1987 AMC 12 Solutions

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1. $(1+x^2)(1-x^3)$ equals

A $1-x^5$

B $1-x^6$

C $1+x^2-x^3$

D $1+x^2-x^3-x^5$

E $1+x^2-x^3-x^6$

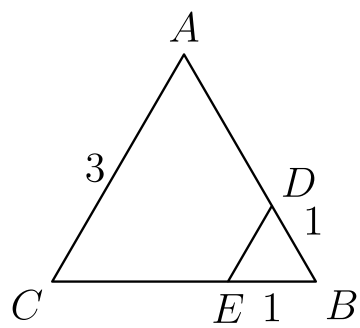
Solution:

Expanding gives

$$(1+x^2)(1-x^3) = 1+x^2-x^3 - x^5.$$

Thus the correct answer is **D**.

2. As shown in the figure, a triangular corner with side lengths $DB = EB = 1$ is cut from equilateral triangle ABC of side length 3. The perimeter of the remaining quadrilateral $ADEC$ is



A 6

B $6\frac{1}{2}$

C 7

D $7\frac{1}{2}$

E 8

Solution:

Since $\angle DBE = 60^\circ$ and $DB = EB = 1$, triangle DBE is equilateral, so $DE = 1$. The original perimeter is 9. Cutting off the corner removes two unit segments and adds one:

$$9 - 1 - 1 + 1 = 8.$$

Thus the correct answer is **E**.

3. How many primes less than 100 have 7 as the ones digit? (Assume the usual base 10 representation.)

- A 4
- B 5
- C 6
- D 7
- E 8

Solution:

The primes are

$$7,\ 17,\ 37,\ 47,\ 67,\ 97.$$

The other candidates are composite: $27, 57, 77,$ and 87. Hence there are 6 such primes.

Thus the correct answer is **C**.

4.

$$\frac{2^1+2^0+2^{-1}}{2^{-2}+2^{-3}+2^{-4}}$$

equals

- A 6
- B 8
- C $\frac{31}{2}$
- D 24
- E 512

Solution:

The numerator is

$$2+1+\frac{1}{2}=\frac{7}{2},$$

while the denominator is

$$\frac{1}{4}+\frac{1}{8}+\frac{1}{16}=\frac{7}{16}.$$

Their quotient is $\frac{\frac{7}{2}}{\frac{7}{16}}=8$.

Thus the correct answer is **B**.

5. A student recorded the exact percentage frequency distribution for a set of measurements, as shown to the right. However, the student neglected to indicate N, the total number of measurements. What is the smallest possible value of N?

measured value	percent frequency
0	12.5
1	0
2	50
3	25
4	12.5
	100

A 5

B 8

C 16

D 25

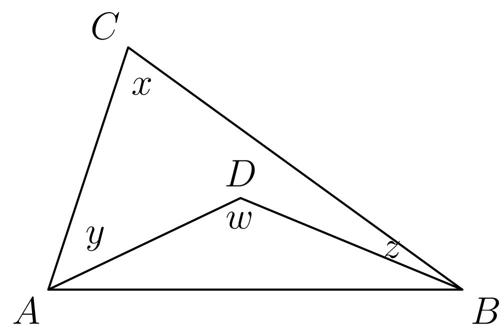
E 50

Solution:

Since $12.5\% = \frac{1}{8}$, the total N must be divisible by 8. Taking $N = 8$ gives frequencies $1, 0, 4, 2, 1,$ all whole numbers. Thus the smallest possible total is 8.

Therefore the correct answer is **B**.

6. In the $\triangle ABC$ shown, D is some interior point, and x, y, z, w are the measures of angles in degrees. Solve for x in terms of y, z, and w.



A $w - y - z$

B $w - 2y - 2z$

C $180 - w - y - z$

D $2w - y - z$

E $180 - w + y + z$

Solution:

Let the unlabelled angles of triangle ADB at A and B be α and β. Then

$$\alpha + \beta + w = 180^\circ.$$

The angle sum of triangle ABC gives

$$(\alpha + y) + (\beta + z) + x = 180^\circ.$$

Subtracting yields $x + y + z = w$, so $x = w - y - z$.

Thus the correct answer is **A**.

