1983 AMC 12 Solutions

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1. If $x \neq 0, \frac{x}{2} = y^2$ and $\frac{x}{4} = 4y$, then x equals

A 8

B 16

C 32

D 64

E 128

Solution:

The second equation gives $x = 16y$. Substituting into the first gives $8y = y^2$, so $y(y-8) = 0$. The condition $x \neq 0$ rules out $y = 0$, leaving $y = 8$ and $x = 128$.

Therefore, the correct answer is **E**.

2. Point P is outside circle C on the plane. At most how many points on C are 3 cm from P?

- A 1
- B 2
- C 3
- D 4
- E 8

Solution:

The points 3 cm from P form a circle centered at P. Two distinct circles intersect in at most two points, and two intersections are possible.

Therefore, the correct answer is **B**.

3. Three primes, p, q and r, satisfy $p + q = r$ and $1 < p < q$. Then p equals

A 2

B 3

C 7

D 13

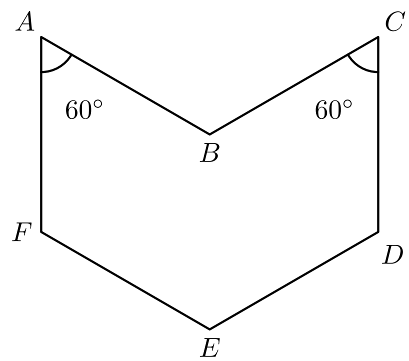
E 17

Solution:

If both p and q were odd, then $r = p + q$ would be even and greater than 2, so it would not be prime. Thus one addend is the only even prime, 2. Since $p < q$, that addend is p.

Therefore, the correct answer is **A**.

4. In the adjoining plane figure, sides AF and CD are parallel, as are sides AB and FE, and sides BC and ED. Each side has length 1. Also, $\angle FAB = \angle BCD = 60^\circ$. The area of the figure is



A $\frac{\sqrt{3}}{2}$

B 1

C $\frac{3}{2}$

D $\sqrt{3}$

E 2

Solution:

The parallel directions and 60° angles place the vertices, up to rigid motion, at

$$\begin{aligned} A &= (0,0), \\ B &= (\tfrac{\sqrt{3}}{2}, -\tfrac{1}{2}), \\ C &= (\sqrt{3}, 0), \\ D &= (\sqrt{3}, -1), \\ E &= (\tfrac{\sqrt{3}}{2}, -\tfrac{3}{2}), \\ F &= (0, -1). \end{aligned}$$

Applying the shoelace formula to these six vertices gives area $\sqrt{3}$.

Therefore, the correct answer is **D**.

5. Triangle ABC has a right angle at C. If $\sin A = \frac{2}{3}$, then $\tan B$ is

A $\frac{3}{5}$

B $\frac{\sqrt{5}}{3}$

C $\frac{2}{\sqrt{5}}$

D $\frac{\sqrt{5}}{2}$

E $\frac{5}{3}$

Solution:

Since $\sin A = \frac{2}{3}$, $\cos A = \sqrt{1 - \frac{4}{9}} = \frac{\sqrt{5}}{3}$. Because $B = 90^\circ - A$, we have $\tan B = \cot A$ and $\frac{\cos A}{\sin A} = \frac{\sqrt{5}}{2}$.

Therefore, the correct answer is **D**.

6. When $x^5, x + \frac{1}{x}$ and $1 + \frac{2}{x} + \frac{3}{x^3}$ are multiplied, the product is a polynomial of degree

- A 2
- B 3
- C 6
- D 7
- E 8

Solution:

The greatest exponent comes from $x^5 \cdot x \cdot 1 = x^6$. Every other term has smaller exponent, and the smallest possible exponent is $5 - 1 - 3 = 1$, so the product is indeed a polynomial. Its degree is 6.

Therefore, the correct answer is **C**.

7. Alice sells an item at $\$10$ less than the list price and receives 10% of her selling price as her commission. Bob sells the same item at $\$20$ less than the list price and receives 20% of his selling price as his commission. If they both get the same commission, then the list price is

A $\$20$

B $\$30$

C $\$50$

D $\$70$

E $\$100$

Solution:

If the list price is L, equality of commissions gives $0.1(L-10)=0.2(L-20)$. Multiplying by 10 and solving gives $L-10=2L-40$, so $L=30$.

