1984 AMC 12 Solutions

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1. $\frac{1000^2}{252^2-248^2}$ equals

- A $62{,}500$
- B 1000
- **C** 500
- D 250
- E $\frac{1}{2}$

Solution:

Factoring the denominator gives

$$\begin{aligned} 252^2 - 248^2 &= (252+248) \\ &\quad \cdot (252-248) \\ &= 500 \cdot 4 = 2000. \end{aligned}$$

Thus the given quotient is $\frac{1000^2}{2000} = 500$. Therefore, the correct answer is **C**.

2. If x, y and $y-\frac{1}{x}$ are not 0, then

$$\frac{x-\frac{1}{y}}{y-\frac{1}{x}}$$

equals

A 1

B $\frac{x}{y}$

C $\frac{y}{x}$

D $\frac{x}{y}-\frac{y}{x}$

E $xy-\frac{1}{xy}$

Solution:

We have $x-\frac{1}{y}=\frac{xy-1}{y}$ and $y-\frac{1}{x}=\frac{xy-1}{x}$. The hypotheses make the needed quantities nonzero, so their quotient is

$$\frac{\frac{xy-1}{y}}{\frac{xy-1}{x}}=\frac{x}{y}.$$

Therefore, the correct answer is **B**.

3. Let n be the smallest nonprime integer greater than 1 with no prime factor less than 10. Then

A $100 < n \leq 110$

B $110 < n \leq 120$

C $120 < n \leq 130$

D $130 < n \leq 140$

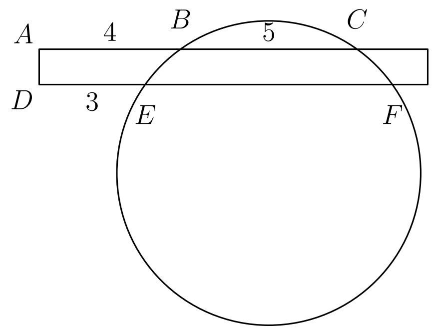
E $140 < n \leq 150$

Solution:

Every prime factor of n is at least 11. The smallest composite with that property is $11 \cdot 11 = 121$, which lies in $120 < n \leq 130$.

Therefore, the correct answer is **C**.

4. A rectangle intersects a circle as shown: $AB = 4, BC = 5$ and $DE = 3$. Then EF equals



A 6

B 7

C $\frac{20}{3}$

D 8

E 9

Solution:

The line through the circle's center perpendicular to the parallel chords BC and EF bisects both. Measured from the rectangle's left side, the midpoint of BC is $4 + \frac{5}{2}$, while the midpoint of EF is $3 + \frac{EF}{2}$. Thus

$$4 + \frac{5}{2} = 3 + \frac{EF}{2},$$

giving $EF = 7$.

Therefore, the correct answer is **B**.

5. The largest integer n for which $n^{200} < 5^{300}$ is

- A 8
- B 9
- C 10
- D 11
- E 12

Solution:

Taking positive one-hundredth roots gives $n^2 < 5^3 = 125.$ Since $11^2 = 121 < 125$ but $12^2 = 144 > 125,$ the largest possible integer is $11.$

Therefore, the correct answer is **D**.

6. In a certain school, there are three times as many boys as girls and nine times as many girls as teachers. Using the letters b, g, t to represent the number of boys, girls and teachers, respectively, then the total number of boys, girls and teachers can be represented by the expression

- A $31b$
- B $\frac{37}{27}b$
- C $13g$
- D $\frac{37}{27}g$
- E $\frac{37}{27}t$

Solution:

Because $b = 3g$ and $g = 9t$, we have $g = \frac{b}{3}$ and $t = \frac{b}{27}$. Hence

$$b + g + t = b + \frac{b}{3} + \frac{b}{27} = \frac{37}{27}b.$$

Therefore, the correct answer is **B**.

7. When Dave walks to school, he averages 90 steps per minute, each of his steps 75 cm long. It takes him 16 minutes to get to school. His brother, Jack, going to the same school by the same route, averages 100 steps per minute, but his steps are only 60 cm long. How long does it take Jack to get to school?

