1981 AMC 12 Solutions

Typeset by: LIVE by Po-Shen Loh
https://live.poshenloh.com/past-contests/amc12/1981/solutions

1. If $\sqrt{x+2}=2$, then $(x+2)^2$ equals

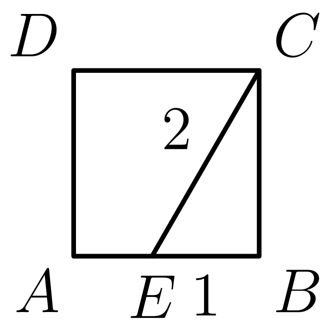
- A $\sqrt{2}$
- B 2
- C 4
- D 8
- E 16

Solution:

From $\sqrt{x+2}=2$, we get $x+2=4$. Therefore $(x+2)^2=4^2=16.$

Therefore, the correct answer is **E**.

2. Point E is on side AB of square $ABCD$. If EB has length one and EC has length two, then the area of the square is



A $\sqrt{3}$

B $\sqrt{5}$

C 3

D $2\sqrt{3}$

E 5

Solution:

Let the square's side length be s. In right triangle EBC,

$$s^2 + 1^2 = 2^2.$$

Hence $s^2 = 3$, which is exactly the area of the square.

Therefore, the correct answer is **C**.

3. For $x \neq 0$, $\frac{1}{x} + \frac{1}{2x} + \frac{1}{3x}$ equals

A $\frac{1}{2x}$

B $\frac{1}{6x}$

C $\frac{5}{6x}$

D $\frac{11}{6x}$

E $\frac{1}{6x^3}$

Solution:

Using the common denominator $6x$ gives

$$\begin{aligned}\frac{1}{x} + \frac{1}{2x} + \frac{1}{3x} &= \frac{6}{6x} + \frac{3}{6x} \\ &\quad + \frac{2}{6x} \\ &= \frac{11}{6x}.\end{aligned}$$

Therefore, the correct answer is **D**.

4. If three times the larger of two numbers is four times the smaller and the difference between the numbers is 8, then the larger of the two numbers is

- A 16
- B 24
- C 32
- D 44
- E 52

Solution:

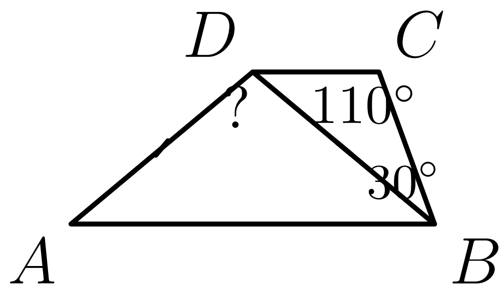
The relation $3L = 4S$ gives $S = \frac{3L}{4}$. Thus

$$L - S = L - \frac{3L}{4} = \frac{L}{4} = 8,$$

so $L = 32.$

Therefore, the correct answer is **C**.

5. In trapezoid $ABCD$, sides AB and CD are parallel, and diagonal BD and side AD have equal length. If $\angle DCB = 110^\circ$ and $\angle CBD = 30^\circ$, then $\angle ADB =$



A 80°

B 90°

C 100°

D 110°

E 120°

Solution:

Since $AB \parallel CD$, consecutive interior angles give $\angle ABC = 180^\circ - 110^\circ = 70^\circ$. Hence $\angle ABD = 70^\circ - 30^\circ = 40^\circ$. Because $AD = BD$, triangle ABD has $\angle DAB = \angle ABD = 40^\circ$. Therefore

$$\begin{aligned}\angle ADB &= 180^\circ - 40^\circ - 40^\circ \\ &= 100^\circ.\end{aligned}$$

Therefore, the correct answer is **C**.

6. If $\frac{x}{x-1} = \frac{y^2+2y-1}{y^2+2y-2}$, then x equals

A $y^2 + 2y - 1$

B $y^2 + 2y - 2$

C $y^2 + 2y + 2$

D $y^2 + 2y + 1$

E $-y^2 - 2y + 1$

Solution:

Let $A = y^2 + 2y - 1$. The equation becomes $\frac{x}{x-1} = \frac{A}{A-1}$. Cross-multiplication gives $xA - x = Ax - A$, so $x = A = y^2 + 2y - 1$.