Therefore, the correct answer is **B**.

8. Let $f(x) = \frac{x+1}{x-1}$. Then for $x^2 \neq 1, f(-x)$ is

A $\frac{1}{f(x)}$

B $-f(x)$

C $\frac{1}{f(-x)}$

D $-f(-x)$

E $f(x)$

Solution:

We have

$$\begin{aligned} f(-x) &= \frac{-x+1}{-x-1} \\ &= \frac{x-1}{x+1} = \frac{1}{f(x)}. \end{aligned}$$

The restriction $x^2 \neq 1$ makes all displayed quantities defined.

Therefore, the correct answer is **A**.

9. In a certain population the ratio of the number of women to the number of men is 11 to 10. If the average (arithmetic mean) age of the women is 34 and the average age of the men is 32, then the average age of the population is

A $32\frac{9}{10}$

B $32\frac{20}{21}$

C 33

D $33\frac{1}{21}$

E $33\frac{1}{10}$

Solution:

Using 11 women and 10 men, the total of all ages is $11 \cdot 34 + 10 \cdot 32 = 694$. Dividing by 21 people gives $\frac{694}{21} = 33\frac{1}{21}$.

Therefore, the correct answer is **D**.

10. Segment AB is both a diameter of a circle of radius 1 and a side of an equilateral triangle ABC. The circle also intersects AC and BC at points D and E, respectively. The length of AE is

A $\frac{3}{2}$

B $\frac{5}{3}$

C $\frac{\sqrt{3}}{2}$

D $\sqrt{3}$

E $\frac{2+\sqrt{3}}{2}$

Solution:

Since E is on the circle with diameter AB, $\angle AEB = 90^\circ$. Also $\angle ABE = 60^\circ$ because ABC is equilateral. Thus ABE is a 30°-60°-90° triangle with hypotenuse $AB = 2$, so $AE = \sqrt{3}$.

Therefore, the correct answer is **D**.

11. Simplify $\sin(x - y)\cos y + \cos(x - y)\sin y$.

A 1

B $\sin x$

C $\cos x$

D $\sin x \cos 2y$

E $\cos x \cos 2y$

Solution:

Taking $u = x - y$ and $v = y$, the sine addition identity gives

$$\begin{aligned} &\sin(x - y)\cos y \\ &\quad + \cos(x - y)\sin y \\ &= \sin((x - y) + y) = \sin x. \end{aligned}$$

Therefore, the correct answer is **B**.

12. If $\log_7(\log_3(\log_2 x)) = 0$, then $x^{-\frac{1}{2}}$ equals

A $\frac{1}{3}$

B $\frac{1}{2\sqrt{3}}$

C $\frac{1}{3\sqrt{3}}$

D $\frac{1}{\sqrt{42}}$

E none of these

Solution:

The equation gives $\log_3(\log_2 x) = 1$, so $\log_2 x = 3$ and $x = 8$. Hence $x^{-\frac{1}{2}} = \frac{1}{\sqrt{8}} = \frac{1}{2\sqrt{2}}$, which is not among choices A–D.

Therefore, the correct answer is **E**.

13. If $xy = a, xz = b$, and $yz = c$, and none of these quantities is 0, then $x^2 + y^2 + z^2$ equals

A $\frac{ab+ac+bc}{abc}$

B $\frac{a^2+b^2+c^2}{abc}$

C $\frac{(a+b+c)^2}{abc}$

D $\frac{(ab+ac+bc)^2}{abc}$

E $\frac{(ab)^2+(ac)^2+(bc)^2}{abc}$

Solution:

From the three products,

$$x^2 = \frac{ab}{c}, \quad y^2 = \frac{ac}{b}, \quad z^2 = \frac{bc}{a}.$$

Putting their sum over denominator abc gives

$$x^2 + y^2 + z^2 = \frac{(ab)^2 + (ac)^2}{abc} + \frac{(bc)^2}{abc}.$$

Therefore, the correct answer is **E**.

