1982 AMC 12 Solutions

Typeset by: LIVE by Po-Shen Loh
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1. When the polynomial $x^3 - 2$ is divided by the polynomial $x^2 - 2$, the remainder is

A 2

B -2

C $-2x - 2$

D $2x + 2$

E $2x - 2$

Solution:

Modulo $x^2 - 2$, we have $x^2 \equiv 2$, so $x^3 - 2 \equiv 2x - 2$. This linear polynomial is the remainder.

Therefore, the correct answer is **E**.

2. If a number eight times as large as x is increased by two, then one fourth of the result equals

A $2x + \frac{1}{2}$

B $x + \frac{1}{2}$

C $2x + 2$

D $2x + 4$

E $2x + 16$

Solution:

Eight times x, increased by two, is $8x + 2$. One fourth is $\frac{8x+2}{4} = 2x + \frac{1}{2}$.

Therefore, the correct answer is **A**.

3. Evaluate $(x^x)^{(x^x)}$ at $x = 2$.

A 16

B 64

C 256

D 1024

E $65{,}536$

Solution:

At $x = 2$, $x^x = 2^2 = 4$. Hence $(x^x)^{(x^x)} = 4^4 = 256$.

Therefore, the correct answer is **C**.

4. The perimeter of a semicircular region, measured in centimeters, is numerically equal to its area, measured in square centimeters. The radius of the semicircle, measured in centimeters, is

A π

B $\frac{2}{\pi}$

C 1

D $\frac{1}{2}$

E $\frac{4}{\pi} + 2$

Solution:

For $r > 0$, equality of perimeter and area gives $\pi r + 2r = \frac{\pi r^2}{2}$. Dividing by r and solving yields $r = \frac{2(\pi+2)}{\pi} = 2 + \frac{4}{\pi}$.

Therefore, the correct answer is **E**.

5. Two positive numbers x and y are in the ratio $a : b$, where $0 < a < b$. If $x + y = c$, then the smaller of x and y is

A $\frac{ac}{b}$

B $\frac{bc-ac}{b}$

C $\frac{ac}{a+b}$

D $\frac{bc}{a+b}$

E $\frac{ac}{b-a}$

Solution:

Write $x = ka$ and $y = kb$. Since $a < b$, x is smaller. From $k(a + b) = c$, $k = \frac{c}{a+b}$, so $x = \frac{ac}{a+b}$.

Therefore, the correct answer is **C**.

6. The sum of all but one of the interior angles of a convex polygon equals 2570°. The remaining angle is

- A 90°
- B 105°
- C 120°
- D 130°
- E 144°

Solution:

The total must be $(n-2)180^\circ$ and the missing angle lies between 0° and 180°. The only multiple of 180° between 2570° and 2750° is 2700°, leaving $2700^\circ - 2570^\circ = 130^\circ$.

Therefore, the correct answer is **D**.

7. If the operation $x * y$ is defined by $x * y = (x+1)(y+1) - 1$, then which one of the following is false?

A $x * y = y * x$ for all real x and y.

B $x * (y + z)$ equals $(x * y) + (x * z)$ for all real x, y, and z.

C $(x - 1) * (x + 1)$ equals $(x * x) - 1$ for all real x.

D $x * 0 = x$ for all real x.

E $x * (y * z) = (x * y) * z$ for all real x, y, and z.

Solution:

Since $x * y = x + y + xy$, the operation is commutative, has identity 0, and becomes ordinary multiplication after adding 1. Thus it is associative, and direct expansion also verifies C. But at $x = 0$, statement B says $y + z = y + z$, while its right side is $y + z$; to expose the extra term take $x = 1, y = z = 0$, giving $1 * 0 = 1$ on the left but $(1 * 0) + (1 * 0) = 2$ on the right. Hence B is false.

Therefore, the correct answer is **B**.

8. By definition $r! = r(r-1)\cdots 1$ and $\binom{j}{k} = \frac{j!}{k!(j-k)!}$, where r, j, k are positive integers and $k < j$. If $\binom{n}{1}, \binom{n}{2}, \binom{n}{3}$ form an arithmetic progression with $n > 3$, then n equals

A 5

B 7

C 9

D 11

E 12

Solution:

The progression condition is $2\binom{n}{2} = \binom{n}{1} + \binom{n}{3}$. Substitution and cancellation give $n^2 - 9n + 14 = 0$, or $(n-2)(n-7) = 0$. Since $n > 3, n = 7$.

