1985 AMC 12 Solutions

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1. If $2x + 1 = 8$, then $4x + 1 =$

A 15

B 16

C 17

D 18

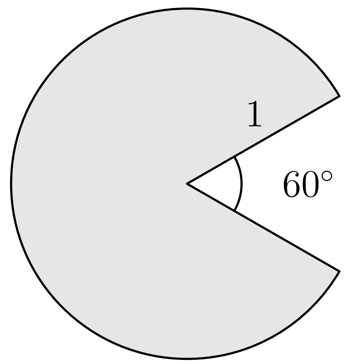
E 19

Solution:

The given equation gives $2x = 7$, so $x = \frac{7}{2}$. Therefore $4x + 1 = 4(\frac{7}{2}) + 1 = 15.$

Thus the correct answer is **A**.

2. In an arcade game, the “monster” is the shaded sector of a circle of radius 1 cm, as shown in the figure. The missing piece (the mouth) has central angle 60°. What is the perimeter of the monster in cm?



A $\pi + 2$

B 2π

C $\frac{5}{3}\pi$

D $\frac{5}{6}\pi + 2$

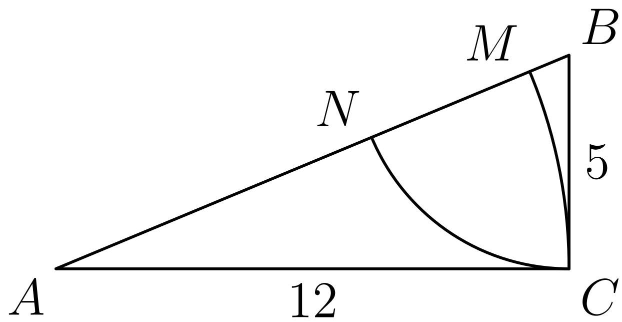
E $\frac{5}{3}\pi + 2$

Solution:

The curved part is $\frac{300^\circ}{360^\circ} = \frac{5}{6}$ of a unit circle, so its length is $\left(\frac{5}{6}\right)(2\pi) = \frac{5\pi}{3}$. The two straight sides of the mouth are radii of total length 2. Hence the perimeter is $\frac{5\pi}{3} + 2$.

Thus the correct answer is **E**.

3. In right $\triangle ABC$ with legs 5 and 12, arcs of circles are drawn, one with center A and radius 12, the other with center B and radius 5. They intersect the hypotenuse in M and N. Then MN has length



A 2

B $\frac{13}{5}$

C 3

D 4

E $\frac{24}{5}$

Solution:

The hypotenuse has length 13. Since $AM = 12$, point M is 12 units from A. Also $BN = 5$, so $AN = AB - BN = 13 - 5 = 8$. Therefore $MN = AM - AN = 12 - 8 = 4$.

Thus the correct answer is **D**.

4. A large bag of coins contains pennies, dimes and quarters. There are twice as many dimes as pennies and three times as many quarters as dimes. An amount of money which could be in the bag is

A $\$306$

B $\$333$

C $\$342$

D $\$348$

E $\$360$

Solution:

If there are p pennies, there are $2p$ dimes and $6p$ quarters. Their total value is $p + 10(2p) + 25(6p) = 171p$ cents. Of the listed dollar amounts, $34200 = 171 \cdot 200$ cents is possible.

Thus the correct answer is **C**.

5. Which terms must be removed from the sum

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{10}+\frac{1}{12}$$

if the sum of the remaining terms is to equal 1?

A $\frac{1}{4}$ and $\frac{1}{8}$

B $\frac{1}{4}$ and $\frac{1}{12}$

C $\frac{1}{8}$ and $\frac{1}{12}$

D $\frac{1}{6}$ and $\frac{1}{10}$

E $\frac{1}{8}$ and $\frac{1}{10}$

Solution:

The full sum is

$$\frac{\substack{60+30+20\\+15+12+10}}{120}=\frac{147}{120}.$$

The removed terms must therefore total $\frac{27}{120}=\frac{9}{40}$. Since $\frac{1}{8}+\frac{1}{10}=\frac{5}{40}+\frac{4}{40}=\frac{9}{40}$, those are the required terms.

Thus the correct answer is **E**.