14. The units digit of $3^{1001}7^{1002}13^{1003}$ is

- A 1
- B 3
- C 5
- D 7
- E 9

Solution:

Modulo 10,

$$3^{1001}13^{1003} \equiv 3^{2004} \equiv 1$$

because powers of 3 repeat every 4. Also $7^{1002} \equiv 7^2 \equiv 9 \pmod{10}$. Thus the product has units digit 9.

Therefore, the correct answer is **E**.

15. Three balls marked $1, 2$ and 3 are placed in an urn. One ball is drawn, its number is recorded, and then the ball is returned to the urn. This process is repeated and then repeated once more, and each ball is equally likely to be drawn on each occasion. If the sum of the numbers recorded is 6, what is the probability that the ball numbered 2 was drawn all three times?

A $\frac{1}{27}$

B $\frac{1}{8}$

C $\frac{1}{7}$

D $\frac{1}{6}$

E $\frac{1}{3}$

Solution:

The ordered triples with sum 6 are the six permutations of $(1, 2, 3)$ and the triple $(2, 2, 2)$. They are equally likely, and exactly one of the seven has three 2's. The conditional probability is therefore $\frac{1}{7}$.

Therefore, the correct answer is **C**.

16. Let

$$x = 0.123456789101112$$
$$\ldots 998999,$$

where the digits are obtained by writing the integers 1 through 999 in order. The 1983 rd digit to the right of the decimal point is

A 2

B 3

C 5

D 7

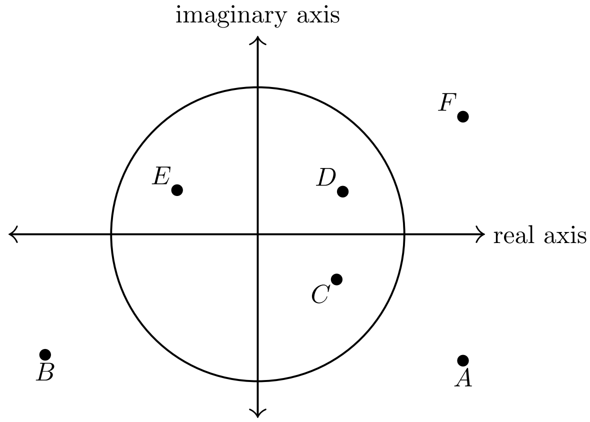
E 8

Solution:

The one-digit integers contribute 9 digits and the two-digit integers contribute $90 \cdot 2 = 180,$ for 189 total. Thus the desired digit is the $1983 - 189 = 1794$th digit among the three-digit integers. Since $1794 = 598 \cdot 3,$ it is the last digit of the 598th three-digit integer, namely $100 + 597 = 697.$ That digit is $7.$

Therefore, the correct answer is **D**.

17. The diagram to the right shows several numbers in the complex plane. The circle is the unit circle centered at the origin. One of these numbers is the reciprocal of F. Which one?



A A

B B

C C

D D

E E

Solution:

If $F = re^{i\theta}$, then $\frac{1}{F} = r^{-1}e^{-i\theta}$. Because F is outside the unit circle in quadrant I, its reciprocal is inside the unit circle in quadrant IV, with the reflected argument. Only point C has those properties.

Therefore, the correct answer is **C**.

18. Let f be a polynomial function such that, for all real x,

$$f(x^2+1) = x^4 + 5x^2 + 3.$$

For all real x, $f(x^2-1)$ is

A $x^4 + 5x^2 + 1$

B $x^4 + x^2 - 3$

C $x^4 - 5x^2 + 1$

D $x^4 + x^2 + 3$

E none of these

Solution:

With $t = x^2 + 1$,

$$\begin{aligned} x^4 + 5x^2 + 3 &= (t-1)^2 \\ &\quad + 5(t-1) + 3 \\ &= t^2 + 3t - 1. \end{aligned}$$

Hence $f(t) = t^2 + 3t - 1$. Substituting $t = x^2 - 1$ gives

$$\begin{aligned} f(x^2-1) &= (x^2-1)^2 \\ &\quad + 3(x^2-1) - 1 \\ &= x^4 + x^2 - 3. \end{aligned}$$

Therefore, the correct answer is **B**.

