1988 AMC 12 Solutions

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1. $\sqrt{8}+\sqrt{18}=$

A $\sqrt{26}$

B $2(\sqrt{2}+\sqrt{3})$

C 7

D $5\sqrt{2}$

E $2\sqrt{13}$

Solution:

We have $\sqrt{8}=2\sqrt{2}$ and $\sqrt{18}=3\sqrt{2}$. Their sum is $5\sqrt{2}$.

Thus the correct answer is **D**.

2. Triangles ABC and XYZ are similar, with A corresponding to X and B to Y. If $AB = 3, BC = 4,$ and $XY = 5,$ then YZ is

A $3\frac{3}{4}$

B 6

C $6\frac{1}{4}$

D $6\frac{2}{3}$

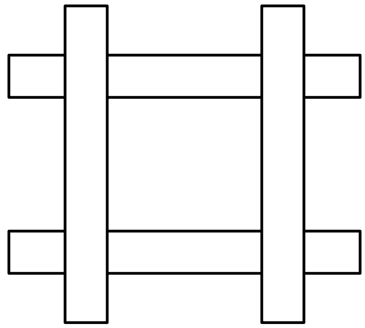
E 8

Solution:

The scale factor is $\frac{XY}{AB} = \frac{5}{3}$. Therefore $YZ = (\frac{5}{3})BC = (\frac{5}{3})(4)$ and $YZ = \frac{20}{3} = 6\frac{2}{3}$.

Thus the correct answer is **D**.

3. Four rectangular paper strips of length 10 and width 1 are put flat on a table and overlap perpendicularly as shown. How much area of the table is covered?



A 36

B 40

C 44

D 96

E 100

Solution:

The strips have total area $4(10) = 40$. There are four 1-by-1 overlaps, each counted twice, so the covered area is $40 - 4 = 36$.

Thus the correct answer is **A**.

4. The slope of the line $\frac{x}{3} + \frac{y}{2} = 1$ is

A $-\frac{3}{2}$

B $-\frac{2}{3}$

C $\frac{1}{3}$

D $\frac{2}{3}$

E $\frac{3}{2}$

Solution:

Multiplying by 2 after isolating $\frac{y}{2}$ gives $y = 2 - \frac{2}{3}x$. The slope is $-\frac{2}{3}$.

Thus the correct answer is **B**.

5. If b and c are constants and

$$(x+2)(x+b) = x^2 + cx + 6,$$

then c is

A -5

B -3

C -1

D 3

E 5

Solution:

Expansion gives $x^2 + (b+2)x + 2b$. Thus $2b = 6$, so $b = 3$, and $c = b + 2 = 5$.

Thus the correct answer is **E**.

6. A figure is an equiangular parallelogram if and only if it is a

- **A** rectangle
- **B** regular polygon
- **C** rhombus
- **D** square
- **E** trapezoid

Solution:

Four equal angles summing to 360° are all right angles. A parallelogram with four right angles is exactly a rectangle.

Thus the correct answer is **A**.

7. Estimate the time it takes to send 60 blocks of data over a communications channel if each block consists of 512 “chunks” and the channel can transmit 120 chunks per second.

- **A** 0.04 seconds
- **B** 0.4 seconds
- **C** 4 seconds
- **D** 4 minutes
- **E** 4 hours

Solution:

The transmission takes $\frac{60 \cdot 512}{120} = 256$ seconds, which is a little over 4 minutes.

Thus the correct answer is **D**.

8. If $\frac{b}{a} = 2$ and $\frac{c}{b} = 3,$ what is the ratio of $a + b$ to $b + c$?

A $\frac{1}{3}$

B $\frac{3}{8}$

C $\frac{3}{5}$

D $\frac{2}{3}$

E $\frac{3}{4}$

Solution:

We have $b = 2a$ and $c = 3b = 6a.$ Therefore $\frac{a+b}{b+c} = \frac{3a}{8a} = \frac{3}{8}.$

Thus the correct answer is **B**.