6. One student in a class of boys and girls is to be chosen to represent the class. Each student is equally likely to be chosen and the probability that a boy is chosen is $\frac{2}{3}$ of the probability that a girl is chosen. The ratio of the number of boys to the total number of boys and girls is

- A $\frac{1}{3}$
- B $\frac{2}{5}$
- C $\frac{1}{2}$
- D $\frac{3}{5}$
- E $\frac{2}{3}$

Solution:

Because every student is equally likely to be selected, the number of boys is $\frac{2}{3}$ of the number of girls. Thus the boys-to-girls ratio is $2 : 3$, and boys make up $\frac{2}{2+3} = \frac{2}{5}$ of the class.

Thus the correct answer is **B**.

7. In some computer languages (such as APL), when there are no parentheses in an algebraic expression, the operations are grouped from right to left. Thus, $a \times b - c$ in such languages means the same as $a(b-c)$ in ordinary algebraic notation. If $a \div b - c + d$ is evaluated in such a language, the result in ordinary algebraic notation would be

A $\frac{a}{b} - c + d$

B $\frac{a}{b} - c - d$

C $\frac{d+c-b}{a}$

D $\frac{a}{b-c+d}$

E $\frac{a}{b-c-d}$

Solution:

Grouping from the right first gives $c+d$. Subtracting this result from b gives $b-(c+d) = b - c - d$. Finally, dividing a by that quantity gives $\frac{a}{b-c-d}$.

Thus the correct answer is **E**.

8. Let a, a', b, b' be real numbers with a and a' nonzero. The solution to $ax + b = 0$ is less than the solution to $a'x + b' = 0$ if and only if

- A $a'b < ab'$
- B $ab' < a'b$
- C $ab < a'b'$
- D $\frac{b}{a} < \frac{b'}{a'}$
- E $\frac{b'}{a'} < \frac{b}{a}$

Solution:

The two solutions are $-\frac{b}{a}$ and $-\frac{b'}{a'}$. Thus the given comparison is

$$-\frac{b}{a} < -\frac{b'}{a'}.$$

Multiplying both sides by -1 reverses the inequality and gives $\frac{b}{a} > \frac{b'}{a'}$, equivalently $\frac{b'}{a'} < \frac{b}{a}$. No multiplication by the unknown-sign quantity aa' is valid.

Thus the correct answer is **E**.

9. The odd positive integers, $1, 3, 5, 7, \ldots$, are arranged in five columns continuing with the pattern shown. Counting from the left, the column in which 1985 appears is the

$$\begin{array}{ccccc} & 1 & 3 & 5 & 7 \\ 15 & 13 & 11 & 9 & \\ & 17 & 19 & 21 & 23 \\ 31 & 29 & 27 & 25 & \\ & 33 & 35 & 37 & 39 \\ 47 & 45 & 43 & 41 & \\ & 49 & 51 & 53 & 55 \end{array}$$

- A first
- B second
- C third
- D fourth
- E fifth

Solution:

The last number in each right-to-left row is $16k - 1$, and the next row begins with $16k + 1$ in the second column. Since $1985 = 16 \cdot 124 + 1$, it begins such a row and lies in the second column.

Thus the correct answer is **B**.

10. An arbitrary circle can intersect the graph of $y = \sin x$ in

A at most 2 points

B at most 4 points

C at most 6 points

D at most 8 points

E more than 16 points

Solution:

Take a circle tangent to the x-axis at the origin with its center high above the axis. As its radius grows, its lower arc stays positive but arbitrarily close to the x-axis over an arbitrarily long interval. On each positive arch of $y = \sin x$, the sine curve is 0 at the endpoints and rises well above this circular arc in between, producing two crossings. Choosing a sufficiently large radius therefore produces more than 16 intersections.

Thus the correct answer is **E**.

11. How many distinguishable rearrangements of the letters in CONTEST have both the vowels first? (For instance, OETCNST is one such arrangement, but OTETSNC is not.)

A 60

B 120

C 240

D 720

E 2520

Solution:

The vowels O, E can be ordered first in $2!$ ways. The remaining letters C, N, T, S, T can be arranged in $\frac{5!}{2!} = 60$ distinguishable ways because the two T's agree. The total is $2 \cdot 60 = 120.$

Thus the correct answer is **B**.