7. If $a-1=b+2=c-3=d+4$, which of the four quantities a, b, c, d is the largest?

A a

B b

C c

D d

E no one is always largest

Solution:

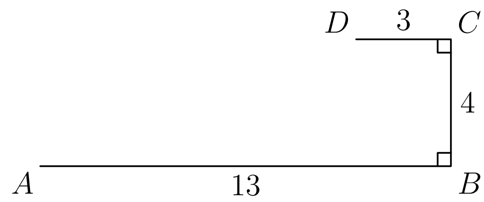
If the common value is k, then

$$\begin{aligned} a &= k+1, \quad & b &= k-2, \\ c &= k+3, \quad & d &= k-4. \end{aligned}$$

Therefore c is always the largest.

Thus the correct answer is **C**.

8. In the figure the sum of the distances AD and BD is



A between 10 and 11

B 12

C between 15 and 16

D between 16 and 17

E 17

Solution:

Triangle BCD is a 3-4-5 right triangle, so $BD = 5$. From A to D, the horizontal displacement is $13 - 3 = 10$ and the vertical displacement is 4, hence

$$AD = \sqrt{10^2 + 4^2} = \sqrt{116}.$$

Since $10 < \sqrt{116} < 11$, the sum $AD + BD$ is between 15 and 16.

Thus the correct answer is **C**.

9. The first four terms of an arithmetic sequence are $a, x, b, 2x$. The ratio of a to b is

A $\frac{1}{4}$

B $\frac{1}{3}$

C $\frac{1}{2}$

D $\frac{2}{3}$

E 2

Solution:

Let the common difference be d. Then $x = a + d,\ b = a + 2d$, and $2x = a + 3d$. Thus

$$2(a + d) = a + 3d,$$

so $a = d$ and $b = 3d$. Therefore $\frac{a}{b} = \frac{1}{3}$.

Thus the correct answer is **B**.

10. How many ordered triples (a, b, c) of nonzero real numbers have the property that each number is the product of the other two?

A 1

B 2

C 3

D 4

E 5

Solution:

Multiplying $a = bc,\ b = ca,\ c = ab$ gives

$$abc = (abc)^2.$$

Since none of the variables is zero, $abc = 1$. Also $abc = a^2 = b^2 = c^2$, so each variable is 1 or -1. Product 1 allows either no negative signs or exactly two, giving $1 + 3 = 4$ ordered triples.

Thus the correct answer is **D**.

11. Let c be a constant. The simultaneous equations

$$\begin{aligned} x - y &= 2, \\ cx + y &= 3. \end{aligned}$$

have a solution (x, y) inside Quadrant I if and only if

- A $c = -1$
- B $c > -1$
- C $c < \frac{3}{2}$
- D $0 < c < \frac{3}{2}$
- E $-1 < c < \frac{3}{2}$

Solution:

Solving the system gives

$$x = \frac{5}{c+1}, \qquad y = \frac{3-2c}{c+1}.$$

The condition $x > 0$ requires $c > -1$. With this positive denominator, $y > 0$ is equivalent to $c < \frac{3}{2}$. Hence $-1 < c < \frac{3}{2}$.

Thus the correct answer is **E**.

12. In an office, at various times during the day the boss gives the secretary a letter to type, each time putting the letter on top of the pile in the secretary's in-box. When there is time, the secretary takes the top letter off the pile and types it. If there are five letters in all, and the boss delivers them in the order $1, 2, 3, 4, 5$, which of the following could not be the order in which the secretary types them?

- A $1, 2, 3, 4, 5$
- B $2, 4, 3, 5, 1$
- C $3, 2, 4, 1, 5$
- D $4, 5, 2, 3, 1$
- E $5, 4, 3, 2, 1$

Solution:

To type 4 first in choice D, letters $1, 2, 3, 4$ must all have been delivered, leaving 3 on top after 4 is removed. Letter 5 can then be delivered and typed, but 3 is still above 2, so 2 cannot be typed next. Thus $4, 5, 2, 3, 1$ is impossible. Each other listed order can be produced by interleaving deliveries and typings.

Therefore the correct answer is **D**.

13. A long piece of paper 5 cm wide is made into a roll for cash registers by wrapping it 600 times around a cardboard tube of diameter 2 cm, forming a roll 10 cm in diameter. Approximate the length of the paper in meters. (Pretend the paper forms 600 concentric circles with diameters evenly spaced from 2 cm to 10 cm.)

