has an odd number of divisors exactly when it is a perfect square. The squares below 50 are

$$1, 4, 9, 16, 25, 36, 49,$$

so there are 7.

Thus the correct answer is **C**.

12. Let f be the function defined by $f(x) = ax^2 - \sqrt{2}$ for some positive a. If $f(f(\sqrt{2})) = -\sqrt{2}$, then $a =$

A $\frac{2-\sqrt{2}}{2}$

B $\frac{1}{2}$

C $2 - \sqrt{2}$

D $\frac{\sqrt{2}}{2}$

E $\frac{2+\sqrt{2}}{2}$

Solution:

Because $a > 0$, the equation $f(y) = ay^2 - \sqrt{2} = -\sqrt{2}$ forces $y = 0$. Hence

$$f(\sqrt{2}) = 2a - \sqrt{2} = 0,$$

so $a = \frac{\sqrt{2}}{2}$.

Thus the correct answer is **D**.

13. If the following instructions are carried out by a computer, which value of X will be printed because of instruction 5?

$1.$ START X AT 3 AND S AT $0.$

$2.$ INCREASE THE VALUE OF X BY $2.$

$3.$ INCREASE THE VALUE OF S BY THE VALUE OF X.

$4.$ IF S IS AT LEAST $10000,$ THEN GO TO INSTRUCTION $5;$ OTHERWISE, GO TO INSTRUCTION 2 AND PROCEED FROM THERE.

$5.$ PRINT THE VALUE OF X.

$6.$ STOP.

A 19

B 21

C 23

D 199

E 201

Solution:

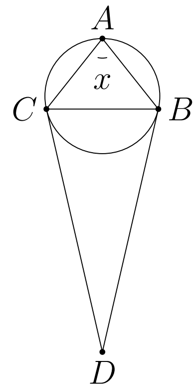
After k passes, $X = 2k + 3,$ and

$$\begin{aligned} S &= 5 + 7 + \cdots + (2k+3) \\ &= k(k+4). \end{aligned}$$

Now $98(102) = 9996 < 10000,$ while $99(103) = 10197 \geq 10000.$ Thus the loop stops at $k = 99,$ when $X = 2(99) + 3 = 201.$

Thus the correct answer is **E**.

14. An acute isosceles triangle, ABC, is inscribed in a circle. Through B and C, tangents to the circle are drawn, meeting at point D. If $\angle ABC = \angle ACB = 2\angle D$ and x is the radian measure of $\angle A$, then $x =$



A $\frac{3}{7}\pi$

B $\frac{4}{9}\pi$

C $\frac{5}{11}\pi$

D $\frac{6}{13}\pi$

E $\frac{7}{15}\pi$

Solution:

The two base angles are each $\frac{\pi-x}{2}$. The minor arc BC has central angle $2x$, so the angle between the tangents is $\pi - 2x$. The given relation yields

$$\frac{\pi - x}{2} = 2(\pi - 2x).$$

Therefore $7x = 3\pi$, or $x = \frac{3\pi}{7}$.

Thus the correct answer is **A**.

15. Four whole numbers, when added three at a time, give the sums $180, 197, 208,$ and $222.$ What is the largest of the four numbers?

A 77

B 83

C 89

D 95

E cannot be determined from the given information

Solution:

If T is the sum of the four numbers, adding the four triple-sums gives

$$\begin{aligned} 3T &= 180 + 197 + 208 + 222 \\ &= 807, \end{aligned}$$

so $T = 269.$ The omitted numbers are $269 - 180, 269 - 197, 269 - 208,$ and $269 - 222,$ whose largest is $89.$

Thus the correct answer is **C**.

16. At one of George Washington's parties, each man shook hands with everyone except his spouse, and no handshakes took place between women. If 13 married couples attended, how many handshakes were there among these 26 people?

A 78

B 185

C 234

D 312

E 325

Solution:

There are $\binom{26}{2} = 325$ possible pairs. Exclude the 13 married pairs and the $\binom{13}{2} = 78$ pairs of women. The number of handshakes is

$$325 - 13 - 78 = 234.$$

Thus the correct answer is **C**.

17. How many of the numbers $100, 101, \ldots, 999$ have three different digits in increasing order or in decreasing order?

