1989 AMC 12 Solutions

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1. $(-1)^{5^2} + 1^{2^5} =$

- A -7
- B -2
- C 0
- D 1
- E 57

Solution:

Because $5^2 = 25$ is odd, $(-1)^{5^2} = -1$. Also $1^{2^5} = 1$, so the sum is 0.

Thus the correct answer is **C**.

2. $\sqrt{\frac{1}{9}+\frac{1}{16}}=$

A $\frac{1}{5}$

B $\frac{1}{4}$

C $\frac{2}{7}$

D $\frac{5}{12}$

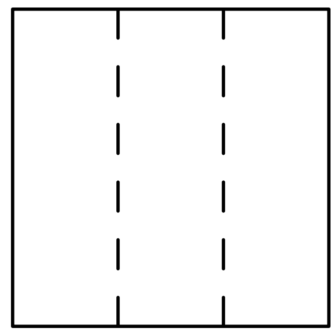
E $\frac{7}{12}$

Solution:

We have $\frac{1}{9}+\frac{1}{16}=\frac{16+9}{144}=\frac{25}{144}$. Its positive square root is $\frac{5}{12}$.

Thus the correct answer is **D**.

3. A square is cut into three rectangles along two lines parallel to a side, as shown. If the perimeter of each of the three rectangles is 24, then the area of the original square is



A 24

B 36

C 64

D 81

E 96

Solution:

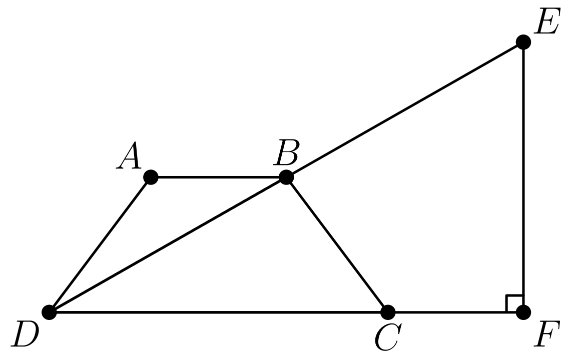
Let the square have side s. Since every rectangle has length s and the same perimeter, their three widths are equal and are each $\frac{s}{3}$. Thus

$$2\left(s+\frac{s}{3}\right)=24,$$

so $s=9$ and the square's area is 81.

Thus the correct answer is **D**.

4. In the figure, $ABCD$ is an isosceles trapezoid with side lengths $AD = BC = 5$, $AB = 4$, and $DC = 10$. The point C is on $\overline{DF}$ and B is the midpoint of hypotenuse $\overline{DE}$ in the right triangle DEF. Then $CF =$



A 3.25

B 3.5

C 3.75

D 4.0

E 4.25

Solution:

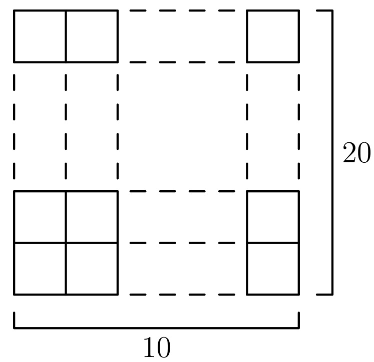
Drop perpendiculars AG and BH to DF. In the isosceles trapezoid, $GH = AB = 4$ and

$$DG = HC = \frac{DC - AB}{2} = 3.$$

Since B is the midpoint of DE and $BH \parallel EF$, the midpoint theorem makes H the midpoint of DF. Hence $DH = DG + GH = 7$, so $DF = 14$ and $CF = 14 - 10 = 4$.

Thus the correct answer is **D**.

5. Toothpicks of equal length are used to build a rectangular grid as shown. If the grid is 20 toothpicks high and 10 toothpicks wide, then the number of toothpicks used is



A 30

B 200

C 410

D 420

E 430

Solution:

There are 11 vertical grid lines with 20 toothpicks each, and 21 horizontal grid lines with 10 toothpicks each. The total is

$$11 \cdot 20 + 21 \cdot 10 = 430.$$

Thus the correct answer is **E**.