Therefore, the correct answer is **A**.

7. How many of the first one hundred positive integers are divisible by all of the numbers $2, 3, 4, 5$?

A 0

B 1

C 2

D 3

E 4

Solution:

The least common multiple is $\text{lcm}(2, 3, 4, 5) = 60$. Among the positive integers through 100, only 60 is a multiple of 60. Thus the count is 1.

Therefore, the correct answer is **B**.

8. For all positive numbers x, y, z, the product

$$\begin{aligned}&(x+y+z)^{-1}\\ &\quad\cdot\left(x^{-1}+y^{-1}+z^{-1}\right)\\ &\quad\cdot(xy+yz+zx)^{-1}\\ &\quad\cdot\left((xy)^{-1}+(yz)^{-1}+(zx)^{-1}\right)\end{aligned}$$

equals

A $x^{-2}y^{-2}z^{-2}$

B $x^{-2}+y^{-2}+z^{-2}$

C $(x+y+z)^{-2}$

D $\frac{1}{xyz}$

E $\frac{1}{xy+yz+zx}$

Solution:

We have

$$x^{-1}+y^{-1}+z^{-1}=\frac{xy+yz+zx}{xyz}$$

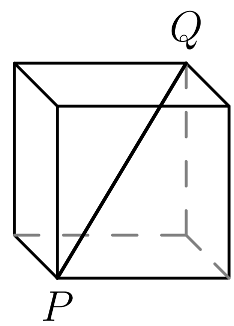
and

$$\begin{aligned}&(xy)^{-1}+(yz)^{-1}\\ &\quad+(zx)^{-1}=\frac{x+y+z}{xyz}.\end{aligned}$$

All symmetric sum factors cancel, leaving $\frac{1}{x^2y^2z^2}=x^{-2}y^{-2}z^{-2}$.

Therefore, the correct answer is **A**.

9. In the adjoining figure, PQ is a diagonal of the cube. If PQ has length a, then the surface area of the cube is



A $2a^2$

B $2\sqrt{2}a^2$

C $2\sqrt{3}a^2$

D $3\sqrt{3}a^2$

E $6a^2$

Solution:

If the side length is s, then the space diagonal is $s\sqrt{3} = a$, so $s^2 = \frac{a^2}{3}$. The surface area is $6s^2 = 6(\frac{a^2}{3}) = 2a^2$.

Therefore, the correct answer is **A**.

10. The lines L and K are symmetric to each other with respect to the line $y = x$. If the equation of line L is $y = ax + b$ with $a \neq 0$ and $b \neq 0$, then the equation of K is $y =$

- A $\frac{1}{a}x + b$
- B $-\frac{1}{a}x + b$
- C $-\frac{1}{a}x - \frac{b}{a}$
- D $\frac{1}{a}x + \frac{b}{a}$
- E $\frac{1}{a}x - \frac{b}{a}$

Solution:

Interchanging x and y in $y = ax + b$ gives $x = ay + b$. Solving,

$$y = \frac{x - b}{a} = \frac{1}{a}x - \frac{b}{a}.$$

Therefore, the correct answer is **E**.

11. The three sides of a right triangle have integral lengths which form an arithmetic progression. One of the sides could have length

- A 22
- B 58
- C 81
- D 91
- E 361

Solution:

If the sides are $s-d, s, s+d$, the Pythagorean equation gives $(s-d)^2+s^2=(s+d)^2$, hence $s=4k$ and $d=k$ for some positive integer k. Thus the sides are $3k, 4k, 5k$. Of the choices, $81=3\cdot 27$ can occur.

Therefore, the correct answer is **C**.

12. If $p, q,$ and M are positive numbers and $q < 100,$ then the number obtained by increasing M by $p\%$ and decreasing the result by $q\%$ exceeds M if and only if

- A $p > q$
- B $p > \frac{q}{100-q}$
- C $p > \frac{q}{1-q}$
- D $p > \frac{100q}{100+q}$
- E $p > \frac{100q}{100-q}$

Solution:

The final amount is

$$M\left(1+\frac{p}{100}\right)\left(1-\frac{q}{100}\right).$$

It exceeds M exactly when $(100+p)(100-q) > 10000.$ Expanding gives $p(100-q) > 100q.$ Since $100-q > 0,$ this is $p > \frac{100q}{100-q}.$

Therefore, the correct answer is **E**.