A 36π

B 45π

C 60π

D 72π

E 90π

Solution:

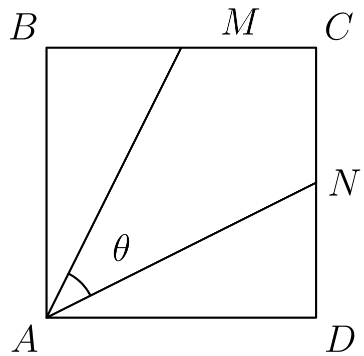
The average of the first and last diameters is

$$\frac{2+10}{2} = 6 \text{ cm.}$$

Hence the total length is approximately $600(6\pi) = 3600\pi$ cm, or 36π meters.

Thus the correct answer is **A**.

14. $ABCD$ is a square and M and N are the midpoints of BC and CD, respectively. Then $\sin\theta =$



A $\frac{\sqrt{5}}{5}$

B $\frac{3}{5}$

C $\frac{\sqrt{10}}{5}$

D $\frac{4}{5}$

E none of these

Solution:

Take the square to have side 2, with $A = (0, 0),\ M = (1, 2)$, and $N = (2, 1)$. Then

$$\begin{aligned}\sin\theta &= \frac{|\det((1,2),(2,1))|}{\sqrt{5}\sqrt{5}} \\ &= \frac{|1-4|}{5} = \frac{3}{5}.\end{aligned}$$

Thus the correct answer is **B**.

15. If (x, y) is a solution to the system

$$\begin{aligned} xy &= 6, \\ x^2y + xy^2 + x + y &= 63, \end{aligned}$$

find $x^2 + y^2$.

A 13

B $\frac{1173}{32}$

C 55

D 69

E 81

Solution:

The second equation becomes

$$\begin{aligned} xy(x+y) + (x+y) \\ &= (xy+1)(x+y) \\ &= 63. \end{aligned}$$

Since $xy = 6$, we get $x + y = 9$. Therefore

$$\begin{aligned} x^2 + y^2 &= (x+y)^2 - 2xy \\ &= 81 - 12 = 69. \end{aligned}$$

Thus the correct answer is **D**.

16. A cryptographer devises the following method for encoding positive integers. First, the integer is expressed in base 5. Second, a 1-to-1 correspondence is established between the digits that appear in the expressions in base 5 and the elements of the set $\{V, W, X, Y, Z\}$. Using this correspondence, the cryptographer finds that three consecutive integers in increasing order are coded as VYZ, VYX, VVW, respectively. What is the base-10 expression for the integer coded as XYZ?

A 48

B 71

C 82

D 108

E 113

Solution:

Because VYX is one more than VYZ, the digit for X is one more than the digit for Z. The next increment changes VYX to VVW, so $X = 4,\ W = 0$, and $Z = 3$. The carry changes Y to V, leaving $Y = 1,\ V = 2$. Thus

$$\begin{aligned} XYZ_5 &= 413_5, \\ 413_5 &= 4 \cdot 25 + 5 + 3 \\ &= 108. \end{aligned}$$

Thus the correct answer is **D**.

17. In a mathematics competition, the sum of the scores of Bill and Dick equalled the sum of the scores of Ann and Carol. If the scores of Bill and Carol had been interchanged, then the sum of the scores of Ann and Carol would have exceeded the sum of the scores of the other two. Also, Dick's score exceeded the sum of the scores of Bill and Carol. Determine the order in which the four contestants finished, from highest to lowest. Assume all scores were nonnegative.

- A Dick, Ann, Carol, Bill
- B Dick, Ann, Bill, Carol
- C Dick, Carol, Bill, Ann
- D Ann, Dick, Carol, Bill
- E Ann, Dick, Bill, Carol

Solution:

Let A, B, C, D be the scores of Ann, Bill, Carol, and Dick. The conditions are

$$\begin{aligned} A + C &= B + D, \\ A + B &> C + D, \\ D &> B + C. \end{aligned}$$

Adding the first two comparisons gives $A > D$. Subtracting the equality from the inequality gives $B > C$. Finally, $D > B + C \geq B$. Hence

$$A > D > B > C.$$

Thus the correct answer is **E**.