Therefore, the correct answer is **B**.

9. A vertical line divides the triangle with vertices $(0, 0)$, $(1, 1)$ and $(9, 1)$ in the xy-plane into two regions of equal area. The equation of the line is $x =$

A 2.5

B 3.0

C 3.5

D 4.0

E 4.5

Solution:

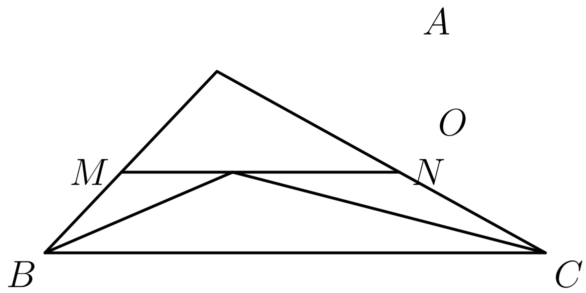
The triangle has area 4. The portion with $0 \leq x \leq 1$ has area

$$\int_0^1 \left(x - \frac{x}{9}\right)\, dx = \frac{4}{9}.$$

For a cut $x = t \geq 1$, the area to its left is therefore $\frac{4}{9} + \int_1^t (1 - \frac{x}{9})\, dx$. Setting this equal to 2 gives $t - \frac{t^2}{18} = \frac{5}{2}$, whose relevant solution is $t = 3$.

Therefore, the correct answer is **B**.

10. In the adjoining diagram, BO bisects $\angle CBA$, CO bisects $\angle ACB$, and MN is parallel to BC. If $AB = 12, BC = 24,$ and $AC = 18,$ then the perimeter of $\triangle AMN$ is



A 30

B 33

C 36

D 39

E 42

Solution:

The sides $12, 18, 24$ have semiperimeter 27 and area

$$\sqrt{27 \cdot 15 \cdot 9 \cdot 3} = 27\sqrt{15},$$

so the inradius is $r = \sqrt{15}$. Taking BC as base, the altitude is $h = \frac{2(27\sqrt{15})}{24} = \frac{9\sqrt{15}}{4}$. Thus the similarity scale from $\triangle ABC$ to $\triangle AMN$ is $1 - \frac{r}{h} = 1 - \frac{4}{9} = \frac{5}{9}$. Its perimeter is $(\frac{5}{9})(12 + 18 + 24) = 30$.

Therefore, the correct answer is **A**.

11. How many integers with four different digits are there between $1{,}000$ and $9{,}999$ such that the absolute value of the difference between the first digit and the last digit is 2?

A 672

B 784

C 840

D 896

E $1{,}008$

Solution:

For leading digits $1, \ldots, 9$, the number of possible last digits differing by 2 is $1 + 1 + 2 + 2 + 2$ plus $2 + 2 + 1 + 1$, or 14. In addition, leading digit 2 may end in 0, giving 15 endpoint pairs. The middle digits can then be chosen in $8 \cdot 7$ ordered ways. Thus the count is $15 \cdot 56 = 840$.

Therefore, the correct answer is **C**.

12. Let $f(x) = ax^7 + bx^3 + cx - 5$, where a, b, and c are constants. If $f(-7) = 7$, then $f(7)$ equals

- A -17
- B -7
- C 14
- D 21
- E not uniquely determined

Solution:

Let $g(x) = ax^7 + bx^3 + cx$, which is odd. Since $f(-7) = g(-7) - 5 = 7$, $g(-7) = 12$. Therefore $g(7) = -12$, and $f(7) = -12 - 5 = -17$.

Therefore, the correct answer is **A**.