6. If $a, b > 0$ and the triangle in the first quadrant bounded by the coordinate axes and the graph of $ax + by = 6$ has area 6, then $ab =$

A 3

B 6

C 12

D 108

E 432

Solution:

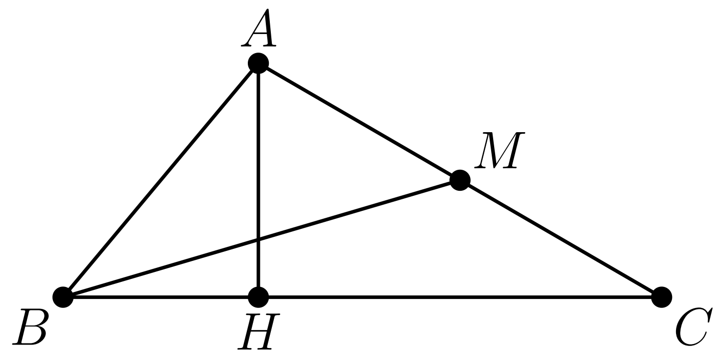
The intercepts are $\frac{6}{a}$ and $\frac{6}{b}$. Therefore the triangle's area is

$$\frac{1}{2} \cdot \frac{6}{a} \cdot \frac{6}{b} = \frac{18}{ab} = 6,$$

which gives $ab = 3$.

Thus the correct answer is **A**.

7. In $\triangle ABC, \angle A = 100°, \angle B = 50°, \angle C = 30°$, AH is an altitude, and BM is a median. Then $\angle MHC =$



A $15°$

B $22.5°$

C $30°$

D $40°$

E $45°$

Solution:

Because M is the midpoint of AC, the horizontal change from H to M is $\frac{HC}{2}$, while the vertical change is $\frac{AH}{2}$. Thus

$$\begin{aligned} \tan \angle MHC &= \frac{\frac{AH}{2}}{\frac{HC}{2}} \\ &= \frac{AH}{HC} \\ &= \tan \angle ACH. \end{aligned}$$

Hence $\angle MHC = \angle C = 30°$.

Thus the correct answer is **C**.

8. For how many integers n between 1 and 100 does $x^2 + x - n$ factor into the product of two linear factors with integer coefficients?

- A 0
- B 1
- C 2
- D 9
- E 10

Solution:

An integer factorization must have the form $(x - k)(x + k + 1)$ for some positive integer k. Expanding gives $x^2 + x - k(k + 1)$, so $n = k(k + 1)$. The values $k = 1, 2, \ldots, 9$ give $n \leq 100$, while $10 \cdot 11 > 100$. There are 9 values.

Thus the correct answer is **D**.

9. Mr. and Mrs. Zeta want to name their baby Zeta so that its monogram (first, middle, and last initials) will be in alphabetical order with no letters repeated. How many such monograms are possible?

- A 276
- B 300
- C 552
- D 600
- E 15600

Solution:

The first two initials must be two distinct letters chosen from A through Y. Once chosen, their order is forced. Therefore the number of monograms is

$$\binom{25}{2} = 300.$$

Thus the correct answer is **B**.

10. Consider the sequence defined recursively by $u_1 = a$ (any positive number), and $u_{n+1} = -\frac{1}{u_n+1}, n = 1, 2, 3, \ldots$. For which of the following values of n must $u_n = a$?

A 14

B 15

C 16

D 17

E 18

Solution:

Direct substitution gives

$$\begin{aligned} u_2 &= -\frac{1}{a+1}, \\ u_3 &= -\frac{a+1}{a}, \\ u_4 &= a. \end{aligned}$$

The sequence therefore repeats every 3 terms, so $u_n = a$ whenever $n \equiv 1 \pmod 3$. Among the choices, only 16 has that form.

Thus the correct answer is **C**.

11. Let a, b, c and d be integers with $a < 2b, b < 3c,$ and $c < 4d.$ If $d < 100,$ the largest possible value for a is

A 2367

B 2375

C 2391

D 2399

E 2400

Solution:

The largest possible values are $d = 99,\ c = 395,\ b = 1184,$ and finally $a = 2367.$ Each value satisfies its strict inequality, so this maximum is attainable.

Thus the correct answer is **A**.