13. Suppose that at the end of any year, a unit of money has lost 10% of the value it had at the beginning of that year. Find the smallest integer n such that after n years the unit of money will have lost at least 90% of its value. (To the nearest thousandth $\log_{10} 3$ is 0.477.)

A 14

B 16

C 18

D 20

E 22

Solution:

At least 90% lost means $0.9^n \leq 0.1$. Now

$$\begin{aligned} \log_{10} 0.9 &= \log_{10} 9 - 1 \\ &= 2(0.477) - 1 \\ &= -0.046. \end{aligned}$$

Thus $n(-0.046) \leq -1$, so $n \geq 21.739\ldots$. The least integer is 22.

Therefore, the correct answer is **E**.

14. In a geometric sequence of real numbers, the sum of the first two terms is 7, and the sum of the first six terms is 91. The sum of the first four terms is

A 28

B 32

C 35

D 49

E 84

Solution:

If the first term is a and ratio is r, then

$$91 = a(1+r)(1+r^2+r^4).$$

Since $a(1+r) = 7$, we have $1 + r^2 + r^4 = 13$. With $t = r^2 \geq 0, t^2 + t - 12 = 0$, so $t = 3$. The first four terms sum to $a(1+r)(1+r^2) = 7(4) = 28$.

Therefore, the correct answer is **A**.

15. If $b > 1, x > 0$, and $(2x)^{\log_b 2} - (3x)^{\log_b 3} = 0$, then x is

A $\frac{1}{216}$

B $\frac{1}{6}$

C 1

D 6

E not uniquely determined

Solution:

Let $A = \log_b 2, B = \log_b 3$, and $u = \log_b x$. Taking base-b logarithms gives $A(A + u) = B(B + u)$. Since $A \neq B$,

$$A + B + u = 0.$$

Therefore $u = -\log_b 6 = \log_b(\frac{1}{6})$, so $x = \frac{1}{6}$.

Therefore, the correct answer is **B**.

16. The base three representation of x is

$$12112211122211112222.$$

The first digit (on the left) of the base nine representation of x is

A 1

B 2

C 3

D 4

E 5

Solution:

The representation has an even number of base-three digits, so its leftmost base-nine digit comes from 12_3. That pair has value $1 \cdot 3 + 2 = 5$, so the first base-nine digit is 5.

Therefore, the correct answer is **E**.

17. The function f is not defined for $x = 0$, but, for all nonzero real numbers x,

$$f(x) + 2f\left(\frac{1}{x}\right) = 3x.$$

The equation $f(x) = f(-x)$ is satisfied by

A exactly one real number

B exactly two real numbers

C no real numbers

D infinitely many, but not all, nonzero real numbers

E all nonzero real numbers

Solution:

Replacing x by $\frac{1}{x}$ gives $f(\frac{1}{x}) + 2f(x) = \frac{3}{x}$. Solving the two linear equations yields

$$f(x) = \frac{2}{x} - x.$$

Hence $f(-x) = -f(x)$, so equality requires $f(x) = 0$. Thus $\frac{2}{x} - x = 0$, or $x^2 = 2$. Both $x = \sqrt{2}$ and $x = -\sqrt{2}$ work, giving exactly two real numbers.

Therefore, the correct answer is **B**.

18. The number of real solutions to the equation

$$\frac{x}{100} = \sin x$$

is

A 61

B 62

C 63

D 64

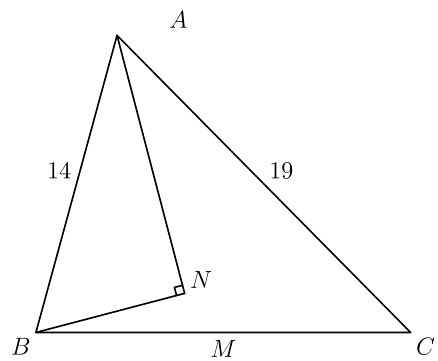
E 65

Solution:

The equation is odd-symmetric and has the solution $x = 0$. On $(0, \pi)$ there is one further positive solution. For each $k = 1, 2, \ldots, 15$, the positive sine hump on $(2k\pi, (2k+1)\pi)$ rises above $\frac{x}{100}$ and then falls below it, giving two solutions. There are no positive solutions beyond 31π, because the next positive hump begins above 100 while $|\sin x| \leq 1$. Thus there are $1 + 2(15) = 31$ positive solutions, the same number negative, and zero itself: $31 + 31 + 1 = 63$.