13. If $a > 1, b > 1$ and $p = \frac{\log_b(\log_b a)}{\log_b a}$, then a^p equals

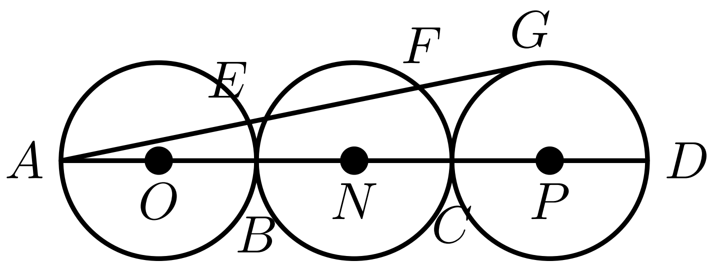
- A 1
- B b
- C $\log_a b$
- D $\log_b a$
- E $a^{\log_b a}$

Solution:

Set $t = \log_b a$, so $a = b^t$ and $p = \frac{\log_b t}{t}$. Then $a^p = (b^t)^{\frac{\log_b t}{t}}$, which equals $b^{\log_b t} = t$. Therefore $a^p = \log_b a$.

Therefore, the correct answer is **D**.

14. In the adjoining figure, points B and C lie on line segment AD, and AB, BC, and CD are diameters of circles O, N, and P, respectively. Circles O, N, and P all have radius 15, and the line AG is tangent to circle P at G. If AG intersects circle N at points E and F, then chord EF has length



A 20

B $15\sqrt{2}$

C 24

D 25

E none of these

Solution:

Here $AP = 75, PG = 15$, and $AG \perp PG$. The distance from N to AG scales with $\frac{AN}{AP} = \frac{45}{75}$, so it is $15(\frac{45}{75}) = 9$. Therefore the chord in the radius-15 circle has length $2\sqrt{15^2 - 9^2} = 2\sqrt{144} = 24$.

Therefore, the correct answer is **C**.

15. Let $\lfloor z \rfloor$ denote the greatest integer not exceeding z. Let x and y satisfy the simultaneous equations

$$y = 2\lfloor x \rfloor + 3,$$
$$y = 3\lfloor x - 2 \rfloor + 5.$$

If x is not an integer, then $x + y$ is

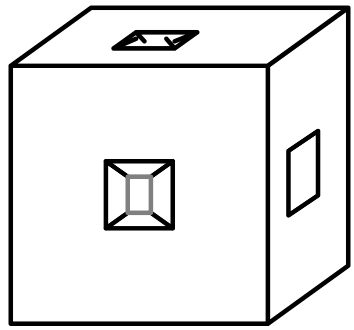
- A an integer
- B between 4 and 5
- C between -4 and 4
- D between 15 and 16
- E 16.5

Solution:

For nonintegral x, let $n = \lfloor x \rfloor$; then $\lfloor x - 2 \rfloor = n - 2$. Thus $2n + 3 = 3(n - 2) + 5$, so $n = 4$ and $y = 11$. Since $4 < x < 5$, we have $15 < x + y < 16$.

Therefore, the correct answer is **D**.

16. In the adjoining figure, a wooden cube has edges of length 3 meters. Square holes of side one meter, centered in each face, are cut through to the opposite face. The edges of the holes are parallel to the edges of the cube. The entire surface area including the inside, in square meters, is



A 54

B 72

C 76

D 84

E 86

Solution:

The six outer faces contribute $6(3^2 - 1^2) = 48$. Each of the three length-3 square tunnels has four inner walls, but the central unit segment of every wall is removed by a perpendicular tunnel, leaving area 2 per wall. Thus the inside contributes $3 \cdot 4 \cdot 2 = 24$. The total is $48 + 24 = 72$.

Therefore, the correct answer is **B**.

17. How many real numbers x satisfy the equation $3^{2x+2} - 3^{x+3} - 3^x + 3 = 0$?

A 0

B 1

C 2

D 3

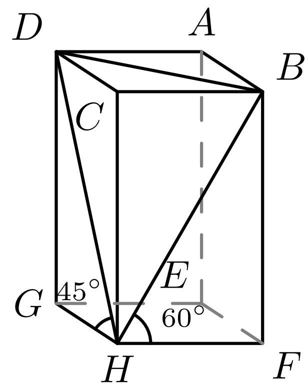
E 4

Solution:

Let $t = 3^x > 0$. Then $9t^2 - 28t + 3 = 0$, whose roots are $t = 3$ and $t = \frac{1}{9}$. Both are positive and each corresponds to one real x, so there are 2 solutions.

Therefore, the correct answer is **C**.

18. In the adjoining figure of a rectangular solid, $\angle DHG = 45^\circ$ and $\angle FHB = 60^\circ$. Find the cosine of $\angle BHD$.