18. It takes A algebra books (all the same thickness) and H geometry books (all the same thickness, which is greater than that of an algebra book) to completely fill a certain shelf. Also, S of the algebra books and M of the geometry books would fill the same shelf. Finally, E of the algebra books alone would fill this shelf. Given that A, H, S, M, E are distinct positive integers, it follows that E is

A $\frac{AM+SH}{M+H}$

B $\frac{AM^2+SH^2}{M^2+H^2}$

C $\frac{AH-SM}{M-H}$

D $\frac{AM-SH}{M-H}$

E $\frac{AM^2-SH^2}{M^2-H^2}$

Solution:

Let the shelf length be 1, and let a, g be the algebra- and geometry-book thicknesses. Then

$$\begin{aligned} Aa + Hg &= 1, \\ Sa + Mg &= 1, \qquad Ea = 1. \end{aligned}$$

Multiplying the first equation by M, the second by H, and subtracting gives

$$(AM - SH)a = M - H.$$

Hence

$$E = \frac{1}{a} = \frac{AM - SH}{M - H}.$$

Thus the correct answer is **D**.

19. Which of the following is closest to $\sqrt{65} - \sqrt{63}$?

A 0.12

B 0.13

C 0.14

D 0.15

E 0.16

Solution:

Rationalizing gives

$$\sqrt{65} - \sqrt{63} = \frac{2}{\sqrt{65} + \sqrt{63}}.$$

The denominator is less than 16, because its square is $128 + 2\sqrt{4095} < 256$. Thus the value is greater than $\frac{2}{16} = 0.125$. Also each radical exceeds 7.5, so the value is less than $\frac{2}{15} < 0.134$. It is therefore closer to 0.13 than to any other choice.

Thus the correct answer is **B**.

20. Evaluate

$$\begin{aligned}&\log_{10}(\tan 1^\circ)\\&\quad+\log_{10}(\tan 2^\circ)\\&\quad+\log_{10}(\tan 3^\circ)+\cdots\\&\quad+\log_{10}(\tan 44^\circ)\\&\quad+\log_{10}(\tan 45^\circ)\\&\quad+\log_{10}(\tan 46^\circ)+\cdots\\&\quad+\log_{10}(\tan 87^\circ)\\&\quad+\log_{10}(\tan 88^\circ)\\&\quad+\log_{10}(\tan 89^\circ).\end{aligned}$$

A 0

B $\frac{1}{2}\log_{10}\left(\frac{\sqrt{3}}{2}\right)$

C $\frac{1}{2}\log_{10} 2$

D 1

E none of these

Solution:

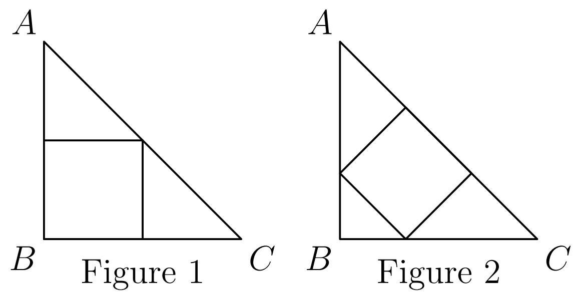
For each $k=1,\ldots,44$,

$$\tan k^\circ \tan(90^\circ-k^\circ)=1.$$

Thus each paired sum of logarithms is $\log_{10} 1=0$. The remaining middle term is $\log_{10}(\tan 45^\circ)=0$. Hence the whole sum is 0.

Thus the correct answer is **A**.

21. There are two natural ways to inscribe a square in a given isosceles right triangle. If it is done as in Figure 1 below, then one finds that the area of the square is $441\ \mathrm{cm}^2$. What is the area (in cm^2) of the square inscribed in the same $\triangle ABC$ as shown in Figure 2 below?



- A 378
- B 392
- C 400
- D 441
- E 484

Solution:

The first square has side 21, so the triangle's legs have length 42. Let s be the side of the tilted square. Its side on the hypotenuse is parallel to the opposite side joining the legs. That opposite side has endpoints $(0, k)$ and $(k, 0)$, so $s = k\sqrt{2}$. The distance between the parallel lines $x + y = k$ and $x + y = 42$ is also s, giving

$$s = \frac{42 - k}{\sqrt{2}}, \qquad k = \frac{s}{\sqrt{2}}.$$

Hence $s = 14\sqrt{2}$, and its area is $s^2 = 392$.

Thus the correct answer is **B**.

22. A ball was floating in a lake when the lake froze. The ball was removed (without breaking the ice), leaving a hole 24 cm across at the top and 8 cm deep. What was the radius of the ball (in centimeters)?