12. Let p, q and r be distinct prime numbers, where 1 is not considered a prime. Which of the following is the smallest positive perfect cube having $n = pq^2r^4$ as a divisor?

A $p^8q^8r^8$

B $(pq^2r^2)^3$

C $(p^2q^2r^2)^3$

D $(pqr^2)^3$

E $4p^3q^3r^3$

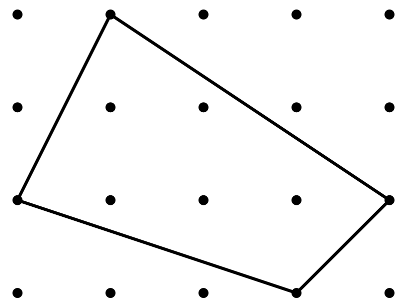
Solution:

A perfect cube has prime exponents divisible by 3. The least multiples of 3 that are at least $1, 2, 4$ are $3, 3, 6,$ respectively. Thus the smallest possible cube is

$$p^3q^3r^6 = (pqr^2)^3.$$

Therefore the correct answer is **D**.

13. Pegs are put in a board 1 unit apart both horizontally and vertically. A rubber band is stretched over 4 pegs as shown in the figure, forming a quadrilateral. Its area in square units is



A 4

B 4.5

C 5

D 5.5

E 6

Solution:

With the lower-left peg as $(0,0)$, the quadrilateral's vertices in order are $(1,3), (4,1), (3,0), (0,1)$. The shoelace formula gives

$$\frac{1}{2}\left|(1+0+3+0) - (12+3+0+1)\right| = 6.$$

Therefore the correct answer is **E**.

14. Exactly three of the interior angles of a convex polygon are obtuse. What is the maximum number of sides of such a polygon?

A 4

B 5

C 6

D 7

E 8

Solution:

For an n-gon, the three obtuse angles have total less than $3 \cdot 180^\circ$, while the other $n-3$ angles have total at most $(n-3)90^\circ$. Hence

$$(n-2)180^\circ < 3(180^\circ) + (n-3)90^\circ,$$

which simplifies to $n < 7$. Six sides are attainable, for example with interior angles $150^\circ, 150^\circ, 150^\circ, 90^\circ, 90^\circ, 90^\circ$. Thus the maximum is 6.

Therefore the correct answer is **C**.

15. If a and b are positive numbers such that $a^b = b^a$ and $b = 9a$, then the value of a is

A 9

B $\frac{1}{9}$

C $\sqrt[9]{9}$

D $\sqrt[3]{9}$

E $\sqrt[4]{3}$

Solution:

Substitution gives $a^{9a} = (9a)^a$. Taking the positive a-th root yields $a^9 = 9a$, so $a^8 = 9$. Therefore $a = 9^{\frac{1}{8}} = 3^{\frac{1}{4}} = \sqrt[4]{3}$.

Thus the correct answer is **E**.

16. If $A = 20°$ and $B = 25°$, then the value of $(1 + \tan A)(1 + \tan B)$ is

A $\sqrt{3}$

B 2

C $1 + \sqrt{2}$

D $2(\tan A + \tan B)$

E none of these

Solution:

Because $\tan(A + B) = \tan 45° = 1$,

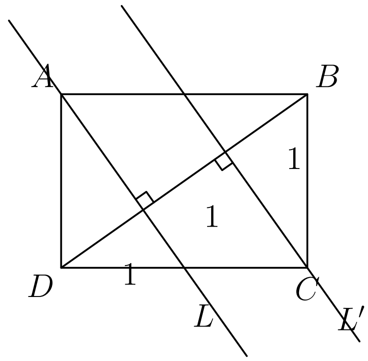
$$\frac{\tan A + \tan B}{1 - \tan A \tan B} = 1,$$

so $\tan A + \tan B = 1 - \tan A \tan B$. Therefore

$$\begin{aligned}&(1 + \tan A)(1 + \tan B)\\ &\quad = 1 + (1 - \tan A \tan B)\\ &\qquad + \tan A \tan B\\ &\quad = 2.\end{aligned}$$

Thus the correct answer is **B**.