A $\frac{\sqrt{3}}{6}$

B $\frac{\sqrt{2}}{6}$

C $\frac{\sqrt{6}}{3}$

D $\frac{\sqrt{6}}{4}$

E $\frac{\sqrt{6}-\sqrt{2}}{4}$

Solution:

Let $H = (0,0,0), F = (u,0,0), G = (0,v,0)$, and $C = (0,0,w)$. Then $B = (u,0,w)$ and $D = (0,v,w)$. The 60° condition gives $w = \sqrt{3}u$, while the 45° condition gives $v = w$. Hence

$$\begin{aligned}\cos\angle BHD &= \frac{w^2}{\sqrt{u^2+w^2}\sqrt{2w^2}} \\ &= \frac{\sqrt{6}}{4}.\end{aligned}$$

Therefore, the correct answer is **D**.

19. Let $f(x) = |x-2| + |x-4| - |2x-6|$, for $2 \leq x \leq 8$. The sum of the largest and smallest values of $f(x)$ is

A 1

B 2

C 4

D 6

E none of these

Solution:

On $[2, 3]$, $f(x) = 2x - 4$; on $[3, 4]$, $f(x) = 8 - 2x$; and on $[4, 8]$, $f(x) = 0$. Thus the minimum is 0 and the maximum is 2, whose sum is 2.

Therefore, the correct answer is **B**.

20. The number of pairs of positive integers (x, y) which satisfy the equation $x^2 + y^2 = x^3$ is

A 0

B 1

C 2

D not finite

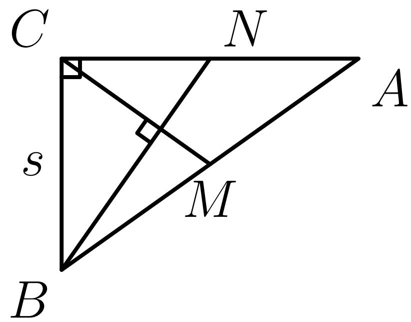
E none of these

Solution:

For every positive integer k, take $x = k^2 + 1$ and $y = k(k^2 + 1) = kx$. Then $y^2 = k^2x^2 = x^2(x - 1)$, so $x^2 + y^2 = x^3$. This supplies infinitely many positive-integer pairs.

Therefore, the correct answer is **D**.

21. In the adjoining figure, the triangle ABC is a right triangle with $\angle BCA = 90^\circ$. Median CM is perpendicular to median BN, and side $BC = s$. The length of BN is



A $s\sqrt{2}$

B $\frac{3}{2}s\sqrt{2}$

C $2s\sqrt{2}$

D $\frac{s\sqrt{5}}{2}$

E $\frac{s\sqrt{6}}{2}$

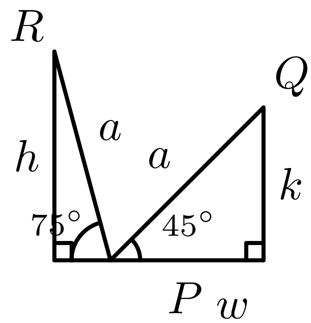
Solution:

Set $C = (0, 0)$, $B = (0, s)$, $A = (t, 0)$. Then $M = (\frac{t}{2}, \frac{s}{2})$ and $N = (\frac{t}{2}, 0)$. Perpendicularity gives $(\frac{t}{2}, \frac{s}{2}) \cdot (\frac{t}{2}, -s) = 0$, hence $t^2 = 2s^2$. Therefore

$$\begin{aligned} BN &= \sqrt{(\frac{t}{2})^2 + s^2} \\ &= \sqrt{\frac{s^2}{2} + s^2} \\ &= \frac{s\sqrt{6}}{2}. \end{aligned}$$

Therefore, the correct answer is **E**.

22. In a narrow alley of width w a ladder of length a is placed with its foot at a point P between the walls. Resting against one wall at Q, a distance k above the ground, the ladder makes a 45° angle with the ground. Resting against the other wall at R, a distance h above the ground, the ladder makes a 75° angle with the ground. The width w is equal to



A a

B RQ

C k

D $\frac{h+k}{2}$

E h

Solution:

The horizontal distances from P to the walls are $a\cos 75^\circ$ and $a\cos 45^\circ$, so $w = a(\cos 75^\circ + \cos 45^\circ)$. The sum-to-product identity gives $\cos 75^\circ + \cos 45^\circ = \sin 75^\circ$. Since $h = a\sin 75^\circ, w = h$.