9. An $8' \times 10'$ table sits in the corner of a square room, as in Figure 1 below. The owners desire to move the table to the position shown in Figure 2. The side of the room is S feet. What is the smallest integer value of S for which the table can be moved as desired without tilting it or taking it apart?

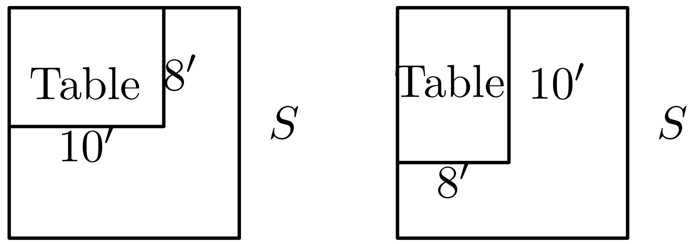


Figure 1 Figure 2

A 11

B 12

C 13

D 14

E 15

Solution:

The table diagonal is $\sqrt{8^2+10^2} = \sqrt{164}$, which is between 12 and 13. During the turn this diagonal must fit across the room, so $S \geq \sqrt{164}$. Conversely, a rectangle can rotate inside a circle whose diameter is its diagonal, so that bound is sufficient. The least integer S is 13.

Thus the correct answer is **C**.

10. In an experiment, a scientific constant C is determined to be 2.43865 with an error of at most ± 0.00312. The experimenter wishes to announce a value for C in which every digit is significant. That is, whatever C is, the announced value must be the correct result when C is rounded to that number of digits. The most accurate value the experimenter can announce for C is

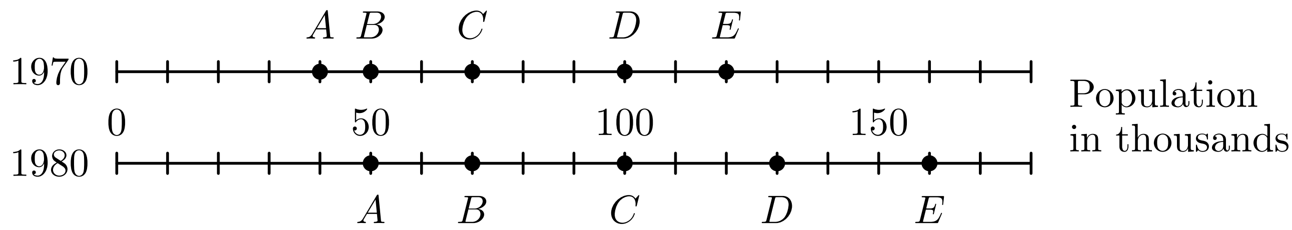
- A 2
- B 2.4
- C 2.43
- D 2.44
- E 2.439

Solution:

The possible interval is $[2.43553, 2.44177]$. Every number in it rounds to 2.44 at three significant digits, but the endpoints round differently at four significant digits. Hence 2.44 is the most accurate guaranteed announcement.

Thus the correct answer is **D**.

11. On each horizontal line in the figure below, the five large dots indicate the populations of cities A, B, C, D, and E in the year indicated. Which city had the greatest percentage increase in population from 1970 to 1980?



A A

B B

C C

D D

E E

Solution:

The percentage increases for A, B, C, D, E are respectively $\frac{10}{40}, \frac{20}{50}, \frac{30}{70}, \frac{30}{100}$, and $\frac{40}{120}$. These are $25\%, 40\%, 42\frac{6}{7}\%, 30\%$, and $33\frac{1}{3}\%$, so city C has the greatest percentage increase.

Thus the correct answer is **C**.

12. Each integer 1 through 9 is written on a separate slip of paper and all nine slips are put into a hat. Jack picks one of these slips at random and puts it back. Then Jill picks a slip at random. Which digit is most likely to be the units digit of the sum of Jack's integer and Jill's integer?

A 0

B 1

C 8

D 9

E each digit is equally likely

Solution:

For a units digit of 0, the ordered pairs are $(1,9), (2,8), \ldots, (9,1)$, giving 9 pairs. Each other units digit occurs 8 times among the 81 ordered pairs. Therefore 0 is most likely.

