1986 AMC 12 Solutions

Typeset by: LIVE by Po-Shen Loh
https://live.poshenloh.com/past-contests/amc12/1986/solutions

1. $[x-(y-x)]-[(x-y)-x]=$

A $2y$

B $2x$

C $-2y$

D $-2x$

E 0

Solution:

Expanding the two bracketed expressions gives

$$\begin{aligned}&[x-(y-x)]-[(x-y)-x]\\&=(2x-y)-(-y)=2x.\end{aligned}$$

Thus the correct answer is **B**.

2. If the line L in the xy-plane has half the slope and twice the y-intercept of the line $y=\frac{2}{3}x+4$, then an equation for L is

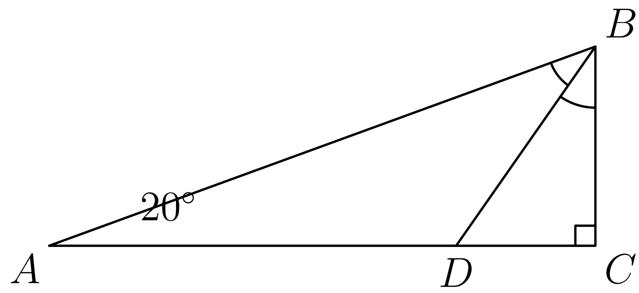
- A $y=\frac{1}{3}x+8$
- B $y=\frac{4}{3}x+2$
- C $y=\frac{1}{3}x+4$
- D $y=\frac{4}{3}x+4$
- E $y=\frac{1}{3}x+2$

Solution:

The original line has slope $\frac{2}{3}$ and y-intercept 4. Therefore L has slope $\frac{1}{3}$ and y-intercept 8, so its equation is $y=\frac{1}{3}x+8$.

Thus the correct answer is **A**.

3. In the figure, $\triangle ABC$ has a right angle at C and $\angle A = 20°$. If BD is the bisector of $\angle ABC$, then $\angle BDC =$



A $40°$

B $45°$

C $50°$

D $55°$

E $60°$

Solution:

Since $\angle C = 90°$ and $\angle A = 20°$, we have $\angle ABC = 70°$. The bisector gives $\angle DBC = 35°$. Therefore, in right triangle BDC,

$$\angle BDC = 90° - 35° = 55°.$$

Thus the correct answer is **D**.

4. Let S be the statement

"If the sum of the digits of the whole number n is divisible by $6,$ then n is divisible by $6.$"

A value of n which shows S to be false is

A 30

B 33

C 40

D 42

E none of these

Solution:

The digit sum of 33 is $6,$ which is divisible by $6.$ However, 33 is odd, so it is not divisible by $6.$ Thus 33 makes the implication false.

Therefore the correct answer is **B**.

5. Simplify $\left(\sqrt[6]{27}-\sqrt{6\frac{3}{4}}\right)^2$.

A $\frac{3}{4}$

B $\frac{\sqrt{3}}{2}$

C $\frac{3\sqrt{3}}{4}$

D $\frac{3}{2}$

E $\frac{3\sqrt{3}}{2}$

Solution:

We have $\sqrt[6]{27}=(3^3)^{\frac{1}{6}}=\sqrt{3}$ and

$$\sqrt{6\frac{3}{4}}=\sqrt{\frac{27}{4}}=\frac{3\sqrt{3}}{2}.$$

Hence the expression is

$$\begin{aligned}\left(\sqrt{3}-\frac{3\sqrt{3}}{2}\right)^2 &= \left(-\frac{\sqrt{3}}{2}\right)^2 \\ &= \frac{3}{4}.\end{aligned}$$

Thus the correct answer is **A**.

6. Using a table of a certain height, two identical blocks of wood are placed as shown in Figure 1. Length r is found to be 32 inches. After rearranging the blocks as in Figure 2, length s is found to be 28 inches. How high is the table?

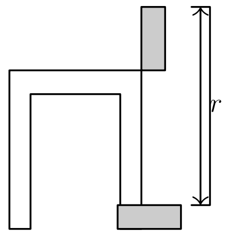


Figure 1

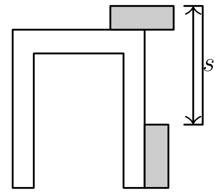


Figure 2

- A 28 inches
- B 29 inches
- C 30 inches
- D 31 inches
- E 32 inches

Solution:

Let h be the table height and let ℓ, w be the long and short block dimensions. Figure 1 gives $h + \ell - w = 32$, while Figure 2 gives $h + w - \ell = 28$. Adding cancels the block dimensions:

$$2h = 60,$$

so $h = 30$ inches.