- A 8
- B 12
- C 13
- D $8\sqrt{3}$
- E $6\sqrt{6}$

Solution:

In a vertical cross-section, half the 24-cm hole is a chord segment of length $12.$ If the sphere radius is $r,$ the distance from its center to the ice plane is $r-8.$ The resulting right triangle gives

$$(r-8)^2+12^2=r^2.$$

Solving yields $16r=208,$ so $r=13.$

Thus the correct answer is **C**.

23. If p is a prime and both roots of $x^2 + px - 444p = 0$ are integers, then

- A $1 < p \leq 11$
- B $11 < p \leq 21$
- C $21 < p \leq 31$
- D $31 < p \leq 41$
- E $41 < p \leq 51$

Solution:

An integer root satisfies

$$x^2 = p(444 - x).$$

Thus $p \mid x$, so write $x = np$. Substitution gives

$$n(n+1)p = 444 = 2^2 \cdot 3 \cdot 37.$$

The consecutive product condition works only with $p = 37$, for which $n = 3$ or -4. The roots are 111 and -148, so $31 < p \leq 41$.

Thus the correct answer is **D**.

24. How many polynomial functions f of degree ≥ 1 satisfy

$$f(x^2) = [f(x)]^2 = f(f(x))?$$

- A 0
- B 1
- C 2
- D finitely many but more than 2
- E infinitely many

Solution:

Let f have degree $n \geq 1$ and leading coefficient a. The three expressions have degrees $2n, 2n, n^2$, so $n^2 = 2n$ and therefore $n = 2$. Comparing leading coefficients in $f(x^2) = [f(x)]^2$ gives $a = a^2$, hence $a = 1$.

Write $f(x) = x^2 + bx + c$. Then

$$x^4 + bx^2 + c = (x^2 + bx + c)^2.$$

The cubic coefficient forces $b = 0$, and then the quadratic coefficient forces $c = 0$. Thus the only candidate is $f(x) = x^2$, which indeed satisfies all three expressions. There is exactly one function.

Thus the correct answer is **B**.

25. ABC is a triangle: $A = (0, 0), B = (36, 15)$, and both the coordinates of C are integers. What is the minimum area $\triangle ABC$ can have?

- A $\frac{1}{2}$
- B 1
- C $\frac{3}{2}$
- D $\frac{13}{2}$
- E there is no minimum

Solution:

For $C = (u, v)$, the area is

$$\frac{1}{2}|36v - 15u|.$$

Since $\gcd(36, 15) = 3$, the smallest possible positive value of the determinant is at least 3. It is attained, for example, by $u = 7,\ v = 3$, since $36(3) - 15(7) = 3$. Therefore the minimum area is $\frac{3}{2}$.

Thus the correct answer is **C**.

26. The amount 2.5 is split into two nonnegative real numbers uniformly at random, for instance, into 2.143 and 0.357, or into $\sqrt{3}$ and $2.5-\sqrt{3}$. Then each number is rounded to its nearest integer, for instance, 2 and 0 in the first case above, 2 and 1 in the second. What is the probability that the two integers sum to 3?

A $\frac{1}{4}$

B $\frac{2}{5}$

C $\frac{1}{2}$

D $\frac{3}{5}$

E $\frac{3}{4}$

Solution:

Let the first part be $x \in [0, 2.5]$, so the second is $2.5-x$. Ignoring endpoints of probability zero, the rounded values sum to 3 exactly when

$$\frac{1}{2} < x < 1$$
$$\text{or}$$
$$\frac{3}{2} < x < 2.$$

These intervals have total length 1, out of a sample interval of length 2.5. The probability is therefore $\frac{1}{2.5} = \frac{2}{5}$.

Thus the correct answer is **B**.

27. A cube of cheese $C = \{(x, y, z) \mid 0 \leq x, y, z \leq 1\}$ is cut along the planes $x = y$, $y = z$, and $z = x$. How many pieces are there? (No cheese is moved until all three cuts are made.)

A 5

B 6

C 7

D 8

E 9

Solution:

The three planes are precisely the boundaries where two coordinates are equal. Away from them, each piece is determined by a strict ordering of x, y, z, such as $x < y < z$. Every one of the $3! = 6$ orderings occurs inside the cube, so there are 6 pieces.

Thus the correct answer is **B**.