A 120

B 168

C 204

D 216

E 240

Solution:

An increasing three-digit number cannot use 0, so there are $\binom{9}{3} = 84$ increasing numbers. Any three digits chosen from 0 through 9 form a valid decreasing three-digit number, because the largest digit comes first; this gives $\binom{10}{3} = 120$. The total is $84 + 120 = 204$.

Thus the correct answer is **C**.

18. First a is chosen at random from the set $\{1, 2, 3, \ldots, 99, 100\}$, and then b is chosen at random from the same set. The probability that the integer $3^a + 7^b$ has units digit 8 is

- A $\frac{1}{16}$
- B $\frac{1}{8}$
- C $\frac{3}{16}$
- D $\frac{1}{5}$
- E $\frac{1}{4}$

Solution:

The units digits of 3^a for $a \equiv 1, 2, 3, 0 \pmod 4$ are $3, 9, 7, 1,$ while those of 7^b are $7, 9, 3, 1.$ A sum ending in 8 occurs for the residue pairs

$$\begin{aligned}(a, b) \equiv (2, 2), (3, 0),& \\ (0, 1) \pmod 4.&\end{aligned}$$

Each residue occurs 25 times among $1, \ldots, 100,$ so the probability is $\frac{3}{16}$.

Thus the correct answer is **C**.

19. For how many integers N between 1 and 1990 is the improper fraction

$$\frac{N^2+7}{N+4}$$

not in lowest terms?

A 0

B 86

C 90

D 104

E 105

Solution:

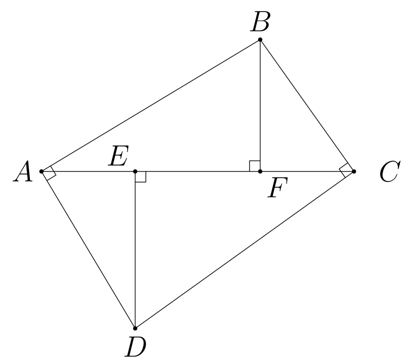
Modulo $N+4$, we have $N \equiv -4$, so

$$N^2+7 \equiv 16+7=23.$$

Thus the fraction is reducible exactly when $23 \mid N+4$, or $N \equiv 19 \pmod{23}$. The values are $19, 42, \ldots, 1974$, a total of 86.

Thus the correct answer is **B**.

20. In the figure, $ABCD$ is a quadrilateral with right angles at A and C. Points E and F are on $\overline{AC}$, and $\overline{DE}$ and $\overline{BF}$ are perpendicular to $\overline{AC}$. If $AE = 3$, $DE = 5$, and $CE = 7$, then $BF =$



A 3.6

B 4

C 4.2

D 4.5

E 5

Solution:

Set $A = (0,0)$, $E = (3,0)$, $C = (10,0)$, $D = (3,-5)$, and $B = (f,h)$, where $h = BF$. Since $AB \perp AD$,

$$(f,h) \cdot (3,-5) = 0,$$

so $3f = 5h$. Since $BC \perp DC$,

$$(f-10,h) \cdot (-7,-5) = 0,$$

so $7f + 5h = 70$. Substitution gives $f = 7$ and $h = \frac{21}{5} = 4.2$.

Thus the correct answer is **C**.

21. Consider a pyramid P-$ABCD$ whose base $ABCD$ is square and whose vertex P is equidistant from $A, B, C,$ and D. If $AB = 1$ and $\angle APB = 2\theta$, then the volume of the pyramid is

A $\frac{\sin\theta}{6}$

B $\frac{\cot\theta}{6}$

C $\frac{1}{6\sin\theta}$

D $\frac{1-\sin 2\theta}{6}$

E $\frac{\sqrt{\cos 2\theta}}{6\sin\theta}$

Solution:

Let $R = PA = PB$. In isosceles triangle APB,

$$1 = AB = 2R\sin\theta,$$

so $R = \frac{1}{2\sin\theta}$. If h is the pyramid height, the horizontal distance from the square's center to A is $\frac{1}{\sqrt{2}}$, hence

$$\begin{aligned} h^2 &= R^2 - \frac{1}{2} \\ &= \frac{\cos 2\theta}{4\sin^2\theta}. \end{aligned}$$

Thus $h = \frac{\sqrt{\cos 2\theta}}{2\sin\theta}$, and the volume is $\frac{1}{3}(1)h = \frac{\sqrt{\cos 2\theta}}{6\sin\theta}$.