Thus the correct answer is **A**.

13. If $\sin x = 3\cos x$, then what is $\sin x \cos x$?

A $\frac{1}{6}$

B $\frac{1}{5}$

C $\frac{2}{9}$

D $\frac{1}{4}$

E $\frac{3}{10}$

Solution:

Squaring gives $\sin^2 x = 9\cos^2 x$. Hence $10\cos^2 x = 1$. Multiplying $\sin x = 3\cos x$ by $\cos x$ gives $\sin x \cos x = 3\cos^2 x = \frac{3}{10}$.

Thus the correct answer is **E**.

14. For any real number a and positive integer k, define

$$\binom{a}{k} = \frac{a(a-1)(a-2)\cdots \cdot (a-(k-1))}{k(k-1)(k-2)\cdots \cdot (2)(1)}.$$

What is

$$\binom{-\frac{1}{2}}{100} \div \binom{\frac{1}{2}}{100}\ ?$$

A -199

B -197

C -1

D 197

E 199

Solution:

After canceling $100!$, the numerator factors have magnitudes $1, 3, \ldots, 199$, while the denominator factors have magnitudes $1, 1, 3, \ldots, 197$. All common odd factors cancel, leaving magnitude 199. The numerator has 100 negative factors and the denominator has 99, so the quotient is -199.

Thus the correct answer is **A**.

15. If a and b are integers such that $x^2 - x - 1$ is a factor of $ax^3 + bx^2 + 1$, then b is

A -2

B -1

C 0

D 1

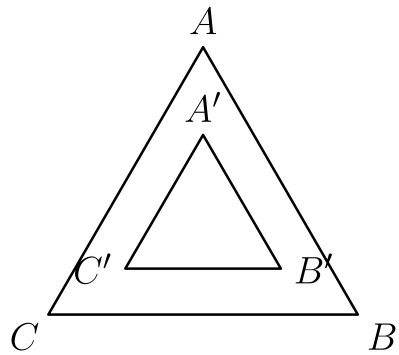
E 2

Solution:

Modulo $x^2 - x - 1$, we have $x^2 \equiv x + 1$ and $x^3 \equiv 2x + 1$. Thus the polynomial is congruent to $(2a + b)x + (a + b + 1)$. Both coefficients must vanish, so $2a + b = 0$ and $a + b = -1$. Hence $a = 1$ and $b = -2$.

Thus the correct answer is **A**.

16. ABC and $A'B'C'$ are equilateral triangles with parallel sides and the same center, as in the figure. The distance between side BC and side $B'C'$ is $\frac{1}{6}$ the altitude of $\triangle ABC$. The ratio of the area of $\triangle A'B'C'$ to the area of $\triangle ABC$ is



A $\frac{1}{36}$

B $\frac{1}{6}$

C $\frac{1}{4}$

D $\frac{\sqrt{3}}{4}$

E $\frac{9+8\sqrt{3}}{36}$

Solution:

Let the outer and inner altitudes be h and h'. Their common center is $\frac{h}{3}$ above BC and $\frac{h'}{3}$ above $B'C'$. Hence $\frac{h-h'}{3} = \frac{h}{6}$, so $h' = \frac{h}{2}$. Areas scale as the square of corresponding lengths, giving $(\frac{h'}{h})^2 = \frac{1}{4}$.

Thus the correct answer is **C**.

17. If $|x| + x + y = 10$ and $x + |y| - y = 12$, find $x + y$.

A -2

B 2

C $\frac{18}{5}$

D $\frac{22}{3}$

E 22

Solution:

If $x \leq 0$, the first equation gives $y = 10$, contradicting the second; hence $x > 0$. If $y \geq 0$, the second gives $x = 12$, contradicting the first; hence $y < 0$. The equations become $2x + y = 10$ and $x - 2y = 12$. Thus $x = \frac{32}{5},\ y = -\frac{14}{5}$, and $x + y = \frac{18}{5}$.