Thus the correct answer is **C**.

7. The sum of the greatest integer less than or equal to x and the least integer greater than or equal to x is 5. The solution set for x is

- A $\frac{5}{2}$
- B $\{x \mid 2 \leq x \leq 3\}$
- C $\{x \mid 2 \leq x < 3\}$
- D $\{x \mid 2 < x \leq 3\}$
- E $\{x \mid 2 < x < 3\}$

Solution:

If x is an integer, then $\lfloor x \rfloor + \lceil x \rceil = 2x$, which cannot equal 5. If x is not an integer, then

$$\lfloor x \rfloor + \lceil x \rceil = 2\lfloor x \rfloor + 1.$$

This equals 5 exactly when $\lfloor x \rfloor = 2$, so $2 < x < 3$.

Thus the correct answer is **E**.

8. The population of the United States in 1980 was $226{,}504{,}825$. The area of the country is $3{,}615{,}122$ square miles. There are $(5280)^2$ square feet in one square mile. Which number below best approximates the average number of square feet per person?

A $5{,}000$

B $10{,}000$

C $50{,}000$

D $100{,}000$

E $500{,}000$

Solution:

The average is approximately

$$\frac{(3.615 \times 10^6)(5280)^2}{2.265 \times 10^8} \approx 4.45 \times 10^5$$

square feet per person. Of the choices, this is closest to $500{,}000$.

Thus the correct answer is **E**.

9. The product

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{9^2}\right)\left(1-\frac{1}{10^2}\right)$$

equals

- A $\frac{5}{12}$
- B $\frac{1}{2}$
- C $\frac{11}{20}$
- D $\frac{2}{3}$
- E $\frac{7}{10}$

Solution:

For each n,

$$1-\frac{1}{n^2}=\frac{n-1}{n}\cdot\frac{n+1}{n}.$$

Therefore the product telescopes:

$$\prod_{n=2}^{10}\left(1-\frac{1}{n^2}\right)=\left(\frac{1}{2}\right)\left(\frac{11}{10}\right)=\frac{11}{20}.$$

Thus the correct answer is **C**.

10. The 120 permutations of AHSME are arranged in dictionary order, as if each were an ordinary five-letter word. The last letter of the 86th word in this list is

A A

B H

C S

D M

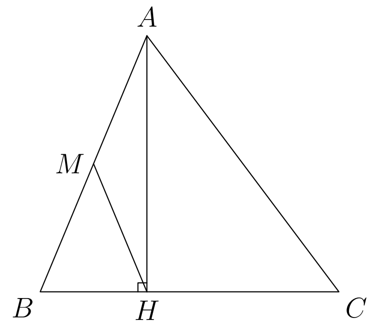
E E

Solution:

Each first letter occupies a block of $4! = 24$ words. Positions 73 through 96 begin with M, so the 86th word is the 14th word in that block. The first 6 begin with MA, the next 6 with ME, and the 13th through 18th begin with MH. The first two of those are $MHAES$ and $MHASE$. Thus the 86th word ends in E.

Therefore the correct answer is **E**.

11. In $\triangle ABC$, $AB = 13$, $BC = 14$ and $CA = 15$. Also, M is the midpoint of side AB and H is the foot of the altitude from A to BC. The length of HM is



- A 6
- B 6.5
- C 7
- D 7.5
- E 8

Solution:

Triangle AHB is right at H, and M is the midpoint of its hypotenuse AB. The midpoint of a right triangle's hypotenuse is equidistant from all three vertices, so

$$HM = \frac{AB}{2} = \frac{13}{2} = 6.5.$$

Thus the correct answer is **B**.

12. John scores 93 on this year's AHSME. Had the old scoring system still been in effect, he would score only 84 for the same answers. How many questions does he leave unanswered? (In the new scoring system that year, one received 5 points for each correct answer, 0 points for each wrong answer, and 2 points for each problem left unanswered. In the previous scoring system, one started with 30 points, received 4 more for each correct answer, lost 1 point for each wrong answer, and neither gained nor lost points for unanswered questions.)