19. Point D is on side CB of triangle ABC. If $\angle CAD = \angle DAB = 60^\circ$, $AC = 3$ and $AB = 6$, then the length of AD is

- A 2
- B 2.5
- C 3
- D 3.5
- E 4

Solution:

By the angle bisector theorem, write $CD = t$ and $DB = 2t$. Let $AD = d$. The law of cosines in triangles ACD and ABD, whose angles at A are both 60°, gives

$$t^2 = 9 + d^2 - 3d,$$
$$4t^2 = 36 + d^2 - 6d.$$

Subtracting four times the first equation from the second yields $0 = -3d^2 + 6d$. Since $d > 0, d = 2$.

Therefore, the correct answer is **A**.

20. If $\tan\alpha$ and $\tan\beta$ are the roots of $x^2 - px + q = 0$, and $\cot\alpha$ and $\cot\beta$ are the roots of $x^2 - rx + s = 0$, then rs is necessarily

A pq

B $\frac{1}{pq}$

C $\frac{p}{q^2}$

D $\frac{q}{p^2}$

E $\frac{p}{q}$

Solution:

Let the first roots be $u = \tan\alpha$ and $v = \tan\beta$. Then $u + v = p$ and $uv = q$. The second roots are $\frac{1}{u}$ and $\frac{1}{v}$, so

$$r = \frac{1}{u} + \frac{1}{v} = \frac{p}{q},$$
$$s = \frac{1}{uv} = \frac{1}{q}.$$

Therefore $rs = \frac{p}{q^2}$.

Therefore, the correct answer is **C**.

21. Find the smallest positive number from the numbers below

A $10-3\sqrt{11}$

B $3\sqrt{11}-10$

C $18-5\sqrt{13}$

D $51-10\sqrt{26}$

E $10\sqrt{26}-51$

Solution:

Since $100>99,$ choice A is positive; since $324<325,$ choice C is negative; and since $2601>2600,$ choice D is positive while E is negative. The two positive values satisfy

$$\begin{aligned}10-3\sqrt{11}&=\frac{1}{10+3\sqrt{11}},\\ 51-10\sqrt{26}&=\frac{1}{51+10\sqrt{26}}.\end{aligned}$$

The second denominator is larger, so choice D is the smaller positive number.

Therefore, the correct answer is **D**.

22. Consider the two functions

$$f(x) = x^2 + 2bx + 1,$$
$$g(x) = 2a(x + b),$$

where the variable x and the constants a and b are real numbers. Each such pair of constants a and b may be considered as a point (a, b) in an ab-plane. Let S be the set of such points (a, b) for which the graphs of $y = f(x)$ and $y = g(x)$ do not intersect (in the xy-plane). The area of S is

A 1

B π

C 4

D 4π

E infinite

Solution:

Intersections correspond to roots of

$$x^2 + 2(b - a)x + (1 - 2ab) = 0.$$

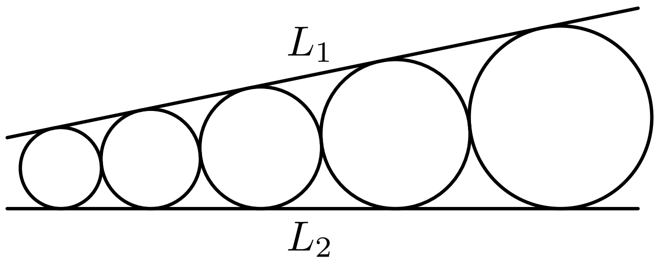
There are no real roots exactly when its discriminant is negative:

$$\begin{aligned} &4(b - a)^2 - 4(1 - 2ab) \\ &= 4(a^2 + b^2 - 1) \\ &< 0. \end{aligned}$$

Thus S is the interior of the unit circle in the ab-plane, with area π.

Therefore, the correct answer is **B**.