A $14\frac{2}{5}$ min.

B 15 min.

C 18 min.

D 20 min.

E $22\frac{2}{3}$ min.

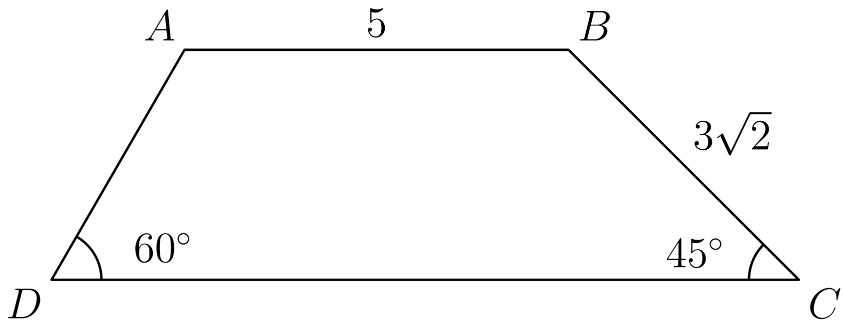
Solution:

The route is $16 \cdot 90 \cdot 75$ centimeters long. Jack covers $100 \cdot 60$ centimeters per minute, so his time is

$$\frac{16 \cdot 90 \cdot 75}{100 \cdot 60} = 18$$

minutes. Therefore, the correct answer is **C**.

8. Figure $ABCD$ is a trapezoid with $AB \parallel DC, AB = 5, BC = 3\sqrt{2}, \angle BCD = 45^\circ$ and $\angle CDA = 60^\circ$. The length of DC is



A $7 + \frac{2}{3}\sqrt{3}$

B 8

C $9\frac{1}{2}$

D $8 + \sqrt{3}$

E $8 + 3\sqrt{3}$

Solution:

Drop perpendiculars from A and B to DC. Since $BC = 3\sqrt{2}$ and the angle at C is 45°, both the height and the right horizontal offset are 3. The left 30°-60°-90° triangle has height 3, so its horizontal offset is $\sqrt{3}$. Therefore

$$DC = \sqrt{3} + AB + 3 = 8 + \sqrt{3}.$$

The correct answer is **D**.

9. The number of digits in $4^{16}5^{25}$ (when written in the usual base 10 form) is

- A 31
- B 30
- C 29
- D 28
- E 27

Solution:

We have

$$\begin{aligned} 4^{16}5^{25} &= 2^{32}5^{25} \\ &= 2^7 \cdot 10^{25} \\ &= 128 \cdot 10^{25}. \end{aligned}$$

This is 128 followed by 25 zeros, so it has $3 + 25 = 28$ digits. Therefore, the correct answer is **D**.

10. Four complex numbers lie at the vertices of a square in the complex plane. Three of the numbers are $1+2i$, $-2+i$ and $-1-2i$. The fourth number is

A $2+i$

B $2-i$

C $1-2i$

D $-1+2i$

E $-2-i$

Solution:

The points $1+2i$ and $-1-2i$ are opposite vertices, so their midpoint 0 is the center of the square. The fourth vertex is therefore the reflection of $-2+i$ through the origin, namely $2-i$.

Therefore, the correct answer is **B**.

11. A calculator has a key which replaces the displayed entry with its square, and another key which replaces the displayed entry with its reciprocal. Let y be the final result if one starts with an entry $x \neq 0$ and alternately squares and reciprocates n times each. Assuming the calculator is completely accurate (e.g., no roundoff or overflow), then y equals

A $x^{(-2)^n}$

B x^{2n}

C x^{-2n}

D x^{-2^n}

E $x^{(-1)^n 2n}$

Solution:

Write the displayed value as x^e. Squaring changes e to $2e$, and then reciprocating changes it to $-2e$. Starting from $e = 1$, after n such pairs the exponent is $(-2)^n$. Thus $y = x^{(-2)^n}$.