A 6

B 9

C 11

D 14

E not uniquely determined

Solution:

Let c, w, u be the numbers correct, wrong and unanswered. The old score and total number of questions give

$$\begin{aligned} 30 + 4c - w &= 84, \\ c + w + u &= 30. \end{aligned}$$

Eliminating w yields $5c + u = 84$. The new score is $5c + 2u = 93$. Subtracting these equations gives $u = 9$.

Thus the correct answer is **B**.

13. A parabola $y = ax^2 + bx + c$ has vertex $(4, 2)$. If $(2, 0)$ is on the parabola, then abc equals

- A -12
- B -6
- C 0
- D 6
- E 12

Solution:

Write the equation as $y = a(x-4)^2 + 2$. Substituting $(2, 0)$ gives $0 = 4a + 2$, so $a = -\frac{1}{2}$. Expanding,

$$y = -\frac{1}{2}x^2 + 4x - 6.$$

Thus $b = 4, c = -6$, and $abc = (-\frac{1}{2})(4)(-6) = 12$.

Therefore the correct answer is **E**.

14. Suppose hops, skips and jumps are specific units of length. If b hops equals c skips, d jumps equals e hops, and f jumps equals g meters, then one meter equals how many skips?

A $\frac{bdg}{cef}$

B $\frac{cdf}{beg}$

C $\frac{cdg}{bef}$

D $\frac{cef}{bdg}$

E $\frac{ceg}{bdf}$

Solution:

From the three relations,

$$\begin{aligned} 1 \text{ meter} &= \frac{f}{g} \text{ jumps}, \\ 1 \text{ jump} &= \frac{e}{d} \text{ hops}, \\ 1 \text{ hop} &= \frac{c}{b} \text{ skips}. \end{aligned}$$

Multiplying the conversion factors gives $\frac{f}{g} \cdot \frac{e}{d} \cdot \frac{c}{b} = \frac{cef}{bdg}$ skips.

Thus the correct answer is **D**.

15. A student attempted to compute the average, A, of x, y and z by computing the average of x and y, and then computing the average of the result and z. Whenever $x < y < z$, the student's final result is

A correct

B always less than A

C always greater than A

D sometimes less than A and sometimes equal to A

E sometimes greater than A and sometimes equal to A

Solution:

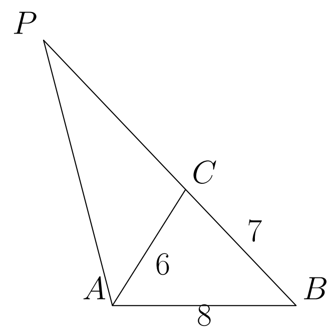
The student's result is $\frac{x+y+2z}{4}$, while the true average is $\frac{x+y+z}{3}$. Their difference is

$$\frac{x+y+2z}{4} - \frac{x+y+z}{3}$$
$$= \frac{2z-x-y}{12}.$$

Since $z > x$ and $z > y$, the numerator is positive. The student's result is therefore always greater than A.

Thus the correct answer is **C**.

16. In $\triangle ABC, AB = 8, BC = 7, CA = 6$ and side BC is extended, as shown in the figure, to a point P so that $\triangle PAB$ is similar to $\triangle PCA$. The length of PC is



A 7

B 8

C 9

D 10

E 11

Solution:

The stated order of similarity gives

$$\frac{PA}{PC} = \frac{PB}{PA} = \frac{AB}{CA} = \frac{4}{3}.$$

Let $PC = t$. Then $PA = \frac{4t}{3}$ and $PB = \frac{16t}{9}$. Since $PB = PC + CB = t + 7$,

$$\frac{16t}{9} = t + 7,$$

which gives $t = 9$.

Thus the correct answer is **C**.

17. A drawer in a darkened room contains 100 red socks, 80 green socks, 60 blue socks and 40 black socks. A youngster selects socks one at a time from the drawer but is unable to see the color of the socks drawn. What is the smallest number of socks that must be selected to guarantee that the selection contains at least 10 pairs? (A pair of socks is two socks of the same color. No sock may be counted in more than one pair.)