Thus the correct answer is **C**.

18. At the end of a professional bowling tournament, the top 5 bowlers have a play-off. First $\#5$ bowls $\#4.$ The loser receives 5th prize and the winner bowls $\#3$ in another game. The loser of this game receives 4th prize and the winner bowls $\#2.$ The loser of this game receives 3rd prize and the winner bowls $\#1.$ The winner of this game gets 1st prize and the loser gets 2nd prize. In how many orders can bowlers $\#1$ through $\#5$ receive the prizes?

A 10

B 16

C 24

D 120

E none of these

Solution:

Each of the four games has two possible winners. A sequence of four outcomes uniquely determines the five prize positions, so there are $2^4 = 16$ orders.

Thus the correct answer is **B**.

19. Simplify

$$\frac{\begin{array}{c} bx(a^2x^2 + 2a^2y^2 + b^2y^2) \\ + ay(a^2x^2 + 2b^2x^2 + b^2y^2) \end{array}}{bx + ay}.$$

A $a^2x^2 + b^2y^2$

B $(ax + by)^2$

C $(ax + by)(bx + ay)$

D $2(a^2x^2 + b^2y^2)$

E $(bx + ay)^2$

Solution:

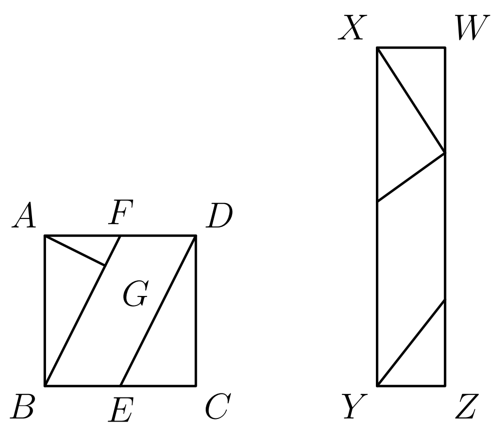
Regrouping the numerator gives

$$\begin{aligned} &(bx + ay)(a^2x^2 + b^2y^2) \\ &\quad + 2abxy(ay + bx) \\ &= (bx + ay)(ax + by)^2. \end{aligned}$$

Dividing by $bx + ay$ yields $(ax + by)^2$.

Thus the correct answer is **B**.

20. In one of the adjoining figures a square of side 2 is dissected into four pieces so that E and F are the midpoints of opposite sides and AG is perpendicular to BF. These four pieces can then be reassembled into a rectangle as shown in the second figure. The ratio of height to base, $\frac{XY}{YZ}$, in this rectangle is



A 4

B $1+2\sqrt{3}$

C $2\sqrt{5}$

D $\frac{8+4\sqrt{3}}{3}$

E 5

Solution:

Both BF and DE have horizontal and vertical changes 1 and 2, so each has length $\sqrt{5}$. Thus $XY = BF + DE = 2\sqrt{5}$. The rectangle has area 4, so $YZ = \frac{4}{2\sqrt{5}} = \frac{2}{\sqrt{5}}$. Therefore $\frac{XY}{YZ} = 5$.

Thus the correct answer is **E**.

21. The complex number z satisfies $z + |z| = 2 + 8i$. What is $|z|^2$? Note: if $z = a + bi$, then $|z| = \sqrt{a^2 + b^2}$.

A 68

B 100

C 169

D 208

E 289

Solution:

Writing $z = a + bi$ gives $b = 8$ and $a + \sqrt{a^2 + 64} = 2$. Thus $\sqrt{a^2 + 64} = 2 - a$. Squaring yields $a = -15$. Therefore $|z|^2 = a^2 + b^2 = 225 + 64 = 289$.

Thus the correct answer is **E**.

22. For how many integers x does a triangle with side lengths $10, 24,$ and x have all its angles acute?

- A 4
- B 5
- C 6
- D 7
- E more than 7

Solution:

Acuteness requires $x^2 < 10^2 + 24^2 = 676$ and $24^2 < 10^2 + x^2$, so $x < 26$ and $x^2 > 476$. For integer x, this gives $22 \leq x \leq 25$. All four values also satisfy the triangle inequality, so there are 4 values.