Thus the correct answer is **E**.

22. If the six solutions of $x^6 = -64$ are written in the form $a + bi$, where a and b are real, then the product of those solutions with $a > 0$ is

A -2

B 0

C $2i$

D 4

E 16

Solution:

The roots have modulus 2 and arguments

$$\frac{\pi}{6} + \frac{k\pi}{3}, \qquad k = 0, 1, \ldots, 5.$$

The roots with positive real part have arguments $\frac{\pi}{6}$ and $\frac{11\pi}{6}$. They are conjugates of modulus 2, so their product is $2^2 = 4$.

Thus the correct answer is **D**.

23. If $x, y > 0, \log_y x + \log_x y = \frac{10}{3}$, and $xy = 144$, then $\frac{x+y}{2} =$

- A $12\sqrt{2}$
- B $13\sqrt{3}$
- C 24
- D 30
- E 36

Solution:

Let $t = \log_y x$. Then

$$t + \frac{1}{t} = \frac{10}{3},$$

so $3t^2 - 10t + 3 = 0$, giving $t = 3$ or $\frac{1}{3}$. Thus one of x, y is the cube of the other. Let the smaller be u. Then $u^4 = xy = 144$, so $u = 2\sqrt{3}$ and $u^3 = 24\sqrt{3}$. Therefore

$$\frac{x+y}{2} = \frac{2\sqrt{3} + 24\sqrt{3}}{2} = 13\sqrt{3}.$$

Thus the correct answer is **B**.

24. All students at Adams High School and at Baker High School take a certain exam. The average scores for boys, for girls, and for boys and girls combined, at Adams HS and Baker HS are shown in the table, as is the average for boys at the two schools combined. What is the average score for the girls at the two schools combined?

	Adams	**Baker**	**Adams & Baker**
Boys:	71	81	79
Girls:	76	90	$?$
Boys & Girls:	74	84	

A 81

B 82

C 83

D 84

E 85

Solution:

Let Adams have x boys and y girls. From its average,

$$71x + 76y = 74(x + y),$$

so $y = \frac{3x}{2}$. If Baker has u boys and v girls, its average gives $u = 2v$. The combined boys' average gives

$$71x + 81u = 79(x + u),$$

so $u = 4x$ and $v = 2x$. Hence the combined girls' average is

$$\frac{76(\frac{3x}{2}) + 90(2x)}{\frac{3x}{2} + 2x} = 84.$$

Thus the correct answer is **D**.

25. Nine congruent spheres are packed inside a unit cube in such a way that one of them has its center at the center of the cube and each of the others is tangent to the center sphere and to three faces of the cube. What is the radius of each sphere?

A $1-\frac{\sqrt{3}}{2}$

B $\frac{2\sqrt{3}-3}{2}$

C $\frac{\sqrt{2}}{6}$

D $\frac{1}{4}$

E $\frac{\sqrt{3}(2-\sqrt{2})}{4}$

Solution:

If the radius is r, a corner sphere has center (r, r, r) and the central sphere has center $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Tangency gives

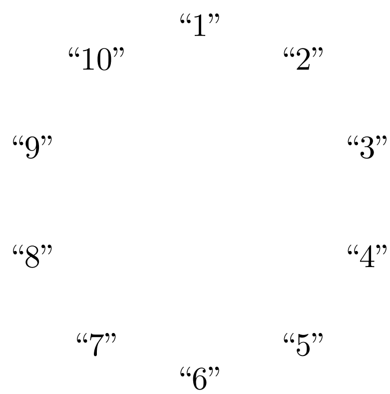
$$\sqrt{3}\left(\frac{1}{2}-r\right)=2r.$$

Solving and rationalizing,

$$r=\frac{\sqrt{3}}{2(2+\sqrt{3})}=\frac{2\sqrt{3}-3}{2}.$$

Thus the correct answer is **B**.