17. Diagonal DB of rectangle $ABCD$ is divided into three segments of length 1 by parallel lines L and L' that pass through A and C and are perpendicular to DB. The area of $ABCD$, rounded to one decimal place, is



A 4.1

B 4.2

C 4.3

D 4.4

E 4.5

Solution:

Let $D = (0,0), B = (w,h), A = (0,h)$ and $C = (w,0)$. Since $DB = 3$, we have $w^2 + h^2 = 9$. The projection of DA onto DB has length $\frac{h^2}{3}$, and the first marked segment has length 1, so $h^2 = 3$. Similarly, the projection of DC reaches the second division point, so $\frac{w^2}{3} = 2$ and $w^2 = 6$. Hence the area is

$$wh = \sqrt{18} = 3\sqrt{2} \approx 4.2426,$$

which rounds to 4.2.

Therefore the correct answer is **B**.

18. Six bags of marbles contain $18, 19, 21, 23, 25$ and 34 marbles, respectively. One bag contains chipped marbles only. The other 5 bags contain no chipped marbles. Jane takes three of the bags and George takes two of the others. Only the bag of chipped marbles remains. If Jane gets twice as many marbles as George, how many chipped marbles are there?

A 18

B 19

C 21

D 23

E 25

Solution:

The six bags total $140.$ If the chipped bag contains c marbles, then the other bags total three times George's amount, so $140 - c$ is divisible by $3.$ Of the choices, only $c = 23$ has this property. It is attainable: George can take $18 + 21 = 39,$ while Jane takes $19 + 25 + 34 = 78.$

Thus the correct answer is **D**.

19. Consider the graphs of $y = Ax^2$ and $y^2 + 3 = x^2 + 4y$, where A is a positive constant and x and y are real variables. In how many points do the two graphs intersect?

A exactly 4

B exactly 2

C at least 1, but the number varies for different positive values of A

D 0 for at least one positive value of A

E none of these

Solution:

Substituting $x^2 = \frac{y}{A}$ into the second equation gives

$$Ay^2 - (4A+1)y + 3A = 0.$$

Its discriminant is $4A^2 + 8A + 1 > 0$. Its roots have positive product 3 and positive sum $\frac{4A+1}{A}$, so both roots are positive and distinct. For each root y, the equation $x^2 = \frac{y}{A}$ gives two distinct values of x. Thus there are exactly four intersection points.

Therefore the correct answer is **A**.

20. A wooden cube with edge length n units (where n is an integer > 2) is painted black all over. By slices parallel to its faces, the cube is cut into n^3 smaller cubes each of unit edge length. If the number of smaller cubes with just one face painted black is equal to the number of smaller cubes completely free of paint, what is n?

- A 5
- B 6
- C 7
- D 8
- E none of these

Solution:

There are $(n-2)^3$ completely unpainted cubes. Each of the six faces contributes $(n-2)^2$ cubes with exactly one painted face, so equality gives

$$6(n-2)^2 = (n-2)^3.$$

Since $n > 2$, division by $(n-2)^2$ gives $n - 2 = 6$, hence $n = 8$.

Therefore the correct answer is **D**.

21. How many integers x satisfy the equation

$$(x^2 - x - 1)^{x+2} = 1?$$

A 2

B 3

C 4

D 5

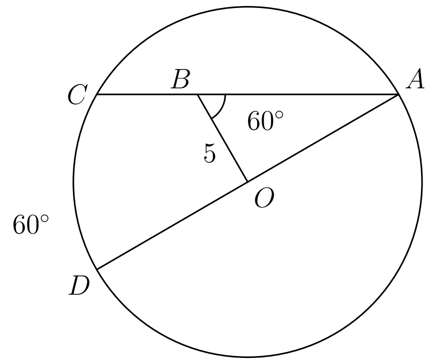
E none of these

Solution:

If the base is $1,$ then $x^2 - x - 1 = 1,$ giving $x = -1, 2.$ If the base is $-1,$ then $x(x - 1) = 0;$ only $x = 0$ makes the exponent $x + 2$ even. If the exponent is $0,$ then $x = -2,$ and its base is $5,$ so it also works. Thus the four solutions are $-2, -1, 0, 2.$

Therefore the correct answer is **C**.