Thus the correct answer is **A**.

23. The six edges of tetrahedron $ABCD$ measure $7, 13, 18, 27, 36,$ and 41 units. If the length of edge AB is $41,$ then the length of edge CD is

A 7

B 13

C 18

D 27

E 36

Solution:

For a face containing the edge $7,$ its other two sides must differ by less than $7.$ Among the remaining lengths, the only disjoint pairs with this property are $(13, 18)$ and $(36, 41),$ so edge 27 is opposite edge $7.$ With $AB = 41,$ the two possible placements of these pairs can be checked by the triangle inequalities; the placement making the edge opposite AB equal to 18 fails, while the valid placement has $CD = 13.$

Thus the correct answer is **B**.

24. An isosceles trapezoid is circumscribed around a circle. The longer base of the trapezoid is 16, and one of the base angles is $\arcsin(0.8)$. Find the area of the trapezoid.

A 72

B 75

C 80

D 90

E not uniquely determined

Solution:

Let the shorter base be b and each leg be ℓ. Tangency gives $16 + b = 2\ell$. If the base angle is α, then $\sin\alpha = \frac{4}{5},\ \cos\alpha = \frac{3}{5}$, and $16 - b = 2\ell\cos\alpha = (\frac{6}{5})\ell$. Solving gives $b = 4,\ \ell = 10$, and height $\ell\sin\alpha = 8$. The area is $\frac{1}{2}(16+4)(8) = 80$.

Thus the correct answer is **C**.

25. X, Y, and Z are pairwise disjoint sets of people. The average ages of people in the sets $X, Y, Z, X \cup Y, X \cup Z$, and $Y \cup Z$ are given in the table below.

Set	X	Y	Z	$X \cup Y$	$X \cup Z$	$Y \cup Z$
Average age of people in the set	37	23	41	29	39.5	33

Find the average age of the people in the set $X \cup Y \cup Z$.

A 33

B 33.5

C $33.\overline{66}$

D $33.83\overline{3}$

E 34

Solution:

Let the set sizes be x, y, z. The three union averages give

$$37x + 23y = 29(x + y),$$
$$37x + 41z = 39.5(x + z),$$

and $23y + 41z = 33(y + z)$. Thus $y = \frac{4x}{3}$ and $z = \frac{5x}{3}$. The total average is

$$\frac{37x + 23(\dfrac{4x}{3}) + 41(\dfrac{5x}{3})}{x + \frac{4x}{3} + \frac{5x}{3}} = 34.$$

Thus the correct answer is **E**.

26. Suppose that p and q are positive numbers for which

$$\begin{aligned} \log_9(p) &= \log_{12}(q) \\ &= \log_{16}(p+q). \end{aligned}$$

What is the value of $\frac{q}{p}$?

A $\frac{4}{3}$

B $\frac{1}{2}(1+\sqrt{3})$

C $\frac{8}{5}$

D $\frac{1}{2}(1+\sqrt{5})$

E $\frac{16}{9}$

Solution:

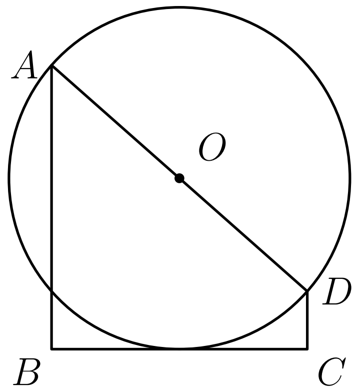
Let the common value be t. Then $p = 9^t,\ q = 12^t$, and $p+q = 16^t$. Put $r = \frac{q}{p} = (\frac{4}{3})^t$. Dividing the last equation by p gives

$$\begin{aligned} 1 + r &= (\frac{16}{9})^t \\ &= ((\frac{4}{3})^t)^2 = r^2. \end{aligned}$$

Since $r > 0, r = \frac{1+\sqrt{5}}{2}$.