12. The traffic on a certain east-west highway moves at a constant speed of 60 miles per hour in both directions. An eastbound driver passes 20 westbound vehicles in a five-minute interval. Assume vehicles in the westbound lane are equally spaced. Which of the following is closest to the number of westbound vehicles present in a 100-mile section of highway?

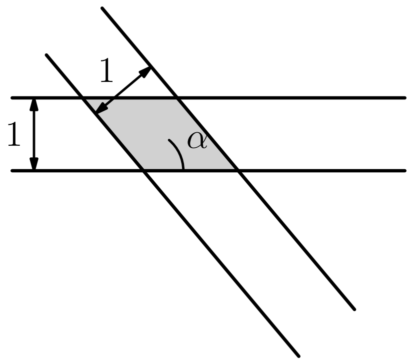
- A 100
- B 120
- C 200
- D 240
- E 400

Solution:

The relative speed is 120 miles per hour, so in five minutes the driver covers 10 relative miles. Passing 20 equally spaced vehicles in that distance means the spacing is about $\frac{10}{20} = \frac{1}{2}$ mile. Thus a 100-mile section contains about $\frac{100}{\frac{1}{2}} = 200$ vehicles.

Thus the correct answer is **C**.

13. Two strips of width 1 overlap at an angle of α as shown. The area of the overlap (shown shaded) is



A $\sin \alpha$

B $\frac{1}{\sin \alpha}$

C $\frac{1}{1-\cos \alpha}$

D $\frac{1}{\sin^2 \alpha}$

E $\frac{1}{(1-\cos \alpha)^2}$

Solution:

The overlap is a parallelogram. Taking a slanted side as its base, the perpendicular height is the width 1 of the slanted strip. If that base has length L, its vertical component is the width of the horizontal strip, so $L \sin \alpha = 1$. Hence $L = \frac{1}{\sin \alpha}$, and the area is $L \cdot 1 = \frac{1}{\sin \alpha}$.

Thus the correct answer is **B**.

14. $\cot 10 + \tan 5 =$

A $\csc 5$

B $\csc 10$

C $\sec 5$

D $\sec 10$

E $\sin 15$

Solution:

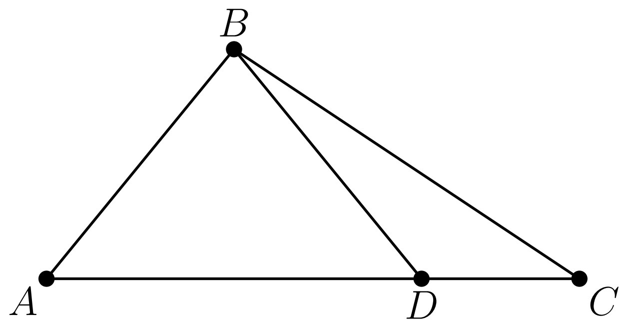
Using $\sin 10 = 2\sin 5 \cos 5$,

$$\cot 10 + \tan 5 = \frac{\cos 10}{\sin 10} + \frac{\sin 5}{\cos 5}.$$

Combining the fractions gives numerator $\cos 10 + 2\sin^2 5$. Since $\cos 10 = \cos^2 5 - \sin^2 5$, this numerator is 1, and the expression is $\csc 10$.

Thus the correct answer is **B**.

15. In $\triangle ABC, AB = 5, BC = 7, AC = 9$ and D is on $\overline{AC}$ with $BD = 5$. Find the ratio $AD : DC$.



A $4 : 3$

B $7 : 5$

C $11 : 6$

D $13 : 5$

E $19 : 8$

Solution:

Let $AD = m$ and $DC = 9 - m$. Stewart's Theorem gives

$$7^2 m + 5^2(9 - m)$$
$$= 9\big(5^2 + m(9 - m)\big).$$

Simplifying yields $9m^2 - 57m = 0$, so $m = \frac{19}{3}$. Then $DC = \frac{8}{3}$, and $AD : DC = 19 : 8$.

Thus the correct answer is **E**.

16. A *lattice point* is a point in the plane with integer coordinates. How many lattice points are on the line segment whose endpoints are $(3, 17)$ and $(48, 281)$? (Include both endpoints of the segment in your count.)