23. In the adjoining figure the five circles are tangent to one another consecutively and to the lines L_1 and L_2. If the radius of the largest circle is 18 and that of the smallest one is 8, then the radius of the middle circle is



A 12

B 12.5

C 13

D 13.5

E 14

Solution:

The centers lie on the angle bisector of L_1 and L_2. Similarity of the configuration for any two consecutive circles shows that consecutive radii have a constant ratio. Thus the five radii form a geometric sequence. If the middle radius is m, symmetry of a five-term geometric sequence gives $m^2 = 8 \cdot 18$, so $m = 12$.

Therefore, the correct answer is **A**.

24. How many non-congruent right triangles are there such that the perimeter in cm and area in cm^2 are numerically equal?

A none

B 1

C 2

D 4

E infinitely many

Solution:

Start with any right triangle having perimeter P and area K. Scaling every side by $k = \frac{P}{K}$ produces perimeter kP and area k^2K, which are equal. Infinitely many nonsimilar right-triangle shapes exist, and the resulting triangles are therefore noncongruent.

Therefore, the correct answer is **E**.

25. If $60^a = 3$ and $60^b = 5$, then $12^{\frac{1-a-b}{2(1-b)}}$ is

A $\sqrt{3}$

B 2

C $\sqrt{5}$

D 3

E $\sqrt{12}$

Solution:

Since $a = \log_{60} 3$ and $b = \log_{60} 5$,

$$\begin{aligned} 1 - a - b &= \log_{60} \frac{60}{15} = \log_{60} 4, \\ 1 - b &= \log_{60} \frac{60}{5} = \log_{60} 12. \end{aligned}$$

Therefore the exponent is $\frac{1}{2} \log_{12} 4$, and

$$12^{\left(\frac{1}{2}\right) \log_{12} 4} = \sqrt{4} = 2.$$

Therefore, the correct answer is **B**.

26. The probability that event A occurs is $\frac{3}{4}$; the probability that event B occurs is $\frac{2}{3}$. Let p be the probability that both A and B occur. The smallest interval necessarily containing p is the interval

- A $\left[\frac{1}{12}, \frac{1}{2}\right]$
- B $\left[\frac{5}{12}, \frac{1}{2}\right]$
- C $\left[\frac{1}{2}, \frac{2}{3}\right]$
- D $\left[\frac{5}{12}, \frac{2}{3}\right]$
- E $\left[\frac{1}{12}, \frac{2}{3}\right]$

Solution:

Inclusion-exclusion gives

$$\begin{aligned} p &= P(A \cap B) \\ &= \frac{3}{4} + \frac{2}{3} - P(A \cup B). \end{aligned}$$

Since $\frac{3}{4} \leq P(A \cup B) \leq 1$, it follows that

$$\frac{5}{12} \leq p \leq \frac{2}{3}.$$

Both endpoints can occur, so this is the smallest necessary interval.

Therefore, the correct answer is **D**.

27. A large sphere is on a horizontal field on a sunny day. At a certain time the shadow of the sphere reaches out a distance of 10 m from the point where the sphere touches the ground. At the same instant a meter stick (held vertically with one end on the ground) casts a shadow of length 2 m. What is the radius of the sphere in meters? (Assume the sun's rays are parallel and the meter stick is a line segment.)

A $\frac{5}{2}$

B $9-4\sqrt{5}$

C $8\sqrt{10}-23$

D $6-\sqrt{15}$

E $10\sqrt{5}-20$

Solution:

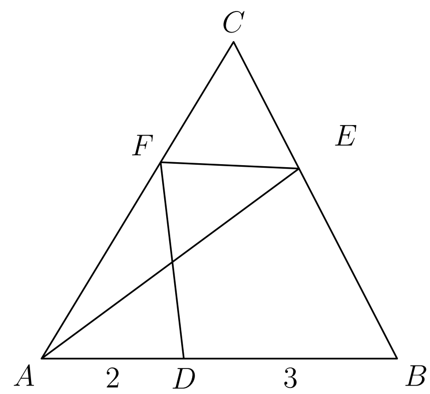
Take the sphere's ground-contact point as $(0,0)$ and its center as $(0,r)$. The limiting sun ray passes through the shadow endpoint $(10,0)$ and has slope $-\frac{1}{2}$, so its equation is $x+2y-10=0$. Tangency means the distance from $(0,r)$ to this line equals r :

$$\frac{10-2r}{\sqrt{5}}=r.$$

Hence $r=\frac{10}{2+\sqrt{5}}=10\sqrt{5}-20.$

Therefore, the correct answer is **E**.