Thus the correct answer is **D**.

27. In the figure, $AB \perp BC, BC \perp CD$, and BC is tangent to the circle with center O and diameter AD. In which one of the following cases is the area of $ABCD$ an integer?



A $AB = 3, CD = 1$

B $AB = 5, CD = 2$

C $AB = 7, CD = 3$

D $AB = 9, CD = 4$

E $AB = 11, CD = 5$

Solution:

The tangent point is the midpoint of BC, and the tangent-secant relation gives $(\frac{BC}{2})^2 = AB \cdot CD$. Thus $BC = 2\sqrt{AB \cdot CD}$, and the trapezoid area is

$$\begin{aligned} \text{Area} &= \frac{AB + CD}{2} BC \\ &= (AB + CD)\sqrt{AB \cdot CD}. \end{aligned}$$

Only $AB = 9,\ CD = 4$ makes the product under the radical a square; the area is $13 \cdot 6 = 78.$

Thus the correct answer is **D**.

28. An unfair coin has probability p of coming up heads on a single toss. Let w be the probability that, in 5 independent tosses of this coin, heads come up exactly 3 times. If $w = \frac{144}{625}$, then

A p must be $\frac{2}{5}$

B p must be $\frac{3}{5}$

C p must be greater than $\frac{3}{5}$

D p is not uniquely determined

E there is no value of p for which $w = \frac{144}{625}$

Solution:

Here $w(p) = 10p^3(1-p)^2$. At $p = \frac{2}{5}$,

$$w = 10\left(\frac{2}{5}\right)^3\left(\frac{3}{5}\right)^2 = \frac{144}{625}.$$

Also $w(\frac{3}{5}) = \frac{216}{625} > \frac{144}{625}$, while $w(1) = 0$. By continuity there is another solution between $\frac{3}{5}$ and 1. Hence p is not unique.

Thus the correct answer is **D**.

29. You plot weight (y) against height (x) for three of your friends and obtain the points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$. If

$$x_1 < x_2 < x_3,$$
$$x_3 - x_2 = x_2 - x_1,$$

which of the following is necessarily the slope of the line which best fits the data? "Best fits" means that the sum of the squares of the vertical distances from the data points to the line is smaller than for any other line.

A $\frac{y_3-y_1}{x_3-x_1}$

B $\frac{(y_2-y_1)-(y_3-y_2)}{x_3-x_1}$

C $\frac{2y_3-y_1-y_2}{2x_3-x_1-x_2}$

D $\frac{y_2-y_1}{x_2-x_1} + \frac{y_3-y_2}{x_3-x_2}$

E none of these

Solution:

Translate and scale so the x-coordinates are $-d, 0, d$. Their mean is 0, so the least-squares slope is

$$\frac{(-d)y_1 + 0y_2 + dy_3}{d^2 + 0 + d^2} = \frac{y_3 - y_1}{2d}$$
$$= \frac{y_3 - y_1}{x_3 - x_1}.$$

Thus the correct answer is **A**.

30. Let $f(x) = 4x - x^2$. Given x_0, consider the sequence defined by $x_n = f(x_{n-1})$ for all $n \geq 1$. For how many real numbers x_0 will the sequence $x_0, x_1, x_2, \ldots$ take on only a finite number of different values?

A 0

B 1 or 2

C $3, 4, 5$ or 6

D more than 6 but finitely many

E infinitely many

Solution:

The starting values $0, 4, 2$ give finite orbits 0, $4 \to 0$, and $2 \to 4 \to 0$. More generally, if a_n begins a finite chain ending at 0, solve

$$4a_{n+1} - a_{n+1}^2 = a_n.$$

Its real solutions are $a_{n+1} = 2 \pm \sqrt{4 - a_n}$. Choosing a preimage not already in the chain extends it by one new value. Repeating produces infinitely many distinct starting values with finite orbits.

Thus the correct answer is **E**.

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