Therefore, the correct answer is **C**.

19. In $\triangle ABC, M$ is the midpoint of side BC, AN bisects $\angle BAC, BN \perp AN,$ and θ is the measure of $\angle BAC$. If sides AB and AC have lengths 14 and $19,$ respectively, then length MN equals



A 2

B $\frac{5}{2}$

C $\frac{5}{2} - \sin\theta$

D $\frac{5}{2} - \frac{1}{2}\sin\theta$

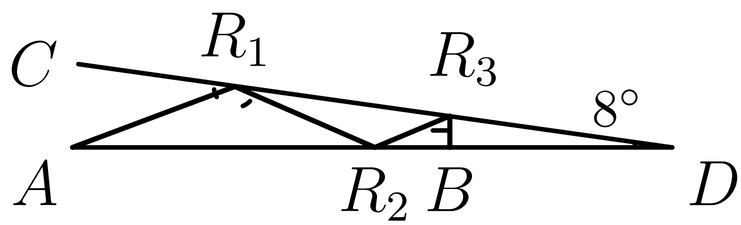
E $\frac{5}{2} - \frac{1}{2}\sin\left(\frac{\theta}{2}\right)$

Solution:

Extend BN to meet AC at E. The right triangles ABN and AEN are congruent because AN is common and bisects the angle at A. Hence BE is bisected by N and $AE = AB = 14$. Since $AC = 19, EC = 5$. In triangle BEC, points N and M are the midpoints of BE and BC, so $MN = \frac{EC}{2} = \frac{5}{2}$.

Therefore, the correct answer is **B**.

20. A ray of light originates from point A and travels in a plane, being reflected n times between lines AD and CD, before striking a point B (which may be on AD or CD) perpendicularly and retracing its path to A. (At each point of reflection the light makes two equal angles as indicated in the adjoining figure. The figure shows the light path for $n = 3$.) If $\angle CDA = 8^\circ$, what is the largest value n can have?



A 6

B 10

C 38

D 98

E There is no largest value.

Solution:

Let $\theta > 0$ be the initial acute angle between the ray and AD. Successive exterior-angle relations increase the corresponding acute angle by 8° at each reflection. At the final perpendicular strike,

$$90^\circ = \theta + (8n + 8)^\circ.$$

Thus $\theta = 82^\circ - 8n^\circ > 0$, so $n < \frac{82}{8} = 10.25$. The greatest integer possible is 10, attained when $\theta = 2^\circ$.

Therefore, the correct answer is **B**.

21. In a triangle with sides of lengths a, b, and c,

$$(a+b+c)(a+b-c) = 3ab.$$

The measure of the angle opposite the side of length c is

A 15°

B 30°

C 45°

D 60°

E 150°

Solution:

The condition gives $(a+b)^2 - c^2 = 3ab$, so

$$c^2 = a^2 + b^2 - ab.$$

If θ is opposite c, the law of cosines says $c^2 = a^2 + b^2 - 2ab\cos\theta$. Therefore $2ab\cos\theta = ab$, so $\cos\theta = \frac{1}{2}$ and $\theta = 60^\circ$.

Therefore, the correct answer is **D**.

22. How many lines in a three-dimensional rectangular coordinate system pass through four distinct points of the form (i, j, k), where i, j, and k are positive integers not exceeding four?