- A 21
- B 23
- C 24
- D 30
- E 50

Solution:

With 23 selected socks, the number of colors having an odd count must itself be odd, so it is at most 3. Thus at most 3 socks are unpaired, leaving at least 20 socks in 10 pairs. But 22 socks do not suffice: color counts $7, 5, 5, 5$ produce only $3 + 2 + 2 + 2 = 9$ pairs. Therefore the minimum is 23.

Thus the correct answer is **B**.

18. A plane intersects a right circular cylinder of radius 1 forming an ellipse. If the major axis of the ellipse is 50% longer than the minor axis, the length of the major axis is

A 1

B $\frac{3}{2}$

C 2

D $\frac{9}{4}$

E 3

Solution:

The minor axis of an elliptical plane section of a right circular cylinder is a diameter of the cylinder. Its length is therefore $2.$ The major axis is 50% longer, so its length is

$$2 + 0.50(2) = 3.$$

Thus the correct answer is **E**.

19. A park is in the shape of a regular hexagon 2 km on a side. Starting at a corner, Alice walks along the perimeter of the park for a distance of 5 km. How many kilometers is she from her starting point?

A $\sqrt{13}$

B $\sqrt{14}$

C $\sqrt{15}$

D $\sqrt{16}$

E $\sqrt{17}$

Solution:

Choose the first side in the horizontal direction. The three directed portions of the walk have vectors

$$(2,0), \qquad (1,\sqrt{3}),$$
$$\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right).$$

Their sum is $\left(\frac{5}{2}, \frac{3\sqrt{3}}{2}\right)$. Its squared length is

$$\left(\frac{5}{2}\right)^2 + \left(\frac{3\sqrt{3}}{2}\right)^2 = 13.$$

The distance is therefore $\sqrt{13}$ km.

Thus the correct answer is **A**.

20. Suppose x and y are inversely proportional and positive. If x increases by $p\%$, then y decreases by

A $p\%$

B $\frac{p}{1+p}\%$

C $\frac{100}{p}\%$

D $\frac{p}{100+p}\%$

E $\frac{100p}{100+p}\%$

Solution:

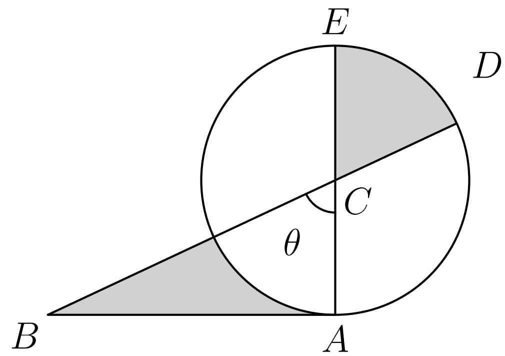
The new value of x is $\frac{x(100+p)}{100}$. Hence the new value of y is

$$y \cdot \frac{100}{100+p}.$$

The fractional decrease is $1 - \frac{100}{100+p} = \frac{p}{100+p}$. Expressed as a percentage, this is $\frac{100p}{100+p}\%$.

Thus the correct answer is **E**.

21. In the configuration below, θ is measured in radians, C is the center of the circle, BCD and ACE are line segments, and AB is tangent to the circle at A.



A necessary and sufficient condition for the equality of the two shaded areas, given $0 < \theta < \frac{\pi}{2}$, is

- A $\tan\theta = \theta$
- B $\tan\theta = 2\theta$
- C $\tan\theta = 4\theta$
- D $\tan 2\theta = \theta$
- E $\tan\frac{\theta}{2} = \theta$

Solution:

Let $r = AC$. The upper shaded sector has area $\frac{\theta r^2}{2}$. The lower shaded region is triangle ABC with an equal sector removed. Thus the two shaded regions are equal exactly when

$$2\left(\frac{\theta r^2}{2}\right) = \frac{1}{2}r \cdot AB.$$

This is equivalent to $\frac{AB}{r} = 2\theta$. Since AB is tangent at A, triangle ABC is right at A, and $\frac{AB}{r} = \tan\theta$. Therefore the condition is $\tan\theta = 2\theta$.

Thus the correct answer is **B**.

22. Six distinct integers are picked at random from $\{1, 2, 3, \ldots, 10\}$. What is the probability that, among those selected, the second smallest is 3?