26. Ten people form a circle. Each picks a number and tells it to the two neighbors adjacent to him in the circle. Then each person computes and announces the average of the numbers of his two neighbors. The figure shows the average announced by each person (**not** the original number the person picked). The number picked by the person who announced the average 6 was



- A 1
- B 5
- C 6
- D 10
- E not uniquely determined from the given information

Solution:

Index the displayed averages $a_0, a_1, \ldots, a_9$ clockwise as $1, 2, \ldots, 10$, and let x_i be the corresponding picked numbers. Write $x_0 = t,\ x_1 = u$. From $x_{i-1} + x_{i+1} = 2a_i$, successive values are

$$\begin{aligned} x_2 &= 4 - t, & x_3 &= 6 - u, \\ x_4 &= 4 + t, & x_5 &= 4 + u, \\ x_6 &= 8 - t, & x_7 &= 10 - u, \\ x_8 &= 8 + t, & x_9 &= 8 + u. \end{aligned}$$

The two closing equations give $t = 6$ and $u = -3$. Hence $x_5 = 4 + u = 1$.

Thus the correct answer is **A**.

27. Which of these triples could **not** be the lengths of the three altitudes of a triangle?

A $1, \sqrt{3}, 2$

B $3, 4, 5$

C $5, 12, 13$

D $7, 8, \sqrt{113}$

E $8, 15, 17$

Solution:

If the altitudes are h_1, h_2, h_3, then the corresponding sides are proportional to $\frac{1}{h_1}, \frac{1}{h_2}, \frac{1}{h_3}$. For $5, 12, 13$,

$$\frac{1}{5} > \frac{1}{12} + \frac{1}{13},$$

so the reciprocals fail the triangle inequality. Direct checking shows that the reciprocals of each other listed triple satisfy all strict triangle inequalities.

Thus the correct answer is **C**.

28. A quadrilateral that has consecutive sides of lengths $70, 90, 130,$ and 110 is inscribed in a circle and also has a circle inscribed in it. The point of tangency of the inscribed circle to the side of length 130 divides that side into segments of lengths x and y. Find $|x - y|$.

A 12

B 13

C 14

D 15

E 16

Solution:

Let the tangent lengths from the four consecutive vertices be u, v, w, z. Then

$$u + v = 70, \qquad v + w = 90,$$
$$w + z = 130, \qquad z + u = 110.$$

Thus $v = 70 - u, w = 20 + u$, and $z = 110 - u$. If the inradius is r, a vertex with angle A has tangent length $r\cot(\frac{A}{2})$. Opposite angles are supplementary, so $uw = vz$. Therefore

$$u(20 + u) = (70 - u)(110 - u),$$

giving $u = 38.5$. Hence the two segments of the 130-side are $w = 58.5$ and $z = 71.5$, whose difference is 13.

Thus the correct answer is **B**.

29. A subset of the integers $1, 2, \ldots, 100$ has the property that none of its members is 3 times another. What is the largest number of members such a subset can have?

A 50

B 66

C 67

D 76

E 78

Solution:

Partition the integers into chains $m, 3m, 9m, \ldots,$ with $3 \nmid m$. In each chain, no two adjacent terms may both be selected, so a maximum selection takes alternating terms starting with m. Equivalently, select the integers whose exponent of 3 is even. There are

$$\begin{aligned}&\left(100-\left\lfloor\frac{100}{3}\right\rfloor\right)\\&\quad+\left(\left\lfloor\frac{100}{9}\right\rfloor-\left\lfloor\frac{100}{27}\right\rfloor\right)\\&\quad+\left\lfloor\frac{100}{81}\right\rfloor\\&=67+8+1=76.\end{aligned}$$

The chain argument also proves no larger selection is possible.

Thus the correct answer is **D**.

30. If $R_n = \frac{1}{2}(a^n + b^n)$, where $a = 3 + 2\sqrt{2}$, $b = 3 - 2\sqrt{2}$, and $n = 0, 1, 2, \ldots$, then R_{12345} is an integer. Its units digit is

- A 1
- B 3
- C 5
- D 7
- E 9

Solution:

Because a, b are roots of $t^2 - 6t + 1 = 0$,

$$R_n = 6R_{n-1} - R_{n-2}.$$

Starting with $R_0 = 1,\ R_1 = 3$, the units digits are

$$1, 3, 7, 9, 7, 3, 1, \ldots,$$

with period 6. Since $12345 \equiv 3 \pmod{6}$, the units digit is the same as that of R_3, namely 9.

Thus the correct answer is **E**.

Problems: https://live.poshenloh.com/past-contests/amc12/1990