Therefore, the correct answer is **A**.

12. If the sequence $\{a_n\}$ is defined by

$$\begin{aligned} a_1 &= 2, \\ a_{n+1} &= a_n + 2n \quad (n \geq 1), \end{aligned}$$

then a_{100} equals

A 9900

B 9902

C 9904

D 10100

E 10102

Solution:

Telescoping the recurrence gives

$$\begin{aligned} a_{100} &= a_1 + 2\sum_{n=1}^{99} n \\ &= 2 + 2 \cdot \frac{99 \cdot 100}{2} \\ &= 9902. \end{aligned}$$

Therefore, the correct answer is **B**.

13. $\frac{2\sqrt{6}}{\sqrt{2}+\sqrt{3}+\sqrt{5}}$ equals

A $\sqrt{2}+\sqrt{3}-\sqrt{5}$

B $4-\sqrt{2}-\sqrt{3}$

C $\sqrt{2}+\sqrt{3}+\sqrt{6}-5$

D $\frac{1}{2}(\sqrt{2}+\sqrt{5}-\sqrt{3})$

E $\frac{1}{3}(\sqrt{3}+\sqrt{5}-\sqrt{2})$

Solution:

Observe that

$$\begin{aligned}&(\sqrt{2}+\sqrt{3}+\sqrt{5})\\ &\quad\cdot(\sqrt{2}+\sqrt{3}-\sqrt{5})\\ &=(\sqrt{2}+\sqrt{3})^2-5\\ &=2\sqrt{6}.\end{aligned}$$

Hence the quotient is $\sqrt{2}+\sqrt{3}-\sqrt{5}$.

Therefore, the correct answer is **A**.

14. The product of all real roots of the equation $x^{\log_{10} x} = 10$ is

A 1

B -1

C 10

D 10^{-1}

E none of these

Solution:

The logarithm requires $x > 0$. Taking base-10 logarithms gives

$$(\log_{10} x)^2 = 1.$$

Thus $\log_{10} x = \pm 1$, so the two roots are 10 and 10^{-1}, whose product is 1.

Therefore, the correct answer is **A**.

15. If $\sin 2x \sin 3x = \cos 2x \cos 3x$, then one value for x is

A 18°

B 30°

C 36°

D 45°

E 60°

Solution:

The equation is equivalent to

$$\cos 2x \cos 3x \\ - \sin 2x \sin 3x \\ = \cos 5x = 0.$$

Therefore $5x = 90^\circ + 180^\circ k$. Taking $k = 0$ gives $x = 18^\circ$, the listed value.

Therefore, the correct answer is **A**.

16. The function $f(x)$ satisfies $f(2+x) = f(2-x)$ for all real numbers x. If the equation $f(x) = 0$ has exactly four distinct real roots, then the sum of these roots is

A 0

B 2

C 4

D 6

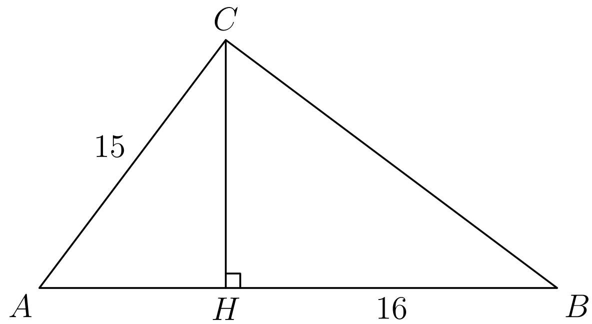
E 8

Solution:

The relation shows that every root $2+r$ is paired with $2-r$. Four distinct roots therefore form two such pairs, and each pair has sum 4. The sum of all four roots is $2 \cdot 4 = 8$.

Therefore, the correct answer is **E**.