22. In a circle with center O, AD is a diameter, ABC is a chord, $BO = 5$ and $\angle ABO = \overset{\frown}{CD} = 60^\circ$. Then the length of BC is



A 3

B $3 + \sqrt{3}$

C $5 - \frac{\sqrt{3}}{2}$

D 5

E none of the above

Solution:

Arc CD gives $\angle CAD = 30^\circ$. Since A, B, C are collinear and A, O, D are collinear, $\angle BAO = 30^\circ$. With $\angle ABO = 60^\circ$, triangle ABO is a 30°-60°-90° triangle. Because $BO = 5$, we get $AB = 10$ and $AO = 5\sqrt{3}$.

Also $\angle ACD = 90^\circ$ because AD is a diameter. In the 30°-60°-90° triangle ACD, the hypotenuse is $AD = 10\sqrt{3}$, so $AC = 15$. Therefore $BC = AC - AB = 5$.

Thus the correct answer is **D**.

23. If

$$x = \frac{-1+i\sqrt{3}}{2},$$
$$y = \frac{-1-i\sqrt{3}}{2},$$

where $i^2 = -1$, then which of the following is not correct?

A $x^5 + y^5 = -1$

B $x^7 + y^7 = -1$

C $x^9 + y^9 = -1$

D $x^{11} + y^{11} = -1$

E $x^{13} + y^{13} = -1$

Solution:

The numbers are the two nonreal cube roots of unity, so

$$x^n + y^n = 2\cos\left(\frac{2\pi n}{3}\right).$$

This equals -1 when $3 \nmid n$, but equals 2 when $3 \mid n$. Of the listed exponents, only 9 is divisible by 3, so its displayed equation is the one that is not correct.

Therefore the correct answer is **C**.

24. A non-zero digit is chosen in such a way that the probability of choosing digit d is $\log_{10}(d+1) - \log_{10} d$. The probability that the digit 2 is chosen is exactly $\frac{1}{2}$ the probability that the digit chosen is in the set

- A $\{2, 3\}$
- B $\{3, 4\}$
- C $\{4, 5, 6, 7, 8\}$
- D $\{5, 6, 7, 8, 9\}$
- E $\{4, 5, 6, 7, 8, 9\}$

Solution:

The probability of 2 is $\log_{10}(\frac{3}{2})$, so twice that probability is $\log_{10}(\frac{9}{4})$. For the digits 4 through 8, the sum telescopes:

$$\sum_{d=4}^{8} \big(\log_{10}(d+1) - \log_{10} d\big) = \log_{10}\frac{9}{4}.$$

Therefore that set has exactly twice the probability of digit 2.

Thus the correct answer is **C**.

25. The volume of a certain rectangular solid is $8\ \mathrm{cm}^3$, its total surface area is $32\ \mathrm{cm}^2$, and its three dimensions are in geometric progression. The sum of the lengths in cm of all the edges of this solid is

A 28

B 32

C 36

D 40

E 44

Solution:

Write the dimensions as $\frac{2}{r}, 2, 2r$, since their product is 8. Half the surface area is the sum of the three pairwise products, so

$$\frac{4}{r} + 4 + 4r = 16,$$

giving $r + \frac{1}{r} = 3$. The sum of the dimensions is $2(r + 1 + \frac{1}{r}) = 8$. Since each dimension occurs on four edges, the sum of all edge lengths is $4 \cdot 8 = 32$.

Thus the correct answer is **B**.

26. Find the least positive integer n for which $\frac{n-13}{5n+6}$ is a non-zero reducible fraction.

A 45

B 68

C 155

D 226

E none of these

Solution:

Euclid's algorithm gives

$$\begin{aligned}&\gcd(n-13, 5n+6)\\ &\quad = \gcd(n-13, 71).\end{aligned}$$

Since 71 is prime, the nonzero fraction is reducible exactly when $n-13$ is a nonzero multiple of 71. The least positive possibility is $n = 13 + 71 = 84$, which is not among the four numerical choices.

Thus the correct answer is **E**.

27. Consider a sequence $x_1, x_2, x_3, \ldots$, defined by

$$x_1 = \sqrt[3]{3},$$
$$x_2 = (\sqrt[3]{3})^{\sqrt[3]{3}}.$$

and in general

$$x_n = (x_{n-1})^{\sqrt[3]{3}} \qquad \text{for } n > 1.$$

What is the smallest value of n for which x_n is an integer?