A 2

B 4

C 6

D 16

E 46

Solution:

The coordinate differences are 45 and 264, whose greatest common divisor is 3. The number of lattice points, including both endpoints, is therefore $\gcd(45, 264) + 1 = 4$.

Thus the correct answer is **B**.

17. The perimeter of an equilateral triangle exceeds the perimeter of a square by 1989 cm. The length of each side of the triangle exceeds the length of each side of the square by d cm. The square has perimeter greater than 0. How many positive integers are **not** possible values for d?

A 0

B 9

C 221

D 663

E infinitely many

Solution:

If the square has side s, the triangle has side $s + d$. The perimeter condition gives

$$3(s + d) - 4s = 1989,$$

so $s = 3d - 1989$. The condition $s > 0$ is equivalent to $d > 663$. Thus the positive integers 1 through 663 are precisely the impossible values.

Thus the correct answer is **D**.

18. The set of all real numbers x for which

$$x+\sqrt{x^2+1}-\frac{1}{x+\sqrt{x^2+1}}$$

is a rational number is the set of all

A integers x

B rational x

C real x

D x for which $\sqrt{x^2+1}$ is rational

E x for which $x+\sqrt{x^2+1}$ is rational

Solution:

Rationalizing gives

$$\frac{1}{x+\sqrt{x^2+1}}=\sqrt{x^2+1}-x.$$

The entire expression is therefore $2x$. It is rational exactly when x is rational.

Thus the correct answer is **B**.

19. A triangle is inscribed in a circle. The vertices of the triangle divide the circle into three arcs of lengths $3, 4,$ and 5. What is the area of the triangle?

A 6

B $\frac{18}{\pi^2}$

C $\frac{9}{\pi^2}(\sqrt{3}-1)$

D $\frac{9}{\pi^2}(\sqrt{3}+1)$

E $\frac{9}{\pi^2}(\sqrt{3}+3)$

Solution:

The radius is $R = \frac{12}{2\pi} = \frac{6}{\pi}$. The arc lengths $3, 4, 5$ give central angles $\frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}$. Their sines sum to $1 + \frac{\sqrt{3}}{2} + \frac{1}{2} = \frac{3+\sqrt{3}}{2}$. Splitting the triangle at the center, its area is

$$\begin{aligned} K &= \frac{1}{2} \cdot \frac{36}{\pi^2} \cdot \frac{3+\sqrt{3}}{2} \\ &= \frac{9}{\pi^2}(\sqrt{3}+3). \end{aligned}$$

Thus the correct answer is **E**.

20. Let x be a real number selected uniformly at random between 100 and 200. If $\lfloor\sqrt{x}\rfloor = 12$, find the probability that $\lfloor\sqrt{100x}\rfloor = 120$. ($\lfloor v\rfloor$ means the greatest integer less than or equal to v.)

A $\frac{2}{25}$

B $\frac{241}{2500}$

C $\frac{1}{10}$

D $\frac{96}{625}$

E 1

Solution:

The given condition is $144 \leq x < 169$, an interval of length 25. The desired condition is

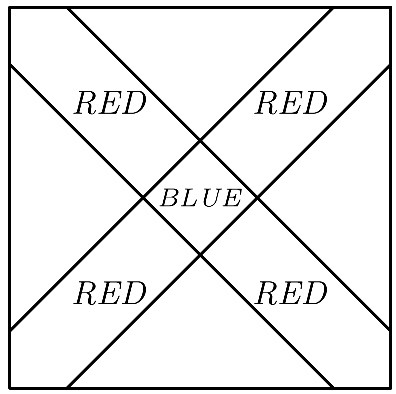
$$120^2 \leq 100x < 121^2,$$

or $144 \leq x < 146.41$. Its length is $2.41 = \frac{241}{100}$, so the conditional probability is

$$\frac{\frac{241}{100}}{25} = \frac{241}{2500}.$$

Thus the correct answer is **B**.

21. A square flag has a red cross of uniform width with a blue square in the center on a white background as shown. (The cross is symmetric with respect to each of the diagonals of the square.) If the entire cross (both the red arms and the blue center) takes up 36% of the area of the flag, what percent of the area of the flag is blue?