28. Triangle ABC in the figure has area 10. Points D, E and F, all distinct from A, B and C, are on sides AB, BC and CA respectively, and $AD = 2, DB = 3$. If triangle ABE and quadrilateral $DBEF$ have equal areas, then that area is



A 4

B 5

C 6

D $\frac{5}{3}\sqrt{10}$

E not uniquely determined

Solution:

Draw DE. From

$$[ABE] = [ADE] + [DBE],$$
$$[DBEF] = [FDE] + [DBE],$$

equality implies $[ADE] = [FDE]$. Their common base DE then gives $AF \parallel DE$, so triangles DBE and ABC are similar. Hence $\frac{BE}{BC} = \frac{DB}{AB} = \frac{3}{5}$. Triangles ABE and ABC share the altitude from A to BC, so $[ABE] = (\frac{3}{5})[ABC] = 6$.

Therefore, the correct answer is **C**.

29. A point P lies in the same plane as a given square of side 1. Let the vertices of the square, taken counterclockwise, be A, B, C and D. Also, let the distances from P to A, B and C, respectively, be u, v and w. What is the greatest distance that P can be from D if $u^2 + v^2 = w^2$?

A $1+\sqrt{2}$

B $2\sqrt{2}$

C $2+\sqrt{2}$

D $3\sqrt{2}$

E $3+\sqrt{2}$

Solution:

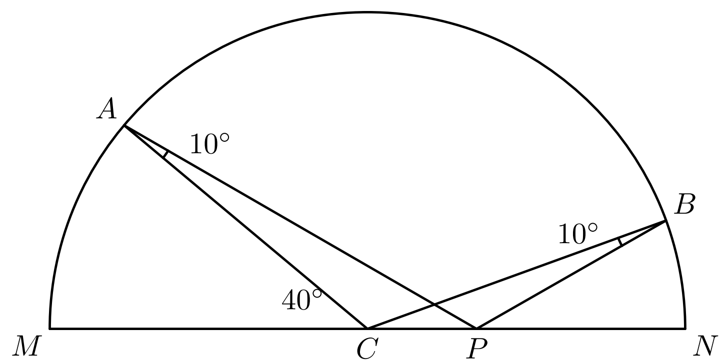
Place $D = (0,0), A = (1,0)$ and $B = (1,1), C = (0,1)$, with $P = (x, y)$. Then

$$\begin{aligned}&(x-1)^2 + y^2 \\ &\quad + (x-1)^2 + (y-1)^2 \\ &= x^2 + (y-1)^2,\end{aligned}$$

which simplifies to $(x-2)^2 + y^2 = 2$. Thus P lies on a circle centered 2 units from D with radius $\sqrt{2}$. Its greatest distance from D is $2+\sqrt{2}$.

Therefore, the correct answer is **C**.

30. Distinct points A and B are on a semicircle with diameter MN and center C. The point P is on CN and $\angle CAP = \angle CBP = 10^\circ$. If $\overset{\frown}{MA} = 40^\circ$, then $\overset{\frown}{BN}$ equals



A 10°

B 15°

C 20°

D 25°

E 30°

Solution:

Since $\overset{\frown}{MA} = 40^\circ$, the central angle $\angle MCA = 40^\circ$, so $\angle ACP = 140^\circ$. Hence $\angle APC = 180^\circ - 140^\circ - 10^\circ = 30^\circ$. Applying the law of sines in triangles ACP and BCP, and using $AC = BC$, gives $\sin \angle BPC = \sin 30^\circ$. Because A and B are distinct, $\angle BPC = 150^\circ$. Therefore

$$\begin{aligned} \angle BCN &= 180^\circ - 150^\circ - 10^\circ \\ &= 20^\circ, \end{aligned}$$

so $\overset{\frown}{BN} = 20^\circ$.

Therefore, the correct answer is **C**.

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