17. A right triangle ABC with hypotenuse AB has side $AC = 15$. Altitude CH divides AB into segments AH and HB, with $HB = 16$. The area of $\triangle ABC$ is



A 120

B 144

C 150

D 216

E $144\sqrt{5}$

Solution:

Let $AH = h$. Similarity in the right triangle gives

$$\begin{aligned} AC^2 &= AH \cdot AB, \\ 225 &= h(h+16), \end{aligned}$$

so $h = 9$. Hence $AB = 25$. Also $CH = \sqrt{AH \cdot HB}$, so $CH = \sqrt{9 \cdot 16} = 12$. The area is $\frac{1}{2} \cdot 25 \cdot 12 = 150$.

Therefore, the correct answer is **C**.

18. A point (x, y) is to be chosen in the coordinate plane so that it is equally distant from the x-axis, the y-axis, and the line $x + y = 2$. Then x is

A $\sqrt{2} - 1$

B $\frac{1}{2}$

C $2 - \sqrt{2}$

D 1

E not uniquely determined

Solution:

Equal distance from the axes gives $y = x$ or $y = -x$. On $y = -x$, the distance to the line $x + y = 2$ is $\sqrt{2}$, so both $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$ satisfy all three distance conditions. Their x-coordinates differ, so x is not uniquely determined.

Therefore, the correct answer is **E**.

19. A box contains 11 balls, numbered $1, 2, 3, \ldots, 11$. If 6 balls are drawn simultaneously at random, what is the probability that the sum of the numbers on the balls drawn is odd?

A $\frac{100}{231}$

B $\frac{115}{231}$

C $\frac{1}{2}$

D $\frac{118}{231}$

E $\frac{6}{11}$

Solution:

An odd sum requires an odd number of the six selected balls to be odd. There are 6 odd and 5 even balls, so the favorable count is

$$\binom{6}{1}\binom{5}{5} + \binom{6}{3}\binom{5}{3} + \binom{6}{5}\binom{5}{1} = 6 + 200 + 30 = 236.$$

Of the $\binom{11}{6} = 462$ selections, the desired probability is $\frac{236}{462} = \frac{118}{231}$.

Therefore, the correct answer is **D**.

20. The number of distinct solutions of the equation $|x - |2x+1|| = 3$ is

A 0

B 1

C 2

D 3

E 4

Solution:

If $x - |2x+1| = 3,$ then $|2x+1| = x - 3,$ which requires $x \geq 3$ and yields no solution. If $x - |2x+1| = -3,$ then $|2x+1| = x + 3.$ Its two linear cases give $x = 2$ and $x = -\frac{4}{3}$, both valid. Thus there are 2 distinct solutions.

Therefore, the correct answer is **C**.

21. The number of triples (a, b, c) of positive integers which satisfy the simultaneous equations

$$ab + bc = 44,$$
$$ac + bc = 23.$$

is

A 0

B 1

C 2

D 3

E 4

Solution:

Since $c(a+b) = 23$ and all variables are positive integers, $c = 1$ and $a + b = 23$. Put $b = 23 - a$ in $ab + b = 44$. Then

$$(23 - a)(a + 1) = 44,$$

or $a^2 - 22a + 21 = 0$, so $a = 1$ or 21. Each gives a positive b, producing exactly two triples.

Therefore, the correct answer is **C**.

22. Let a and c be fixed positive numbers. For each real number t let (x_t, y_t) be the vertex of the parabola $y = ax^2 + tx + c$. If the set of vertices (x_t, y_t) for all real values of t is graphed in the plane, the graph is

- A a straight line
- B a parabola
- C part, but not all, of a parabola
- D one branch of a hyperbola
- E none of these

Solution:

The vertex coordinates are

$$x_t = -\frac{t}{2a}, \qquad y_t = c - \frac{t^2}{4a}.$$

Since $t = -2ax_t$, eliminating t gives $y_t = c - ax_t^2$. As t ranges over all reals, so does x_t, so the entire parabola is traced.

Therefore, the correct answer is **B**.