Therefore, the correct answer is **E**.

23. The lengths of the sides of a triangle are consecutive integers, and the largest angle is twice the smallest angle. The cosine of the smallest angle is

A $\frac{3}{4}$

B $\frac{7}{10}$

C $\frac{2}{3}$

D $\frac{9}{14}$

E none of these

Solution:

Let the consecutive sides be $n, n+1, n+2,$ opposite angles $\theta, \phi, 2\theta$. By the law of sines,

$$\frac{n+2}{n} = \frac{\sin 2\theta}{\sin \theta} = 2\cos\theta.$$

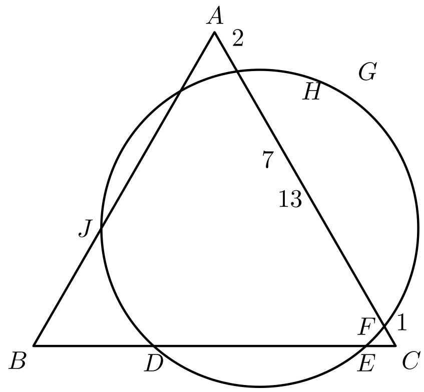
The law of cosines at the smallest angle also gives

$$\cos\theta = \frac{n^2+6n+5}{2(n+1)(n+2)}.$$

Equating these expressions yields $n^2 - 3n - 4 = 0,$ so $n = 4$. Hence the sides are $4, 5, 6$ and $\cos\theta = \frac{6}{2\cdot 4} = \frac{3}{4}$.

Therefore, the correct answer is **A**.

24. In the adjoining figure, the circle meets the sides of an equilateral triangle at six points. If $AG = 2, GF = 13, FC = 1$ and $HJ = 7$, then DE equals



A $2\sqrt{22}$

B $7\sqrt{3}$

C 9

D 10

E 13

Solution:

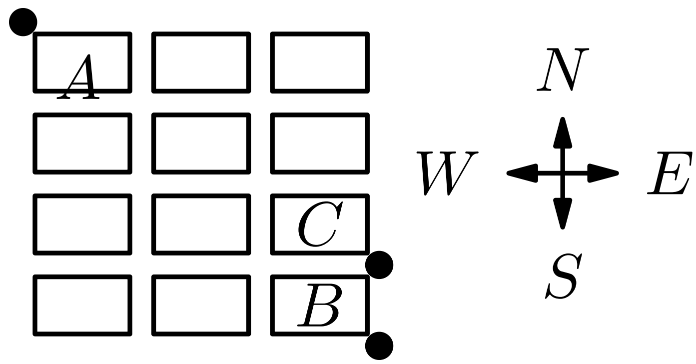
First, power of A gives $AH(AH+7) = AG \cdot AF$. The latter product is $2 \cdot 15 = 30$, so $AH = 3,\ BJ = 6$, and $BH = 13$. Let $BD = u,\ DE = x,\ EC = v$. From the secants at B and C,

$$\begin{aligned} u(u+x) &= BH \cdot BJ \\ &= 13 \cdot 6 = 78, \\ v(v+x) &= CF \cdot CG \\ &= 1 \cdot 14 = 14. \end{aligned}$$

Also $u + x + v = 16$. Subtracting the power equations and using the sum gives $u - v = 4$. Hence $u = 10 - \frac{x}{2}$ and $v = 6 - \frac{x}{2}$. Substitution into $v(v+x) = 14$ gives $x^2 = 88$, so $DE = 2\sqrt{22}$.

Therefore, the correct answer is **A**.

25. The adjoining figure is a map of part of a city: the small rectangles are blocks and the spaces in between are streets. Each morning a student walks from intersection A to intersection B, always walking along streets shown, always going east or south. For variety, at each intersection where he has a choice, he chooses with probability $\frac{1}{2}$ (independent of all other choices) whether to go east or south. Find the probability that, on any given morning, he walks through intersection C.