A	2
B	3
C	4
D	9
E	27

Solution:

Let $c = \sqrt[3]{3}$. Repeated application of the recurrence gives

$$x_n = 3^{\frac{c^{n-1}}{3}},$$

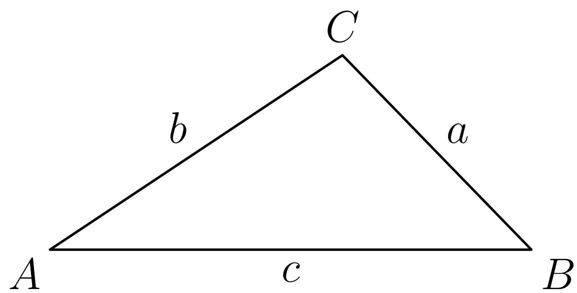
so $x_4 = 3^{\frac{c^3}{3}} = 3$. It remains to rule out the first three terms. The sequence is strictly increasing. Also $c < \frac{3}{2}$, so

$$x_2 = c^c < \left(\frac{3}{2}\right)^{\frac{3}{2}} < 2.$$

Finally $x_3 = 3^{\frac{1}{c}}$. Since $2^c < 2^{\frac{3}{2}} < 3$, we have $x_3 > 2$, while $\frac{1}{c} < 1$ gives $x_3 < 3$. Thus x_1, x_2, x_3 are not integers, and the first integral term is x_4.

Therefore the correct answer is **C**.

28. In $\triangle ABC$, we have $\angle C = 3\angle A$, $a = 27$ and $c = 48$. What is b?



A 33

B 35

C 37

D 39

E not uniquely determined

Solution:

By the sine law and the triple-angle identity,

$$\frac{48}{27} = \frac{\sin 3A}{\sin A} = 3 - 4\sin^2 A.$$

Hence $\sin^2 A = \frac{11}{36}$. Since $A < 60^\circ$, $\cos A = \frac{5}{6}$, and therefore $\cos 2A = \frac{7}{18}$. Also $B = 180^\circ - 4A$, so another application of the sine law gives

$$\begin{aligned} b &= 27\frac{\sin B}{\sin A} \\ &= 27\frac{\sin 4A}{\sin A} \\ &= 27 \cdot 4\cos A\cos 2A \\ &= 35. \end{aligned}$$

Thus the correct answer is **B**.

29. In their base 10 representations, the integer a consists of a sequence of 1985 eights and the integer b consists of a sequence of 1985 fives. What is the sum of the digits of the base 10 representation of the integer $9ab$?

A 15880

B 17856

C 17865

D 17874

E 19851

Solution:

Let $k = 1985$ and $R = \frac{10^k - 1}{9}$. Then $a = 8R, b = 5R,$ and $9ab$ ends in $0.$ Removing that zero does not change the digit sum and leaves

$$N = \frac{9ab}{10} = 36R^2$$
$$= \frac{4}{9}(10^{2k} - 1)$$
$$- \frac{8}{9}(10^k - 1).$$

Thus N is a string of $2k$ fours minus a string of k eights. The subtraction produces $k - 1$ fours, then $3,$ then $k - 1$ fives, then $6.$ Its digit sum is

$$4(k-1) + 3 + 5(k-1) + 6$$
$$= 9k = 17865.$$

Therefore the correct answer is **C**.

30. Let $\lfloor x \rfloor$ be the greatest integer less than or equal to x. Then the number of real solutions to $4x^2 - 40\lfloor x \rfloor + 51 = 0$ is

A 0

B 1

C 2

D 3

E 4

Solution:

Let $n = \lfloor x \rfloor$. A solution must be positive and satisfy

$$x = \sqrt{10n - \frac{51}{4}},$$
$$n \leq x < n + 1.$$

The lower inequality is equivalent to

$$4n^2 - 40n + 51 \leq 0,$$

so $n = 2, 3, \ldots, 8$. The upper inequality is equivalent to

$$4n^2 - 32n + 55 > 0.$$

Among $2, \ldots, 8$, this holds exactly for $n = 2, 6, 7, 8$. Each interval contributes one root, so there are four real solutions.

Therefore the correct answer is **E**.

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