A $\frac{1}{60}$

B $\frac{1}{6}$

C $\frac{1}{3}$

D $\frac{1}{2}$

E none of these

Solution:

There are $\binom{10}{6} = 210$ possible sets. If the second-smallest element is 3, then 3 is selected, one element is chosen from $\{1, 2\}$, and four are chosen from $\{4, 5, \ldots, 10\}$. This gives

$$2\binom{7}{4} = 70$$

favorable sets. The probability is $\frac{70}{210} = \frac{1}{3}$.

Thus the correct answer is **C**.

23. Let

$$N = 69^5 + 5 \cdot 69^4 + 10 \cdot 69^3 + 10 \cdot 69^2 + 5 \cdot 69 + 1.$$

How many positive integers are factors of N?

A 3

B 5

C 69

D 125

E 216

Solution:

By the binomial theorem,

$$N = (69+1)^5 = 70^5 = 2^5 \cdot 5^5 \cdot 7^5.$$

A divisor independently chooses an exponent from 0 through 5 for each of the three primes. Thus N has $6^3 = 216$ positive divisors.

Therefore the correct answer is **E**.

24. Let $p(x) = x^2 + bx + c$, where b and c are integers. If $p(x)$ is a factor of both

$$x^4 + 6x^2 + 25$$

and

$$3x^4 + 4x^2 + 28x + 5,$$

what is $p(1)$?

A 0

B 1

C 2

D 4

E 8

Solution:

The common factor $p(x)$ divides

$$\begin{aligned} &3(x^4 + 6x^2 + 25) \\ &\quad - (3x^4 + 4x^2 + 28x + 5) \\ &\quad = 14(x^2 - 2x + 5). \end{aligned}$$

Because $p(x)$ is monic with integer coefficients, Gauss's lemma implies that it divides $x^2 - 2x + 5$. The two polynomials are monic and have the same degree, so $p(x) = x^2 - 2x + 5$. Hence $p(1) = 1 - 2 + 5 = 4$.

Thus the correct answer is **D**.

25. If $\lfloor x \rfloor$ is the greatest integer less than or equal to x, then

$$\sum_{N=1}^{1024} \lfloor \log_2 N \rfloor =$$

A 8192

B 8204

C 9218

D $\lfloor \log_2(1024!) \rfloor$

E none of these

Solution:

For $0 \leq k \leq 9$, exactly 2^k integers N satisfy $2^k \leq N < 2^{k+1}$, and each contributes k. The final integer $1024 = 2^{10}$ contributes 10. Hence

$$\sum_{N=1}^{1024} \lfloor \log_2 N \rfloor = \sum_{k=0}^{9} k2^k + 10.$$

The finite geometric-sum identity gives $\sum_{k=0}^{9} k2^k = 8194$, so the requested sum is 8204.

Thus the correct answer is **B**.

26. It is desired to construct a right triangle in the coordinate plane so that its legs are parallel to the x and y axes and so that the medians to the midpoints of the legs lie on the lines $y = 3x + 1$ and $y = mx + 2$. The number of different constants m for which such a triangle exists is

A 0

B 1

C 2

D 3

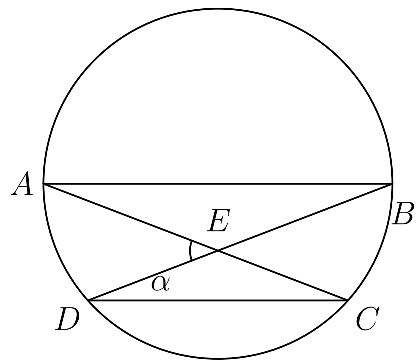
E more than 3

Solution:

Place the right-angle vertex at $(0, 0)$ and the other vertices at $(a, 0)$ and $(0, b)$. The medians to the legs have slopes $-\frac{2b}{a}$ and $-\frac{b}{2a}$, whose ratio is 4. Therefore, if one median has slope 3, the other can have slope 12 or $\frac{3}{4}$. Both occur: choose a triangle with the required pair of slopes and translate its centroid to the intersection of the two specified lines. Thus there are two possible values of m.

Therefore the correct answer is **C**.