A 0.5

B 1

C 2

D 3

E 6

Solution:

Scale the flag to a unit square, and let h be the horizontal distance from its center to a vertex of the blue square. The four congruent white corner triangles have total area $(1-2h)^2$. Since the cross occupies 0.36,

$$(1-2h)^2 = 0.64.$$

With $0 < h < \frac{1}{2}$, this gives $h = 0.1$. The blue square has perpendicular diagonals $2h$ and $2h$, so its area is $\frac{1}{2}(2h)^2 = 2h^2 = 0.02$, or 2% of the flag.

Thus the correct answer is **C**.

22. A child has a set of 96 distinct blocks. Each block is one of 2 materials (*plastic, wood*), 3 sizes (*small, medium, large*), 4 colors (*blue, green, red, yellow*), and 4 shapes (*circle, hexagon, square, triangle*). How many blocks in the set are different from the "plastic medium red circle" in exactly two ways? (The "wood medium red square" is such a block.)

A 29

B 39

C 48

D 56

E 62

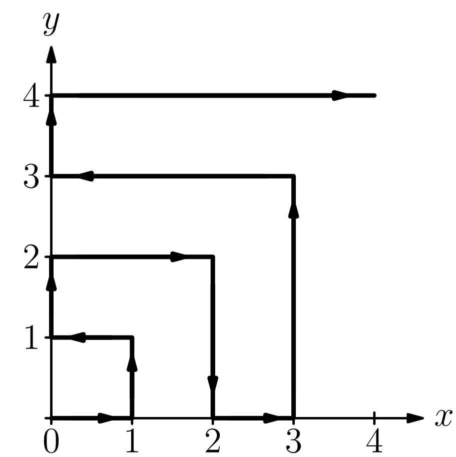
Solution:

The numbers of alternatives for material, size, color, and shape are $1, 2, 3, 3.$ Changing exactly two attributes gives

$$\begin{aligned}1 \cdot 2 + 1 \cdot 3 + 1 \cdot 3 \\ + 2 \cdot 3 + 2 \cdot 3 + 3 \cdot 3 = 29.\end{aligned}$$

Thus the correct answer is **A**.

23. A particle moves through the first quadrant as follows. During the first minute it moves from the origin to $(1,0)$. Thereafter, it continues to follow the directions indicated in the figure, going back and forth between the positive x and y axes, moving one unit of distance parallel to an axis in each minute. At which point will the particle be after exactly 1989 minutes?



A $(35,44)$

B $(36,45)$

C $(37,45)$

D $(44,35)$

E $(45,36)$

Solution:

At time n^2, the particle is at $(n,0)$ for odd n and at $(0,n)$ for even n. Thus at time $44^2 = 1936$ it is at $(0,44)$. It then moves 44 units right, reaching $(44,44)$ at time 1980, and moves downward for the next 9 minutes. At time 1989 it is therefore at $(44,35)$.

Thus the correct answer is **D**.

24. Five people are sitting at a round table. Let $f \geq 0$ be the number of people sitting next to at least one female and $m \geq 0$ be the number of people sitting next to at least one male. The number of possible ordered pairs (f, m) is

- A 7
- B 8
- C 9
- D 10
- E 11

Solution:

For $0, 1, 2$ females, the possible pairs are respectively

$$\begin{gathered}(0,5),\ (2,5),\\(4,5),\ (3,4),\end{gathered}$$

where the two-female cases distinguish adjacent from nonadjacent females. Swapping the sexes gives

$$\begin{gathered}(5,0),\ (5,2),\\(5,4),\ (4,3).\end{gathered}$$

These are 8 distinct ordered pairs.

Thus the correct answer is **B**.

25. In a certain cross-country meet between two teams of five runners each, a runner who finishes in the nth position contributes n to his team's score. The team with the lower score wins. If there are no ties among the runners, how many different winning scores are possible?

A 10

B 13

C 27

D 120

E 126

Solution:

The two scores sum to 55, so the winning score is at most 27. Its minimum is $1+2+3+4+5=15$. Every score from 15 through 20 is obtained by $\{1,2,3,4,k\}$ for $k=5,\ldots,10$. Scores 21 through 25 use $\{1,2,3,k,10\}$ for $k=5,\ldots,9$, and $26, 27$ use $\{1,2,4,9,10\}$ and $\{1,2,5,9,10\}$. Thus all 13 integers from 15 through 27 are possible.