A 60

B 64

C 72

D 76

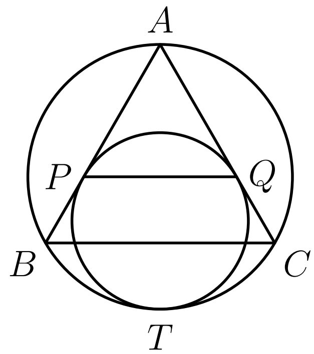
E 100

Solution:

There are $3 \cdot 4^2 = 48$ axis-parallel lines. For face-diagonal directions, choose the pair of varying coordinates in 3 ways, choose one of 2 diagonal slopes, and fix the remaining coordinate in 4 ways, giving $3 \cdot 2 \cdot 4 = 24.$ Finally, the cube has 4 space diagonals. Thus the total is $48 + 24 + 4 = 76.$

Therefore, the correct answer is **D**.

23. Equilateral $\triangle ABC$ is inscribed in a circle. A second circle is tangent internally to the circumcircle at T and tangent to sides AB and AC at points P and Q. If side BC has length 12, then segment PQ has length



A 6

B $6\sqrt{3}$

C 8

D $8\sqrt{3}$

E 9

Solution:

The circumradius of the equilateral triangle is $R = \frac{12}{\sqrt{3}} = 4\sqrt{3}$. If the smaller radius is r, its center lies on the altitude and is $2r$ from A, because its distance to either side is r and the half-angle is $30°$. Internal tangency at the bottom gives $2r = 2R - r$, so $r = \frac{2R}{3} = \frac{8\sqrt{3}}{3}$. The tangency points on the two sides are separated by $PQ = r\sqrt{3} = 8$.

Therefore, the correct answer is **C**.

24. If θ is a constant such that $0 < \theta < \pi$ and $x + \frac{1}{x} = 2\cos\theta$, then for each positive integer n, $x^n + \frac{1}{x^n}$ equals

A $2\cos\theta$

B $2^n \cos\theta$

C $2\cos^n\theta$

D $2\cos n\theta$

E $2^n \cos^n\theta$

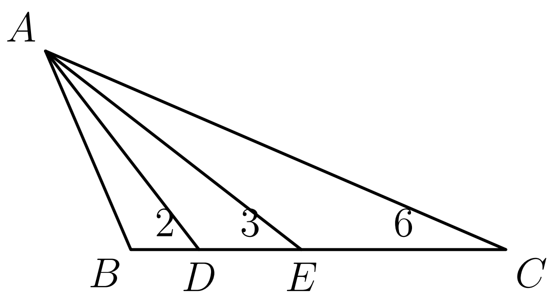
Solution:

The equation is $x^2 - 2x\cos\theta + 1 = 0$, whose roots are $e^{i\theta}$ and $e^{-i\theta}$. Thus $x = e^{\pm i\theta}$ and $x^{-1} = e^{\mp i\theta}$. By De Moivre's theorem,

$$\begin{aligned} x^n + x^{-n} &= e^{in\theta} + e^{-in\theta} \\ &= 2\cos n\theta. \end{aligned}$$

Therefore, the correct answer is **D**.

25. In triangle ABC in the adjoining figure, AD and AE trisect $\angle BAC$. The lengths of $BD, DE,$ and EC are $2, 3,$ and $6,$ respectively. The length of the shortest side of $\triangle ABC$ is



A $2\sqrt{10}$

B 11

C $6\sqrt{6}$

D 6

E not uniquely determined by the given information

Solution:

Let $AB = c$, $AD = y$, and $AC = b$. In triangle ADC, AE bisects $\angle DAC$; the angle-bisector theorem gives

$$\frac{DE}{EC} = \frac{AD}{AC},$$

so $\frac{3}{6} = \frac{y}{b}$ and $b = 2y$. Applying the law of cosines to triangles ADB, ADE, and AEC and equating their common $\cos \alpha$ values gives

$$3c^2 - 2y^2 = 12,$$
$$3c^2 - 4y^2 = -96.$$

Hence $y^2 = 54$ and $c^2 = 40$. The sides are $AB = 2\sqrt{10}$, $BC = 11$, and $AC = 2\sqrt{54} = 6\sqrt{6}$, so the shortest is $2\sqrt{10}$.

Therefore, the correct answer is **A**.

26. Alice, Bob, and Carol repeatedly take turns tossing a die. Alice begins; Bob always follows Alice; Carol always follows Bob; and Alice always follows Carol. Find the probability that Carol will be the first one to toss a six. (The probability of obtaining a six on any toss is $\frac{1}{6}$, independent of the outcome of any other toss.)