27. In the adjoining figure, AB is a diameter of the circle, CD is a chord parallel to AB, and AC intersects BD at E, with $\angle AED = \alpha$. The ratio of the area of $\triangle CDE$ to that of $\triangle ABE$ is



A $\cos\alpha$

B $\sin\alpha$

C $\cos^2\alpha$

D $\sin^2\alpha$

E $1-\sin\alpha$

Solution:

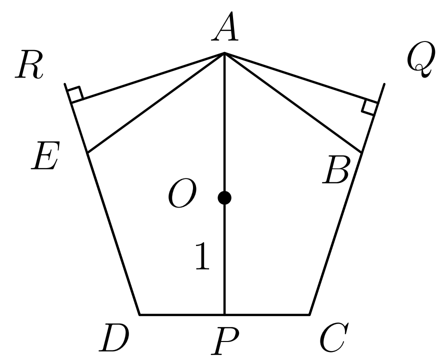
The triangles use the same angle at E, so

$$\frac{[CDE]}{[ABE]} = \frac{CE \cdot DE}{AE \cdot BE}.$$

Intersecting chords give $AE \cdot CE = BE \cdot DE$, so this ratio becomes $(\frac{DE}{AE})^2$. Draw AD. Since AB is a diameter, $\angle ADB = 90^\circ$, and D, E, B are collinear. Thus triangle ADE is right at D, and $\frac{DE}{AE} = \cos\alpha$. The required ratio is $\cos^2\alpha$.

Thus the correct answer is **C**.

28. $ABCDE$ is a regular pentagon. AP, AQ and AR are the perpendiculars dropped from A onto CD, CB extended and DE extended, respectively. Let O be the center of the pentagon. If $OP = 1$, then $AO + AQ + AR$ equals



A 3

B $1 + \sqrt{5}$

C 4

D $2 + \sqrt{5}$

E 5

Solution:

Let the side length be s. Since the apothem $OP = 1$, the five central triangles give pentagon area $\frac{5s}{2}$. The same pentagon is the union of triangles ABC, ACD and ADE, whose respective altitudes to side-length bases are AQ, AP, AR. Hence

$$\frac{s}{2}(AQ + AP + AR) = \frac{5s}{2},$$

so $AQ + AP + AR = 5$. Also $AP = AO + OP = AO + 1$. Therefore $AO + AQ + AR = 4$.

Thus the correct answer is **C**.

29. Two of the altitudes of the scalene triangle ABC have length 4 and 12. If the length of the third altitude is also an integer, what is the biggest it can be?

- A 4
- B 5
- C 6
- D 7
- E none of these

Solution:

Let the third altitude be h. Since each side equals twice the common area divided by its altitude, the side lengths are proportional to

$$\frac{1}{4}, \qquad \frac{1}{12}, \qquad \frac{1}{h}.$$

The two nontrivial triangle inequalities give

$$\begin{aligned} \frac{1}{h} &< \frac{1}{4} + \frac{1}{12} = \frac{1}{3}, \\ \frac{1}{h} &> \frac{1}{4} - \frac{1}{12} = \frac{1}{6}. \end{aligned}$$

Thus $3 < h < 6$. The largest integral possibility is 5, and its three altitudes are distinct as required for a scalene triangle.

Therefore the correct answer is **B**.

30. The number of real solutions (x, y, z, w) of the simultaneous equations

$$\begin{aligned} 2y &= x + \frac{17}{x}, \\ 2z &= y + \frac{17}{y}, \\ 2w &= z + \frac{17}{z}, \\ 2x &= w + \frac{17}{w}. \end{aligned}$$

is

A 1

B 2

C 4

D 8

E 16

Solution:

Each expression $t + \frac{17}{t}$ has the same sign as t, so all four variables have the same sign. Suppose first that they are positive. By AM-GM, every variable is at least $\sqrt{17}$. For $t > \sqrt{17}$,

$$\frac{t + \frac{17}{t}}{2} < t.$$

If any variable exceeded $\sqrt{17}$, the equations would give the impossible strict cycle $x > y > z > w > x$. Hence the only positive solution is

$$x = y = z = w = \sqrt{17}.$$

Negating all four variables preserves the system, giving exactly one negative solution, with all variables equal to $-\sqrt{17}$. Thus there are two real solutions.

Therefore the correct answer is **B**.

Problems: https://live.poshenloh.com/past-contests/amc12/1986