23. $\frac{\sin 10^\circ + \sin 20^\circ}{\cos 10^\circ + \cos 20^\circ}$ equals

A $\tan 10^\circ + \tan 20^\circ$

B $\tan 30^\circ$

C $\frac{1}{2}(\tan 10^\circ + \tan 20^\circ)$

D $\tan 15^\circ$

E $\frac{1}{4}\tan 60^\circ$

Solution:

Let R denote the given ratio. By the sum-to-product identities,

$$R = \frac{\sin 10^\circ + \sin 20^\circ}{\cos 10^\circ + \cos 20^\circ},$$
$$R = \frac{2\sin 15^\circ \cos 5^\circ}{2\cos 15^\circ \cos 5^\circ}$$
$$= \tan 15^\circ.$$

Therefore, the correct answer is **D**.

24. If a and b are positive real numbers and each of the equations

$$x^2 + ax + 2b = 0,$$
$$x^2 + 2bx + a = 0$$

has real roots, then the smallest possible value of $a + b$ is

- A 2
- B 3
- C 4
- D 5
- E 6

Solution:

The two discriminants give $a^2 \geq 8b$ and $b^2 \geq a$. Therefore

$$a^4 \geq 64b^2 \geq 64a.$$

Since $a > 0,$ this yields $a \geq 4,$ and then $b^2 \geq a \geq 4$ gives $b \geq 2.$ Thus $a + b \geq 6.$ Equality occurs at $a = 4, b = 2,$ for which both discriminants are zero.

Therefore, the correct answer is **E**.

25. The total area of all the faces of a rectangular solid is $22\ \mathrm{cm}^2$, and the total length of all its edges is $24\ \mathrm{cm}$. Then the length in cm of any one of its internal diagonals is

A $\sqrt{11}$

B $\sqrt{12}$

C $\sqrt{13}$

D $\sqrt{14}$

E not uniquely determined

Solution:

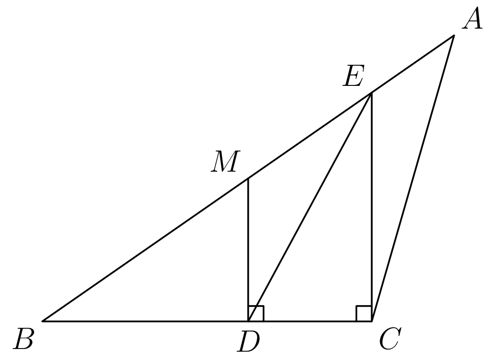
The conditions give $2(xy + xz + yz) = 22$ and $4(x + y + z) = 24$, so $xy + xz + yz = 11$ and $x + y + z = 6$. Hence the square of an internal diagonal is

$$x^2 + y^2 + z^2 = 6^2 - 2(11) = 14.$$

Its length is $\sqrt{14}$.

Therefore, the correct answer is **D**.

26. In the obtuse triangle ABC, $AM = MB, MD \perp BC, EC \perp BC$. If the area of $\triangle ABC$ is 24, then the area of $\triangle BED$ is



A 9

B 12

C 15

D 18

E not uniquely determined

Solution:

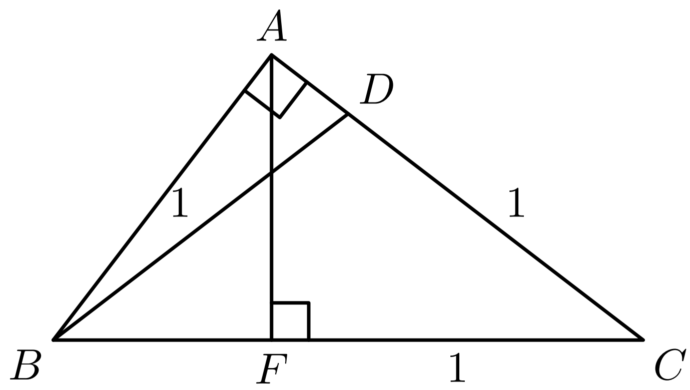
Draw MC. Since $EC \parallel MD$, points E and C have equal perpendicular distances from MD, so $[DME] = [DMC]$. Therefore

$$\begin{aligned}[BED] &= [BMD] + [DME] \\ &= [BMD] + [DMC] \\ &= [BMC].\end{aligned}$$

Because M is the midpoint of AB, triangle BMC has half the area of ABC. Thus $[BED] = 12$.