Thus the correct answer is **B**.

26. A regular octahedron is formed by joining the centers of adjoining faces of a cube. The ratio of the volume of the octahedron to the volume of the cube is

A $\frac{\sqrt{3}}{12}$

B $\frac{\sqrt{6}}{16}$

C $\frac{1}{6}$

D $\frac{\sqrt{2}}{8}$

E $\frac{1}{4}$

Solution:

Take a cube of side 2, so its volume is 8. The face centers are $(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)$. In each octant, the octahedron cuts out a tetrahedron with three perpendicular unit edges and volume $\frac{1}{6}$. Thus its volume is $8(\frac{1}{6}) = \frac{4}{3}$, and the ratio is $\frac{\frac{4}{3}}{8} = \frac{1}{6}$.

Thus the correct answer is **C**.

27. Let n be a positive integer. If the equation $2x + 2y + z = n$ has 28 solutions in positive integers x, y and $z,$ then n must be either

A 14 or 15

B 15 or 16

C 16 or 17

D 17 or 18

E 18 or 19

Solution:

For $s = x + y \geq 2,$ there are $s - 1$ positive ordered pairs $(x, y),$ and $z = n - 2s$ is positive when $s \leq \lfloor \frac{n-1}{2} \rfloor$. Writing $m = \lfloor \frac{n-1}{2} \rfloor,$ the number of solutions is

$$\sum_{s=2}^{m} (s-1) = \frac{m(m-1)}{2}.$$

Setting this equal to 28 gives $m = 8$. Hence $\lfloor \frac{n-1}{2} \rfloor = 8,$ so $n = 17$ or $18.$

Thus the correct answer is **D**.

28. Find the sum of the roots of $\tan^2 x - 9\tan x + 1 = 0$ that are between $x = 0$ and $x = 2\pi$ radians.

A $\frac{\pi}{2}$

B π

C $\frac{3\pi}{2}$

D 3π

E 4π

Solution:

The two roots r, s of $t^2 - 9t + 1 = 0$ are positive and satisfy $rs = 1$. Therefore their acute arctangents add to $\frac{\pi}{2}$. Each tangent value occurs twice between 0 and 2π, with the second occurrence shifted by π. The sum of all four roots is

$$2\left(\frac{\pi}{2}\right) + 2\pi = 3\pi.$$

Thus the correct answer is **D**.

29. Find

$$\sum_{k=0}^{49}(-1)^k\binom{99}{2k},$$

where

$$\binom{n}{j}=\frac{n!}{j!(n-j)!}.$$

A -2^{50}

B -2^{49}

C 0

D 2^{49}

E 2^{50}

Solution:

The sum is the real part of

$$(1+i)^{99}=2^{\frac{99}{2}}e^{\frac{99\pi i}{4}}.$$

Since $99\equiv 3 \pmod 8$, its real part is

$$\begin{aligned}2^{\frac{99}{2}}\cos\frac{3\pi}{4}&=2^{\frac{99}{2}}\left(-\frac{\sqrt{2}}{2}\right)\\&=-2^{49}.\end{aligned}$$

Thus the correct answer is **B**.

30. Suppose that 7 boys and 13 girls line up in a row. Let S be the number of places in the row where a boy and a girl are standing next to each other. For example, for the row $GBBGGGBGBGGGBGBGGBGG$ we have $S = 12$. The average value of S (if all possible orders of these 20 people are considered) is closest to

- A 9
- B 10
- C 11
- D 12
- E 13

Solution:

For each of the 19 adjacent position pairs, the probability of mixed sexes is

$$\frac{7}{20} \cdot \frac{13}{19} + \frac{13}{20} \cdot \frac{7}{19}.$$

By linearity of expectation,

$$\begin{aligned} \mathbb{E}[S] &= 19 \cdot \frac{2 \cdot 7 \cdot 13}{20 \cdot 19} \\ &= \frac{91}{10} = 9.1, \end{aligned}$$

which is closest to 9.

Thus the correct answer is **A**.

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