A $\frac{11}{32}$

B $\frac{1}{2}$

C $\frac{4}{7}$

D $\frac{21}{32}$

E $\frac{3}{4}$

Solution:

If i south moves occur before the third east move, the final step to C is east and the preceding $2+i$ steps contain two east moves. Thus

$$\begin{aligned}\Pr(C) &= \sum_{i=0}^{3}\binom{2+i}{2}2^{-(3+i)}\\ &= \frac{1}{8}+\frac{3}{16}+\frac{6}{32}\\ &\quad+\frac{10}{64}\\ &= \frac{21}{32}.\end{aligned}$$

Therefore, the correct answer is **D**.

26. If the base 8 representation of a perfect square is $ab3c$, where $a \neq 0$, then c is

- A 0
- B 1
- C 3
- D 4
- E not uniquely determined

Solution:

The last two octal digits $3c_8$ represent $24 + c \pmod{64}$. Squares modulo 64 in the range $24, \ldots, 31$ include only 25. Thus $24 + c = 25$, so $c = 1$.

Therefore, the correct answer is **B**.

27. Suppose $z = a + bi$ is a solution of the polynomial equation

$$c_4 z^4 + i c_3 z^3 + c_2 z^2 \\ + i c_1 z + c_0 = 0,$$

where $c_0, c_1, c_2, c_3, c_4, a,$ and b are real constants and $i^2 = -1.$ Which one of the following must also be a solution?

A $-a - bi$

B $a - bi$

C $-a + bi$

D $b + ai$

E none of these

Solution:

Conjugating the equation changes each i to $-i$ and z to $\bar{z}.$ Because the odd-powered terms also change sign when the input is negated, this conjugated equation is precisely the original polynomial evaluated at $-\bar{z}.$ Thus $-\bar{z} = -a + bi$ must be a root.

Therefore, the correct answer is **C**.

28. A set of consecutive positive integers beginning with 1 is written on a blackboard. One number is erased. The average (arithmetic mean) of the remaining numbers is $35\frac{7}{17}$. What number was erased?

A 6

B 7

C 8

D 9

E can not be determined

Solution:

The new average $\frac{602}{17}$ is below the original average $\frac{n+1}{2}$ but differs from it by less than $\frac{n}{n-1}$, forcing $n = 69$ or 70. Since the remaining sum $(n-1)(\frac{602}{17})$ must be integral, $17 \mid n-1$, so $n = 69$. The erased number is

$$\begin{aligned} \text{erased} &= 2415 - 68 \cdot \frac{602}{17} \\ &= 2415 - 2408 \\ &= 7. \end{aligned}$$

Therefore, the correct answer is **B**.

29. Let $x, y,$ and z be three positive real numbers whose sum is $1.$ If no one of these numbers is more than twice any other, then the minimum possible value of the product xyz is

- A $\frac{1}{32}$
- B $\frac{1}{36}$
- C $\frac{4}{125}$
- D $\frac{1}{127}$
- E none of these

Solution:

Order $x \leq y \leq z.$ At a minimum the spread is maximal, so $z = 2x$ and $y = 1 - 3x.$ The ordering requires $\frac{1}{5} \leq x \leq \frac{1}{4}.$ Thus $xyz = 2x^2(1 - 3x).$ Its only interior critical point is a maximum, so compare endpoints: the values are $\frac{4}{125}$ and $\frac{1}{32},$ respectively. The minimum is $\frac{1}{32}.$

Therefore, the correct answer is **A**.

30. Find the units digit in the decimal expansion of

$$(15+\sqrt{220})^{19}+(15+\sqrt{220})^{82}.$$

A 0

B 2

C 5

D 9

E none of these

Solution:

Let $\alpha = 15+\sqrt{220}$ and $\beta = 15-\sqrt{220}$, so $0<\beta<1$. The integers $S_n = \alpha^n + \beta^n$ satisfy $S_n = 30S_{n-1} - 5S_{n-2}$, hence $S_n \equiv 0 \pmod{10}$ for every $n \geq 1$. Therefore

$$\begin{aligned}\alpha^{19}+\alpha^{82} &= S_{19}+S_{82}\\ &\quad -(\beta^{19}+\beta^{82})\end{aligned}$$

is a multiple of 10 minus a positive number less than 1. Its integer part therefore ends in 9.

Therefore, the correct answer is **D**.

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