Therefore, the correct answer is **B**.

27. In $\triangle ABC$, D is on AC and F is on BC. Also, $AB \perp AC$, $AF \perp BC$, and $BD = DC = FC = 1$. Find AC.



A $\sqrt{2}$

B $\sqrt{3}$

C $\sqrt[3]{2}$

D $\sqrt[3]{3}$

E $\sqrt[4]{3}$

Solution:

Let $AC = s$. The right-triangle projection relation gives

$$AC^2 = FC \cdot BC,$$

so $BC = s^2$. Since $BD = DC = 1$, the numerator and denominator in the Law of Cosines satisfy

$$\begin{aligned} BC^2 + DC^2 - BD^2 &= s^4, \\ 2(BC)(DC) &= 2s^2. \end{aligned}$$

Hence $\cos \angle BCD = \frac{s^2}{2}$. But D lies on AC, and in right triangle ABC, $\cos \angle BCA = \frac{AC}{BC} = \frac{1}{s}$. Thus $\frac{s^2}{2} = \frac{1}{s}$, so $s^3 = 2$ and $AC = \sqrt[3]{2}$.

Therefore, the correct answer is **C**.

28. The number of distinct pairs of integers (x, y) such that

$$0 < x < y,$$
$$\sqrt{1984} = \sqrt{x} + \sqrt{y}$$

is

A 0

B 1

C 3

D 4

E 7

Solution:

Because $\sqrt{x} + \sqrt{y} = 8\sqrt{31}$, the two radicals must have common squarefree part 31. Write $x = 31u^2$ and $y = 31v^2$ for positive integers u, v. Then

$$u + v = 8, \qquad u < v.$$

The possibilities are $(u, v) = (1, 7), (2, 6), (3, 5)$, producing three distinct pairs (x, y).

Therefore, the correct answer is **C**.

29. Find the largest value of $\frac{y}{x}$ for pairs of real numbers (x, y) which satisfy

$$(x-3)^2+(y-3)^2=6.$$

A $3+2\sqrt{2}$

B $2+\sqrt{3}$

C $3\sqrt{3}$

D 6

E $6+2\sqrt{3}$

Solution:

The circle lies entirely where $x>0$, so $\frac{y}{x}$ is the slope m of the line $y=mx$ through a point on it. At an extreme, this line is tangent. The distance from the center $(3,3)$ to $mx-y=0$ must equal $\sqrt{6}$, so

$$\frac{|3m-3|}{\sqrt{m^2+1}}=\sqrt{6}.$$

Squaring and simplifying gives $m^2-6m+1=0$, hence $m=3\pm2\sqrt{2}$. The larger value is $3+2\sqrt{2}$.

Therefore, the correct answer is **A**.

30. For any complex number $w = a + bi$, $|w|$ is defined to be the real number $\sqrt{a^2 + b^2}$. If $w = \cos 40^\circ + i \sin 40^\circ$, then

$$|w + 2w^2 + 3w^3 + \cdots + 9w^9|^{-1}$$

equals

A $\frac{1}{9}\sin 40^\circ$

B $\frac{2}{9}\sin 20^\circ$

C $\frac{1}{9}\cos 40^\circ$

D $\frac{1}{18}\cos 20^\circ$

E none of these

Solution:

Let $S = \sum_{k=1}^{9} kw^k$. Since $w^9 = 1$ and $w^{10} = w$,

$$\begin{aligned}(1-w)S &= (w + w^2 + \cdots + w^9) \\ &\quad - 9w^{10} \\ &= -9w.\end{aligned}$$

Therefore $|S| = \frac{9}{|1-w|}$, because $|w| = 1$. Thus

$$|S|^{-1} = \frac{|1-w|}{9} = \frac{2}{9}\sin 20^\circ.$$

Therefore, the correct answer is **B**.

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