28. Let a, b, c, d be real numbers. Suppose that all the roots of

$$z^4 + az^3 + bz^2 + cz + d = 0$$

are complex numbers lying on a circle in the complex plane centered at $0 + 0i$ and having radius 1. The sum of the reciprocals of the roots is necessarily

A a

B b

C c

D $-a$

E $-b$

Solution:

If $|r| = 1$, then $\frac{1}{r} = \overline{r}$. Therefore the sum of the reciprocals is the conjugate of the sum of the roots. By Vieta's formulas, the sum of the roots is $-a$, which is real. Its conjugate is still $-a$.

Thus the correct answer is **D**.

29. Consider the sequence of numbers defined recursively by $t_1 = 1$ and for $n > 1$ by $t_n = 1 + t_{\frac{n}{2}}$ when n is even and by $t_n = \frac{1}{t_{n-1}}$ when n is odd. Given that $t_n = \frac{19}{87}$, the sum of the digits of n is

A 15

B 17

C 19

D 21

E 23

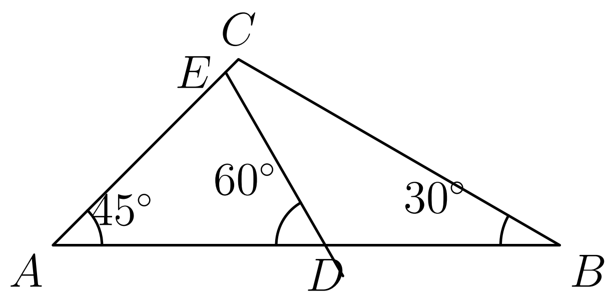
Solution:

Let $N(r)$ be the index at which the value r occurs. Reversing the recursion gives $N(r) = 2N(r-1)$ for $r > 1$, and $N(r) = N(\frac{1}{r}) + 1$ for $0 < r < 1$.

Starting with $N(1) = 1$, repeated use gives $N(2) = 2$, $N(\frac{1}{2}) = 3$, $N(\frac{3}{2}) = 6$, and $N(\frac{2}{3}) = 7$. Continuing gives $N(\frac{5}{3}) = 14$, $N(\frac{8}{3}) = 28$, $N(\frac{3}{8}) = 29$, and $N(\frac{11}{8}) = 58$. Next, $N(\frac{8}{11}) = 59$, $N(\frac{19}{11}) = 118$, $N(\frac{11}{19}) = 119$, $N(\frac{30}{19}) = 238$, $N(\frac{49}{19}) = 476$, and $N(\frac{68}{19}) = 952$. Finally, $N(\frac{87}{19}) = 1904$ and $N(\frac{19}{87}) = 1905$. Thus $n = 1905$, whose digit sum is $1 + 9 + 0 + 5 = 15$.

Therefore the correct answer is **A**.

30. In the figure, $\triangle ABC$ has $\angle A = 45^\circ$ and $\angle B = 30^\circ$. A line DE, with D on AB and $\angle ADE = 60^\circ$, divides $\triangle ABC$ into two pieces of equal area. (Note: the figure may not be accurate; perhaps E is on CB instead of AC.) The ratio $\frac{AD}{AB}$ is



A $\frac{1}{\sqrt{2}}$

B $\frac{2}{2+\sqrt{2}}$

C $\frac{1}{\sqrt{3}}$

D $\frac{1}{\sqrt[3]{6}}$

E $\frac{1}{\sqrt[4]{12}}$

Solution:

Set $AB = 1,\ A = (0,0)$, and $B = (1,0)$. Since $\angle A = 45^\circ$, side AC lies on $y = x$. The line through B making angle 30° with BA meets it at height

$$h_C = \frac{1}{1+\sqrt{3}}.$$

Let $d = AD$. The ray DE has slope $-\sqrt{3}$, so its intersection with $y = x$ has height

$$h_E = \frac{\sqrt{3}\,d}{1+\sqrt{3}}.$$

Equal areas require

$$\frac{1}{2} d h_E = \frac{1}{2}\left(\frac{1}{2} h_C\right).$$

Therefore

$$\frac{\sqrt{3}\,d^2}{2(1+\sqrt{3})} = \frac{1}{4(1+\sqrt{3})},$$
$$d^2 = \frac{1}{2\sqrt{3}} = \frac{1}{\sqrt{12}}.$$

Hence $\frac{AD}{AB} = d = \frac{1}{\sqrt[4]{12}}$.

Thus the correct answer is **E**.

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