A $\frac{1}{3}$

B $\frac{2}{9}$

C $\frac{5}{18}$

D $\frac{25}{91}$

E $\frac{36}{91}$

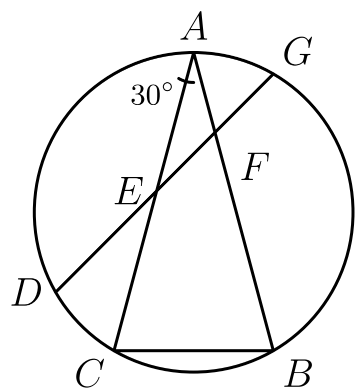
Solution:

Carol wins in the first round with probability $(\frac{5}{6})^2(\frac{1}{6}) = \frac{25}{216}$. A complete round with no six has probability $(\frac{5}{6})^3 = \frac{125}{216}$. Therefore the desired geometric series is

$$\frac{\frac{25}{216}}{1-\frac{125}{216}} = \frac{25}{91}.$$

Therefore, the correct answer is **D**.

27. In the adjoining figure triangle ABC is inscribed in a circle. Point D lies on $\overset{\frown}{AC}$ with $\overset{\frown}{DC} = 30^\circ$, and point G lies on $\overset{\frown}{BA}$ with $\overset{\frown}{BG} > \overset{\frown}{GA}$. Side AB and side AC each have length equal to the length of chord DG, and $\angle CAB = 30^\circ$. Chord DG intersects sides AC and AB at E and F, respectively. The ratio of the area of $\triangle AFE$ to the area of $\triangle ABC$ is



A $\frac{2-\sqrt{3}}{3}$

B $\frac{2\sqrt{3}-3}{3}$

C $7\sqrt{3} - 12$

D $3\sqrt{3} - 5$

E $\frac{9-5\sqrt{3}}{3}$

Solution:

Scale so $AB = AC = DG = 1$. The equal-chord arc relations show that triangle DEC is 30°-60°-90° and $AE = DE$. Put $AE = DE = x$. Then $CE = 1 - x = \frac{2x}{\sqrt{3}}$, so $x = 2\sqrt{3} - 3$. Equal chords also give $AF = FG = EF = \frac{1-x}{2}$. Hence

$$\begin{aligned}\frac{[AFE]}{[ABC]} &= \frac{x(1-x)}{2} \\ &= 7\sqrt{3} - 12.\end{aligned}$$

Therefore, the correct answer is **C**.

28. Consider the set of all equations $x^3 + a_2x^2 + a_1x + a_0 = 0$, where a_2, a_1, a_0 are real constants and $|a_i| < 2$ for $i = 0, 1, 2$. Let r be the largest positive real number which satisfies at least one of these equations. Then

A $1 \leq r < \frac{3}{2}$

B $\frac{3}{2} \leq r < 2$

C $2 \leq r < \frac{5}{2}$

D $\frac{5}{2} \leq r < 3$

E $3 \leq r < \frac{7}{2}$

Solution:

Write an allowed polynomial as g. For $x \geq 0$,

$$\begin{aligned} g(x) &= x^3 + a_2x^2 \\ &\quad + a_1x + a_0 \\ &> f(x), \end{aligned}$$

where $f(x) = x^3 - 2x^2 - 2x - 2$. Thus no positive root can exceed the largest root ρ of f, while coefficients arbitrarily close to -2 make attainable roots arbitrarily close to ρ. Hence ρ is the least upper bound. Since

$$\begin{aligned} f\left(\frac{5}{2}\right) &= -\frac{31}{8} < 0, \\ f(3) &= 1 > 0, \end{aligned}$$

we have $\frac{5}{2} < \rho < 3$.

Therefore, under the intended supremum interpretation, the correct answer is **D**.

29. If $a \geq 1,$ then the sum of the real solutions of

$$\sqrt{a-\sqrt{a+x}}=x$$

is equal to

A $\sqrt{a}-1$

B $\frac{\sqrt{a}-1}{2}$

C $\sqrt{a-1}$

D $\frac{\sqrt{a-1}}{2}$

E $\frac{\sqrt{4a-3}-1}{2}$

Solution:

Since $x \geq 0,$ squaring once gives $a-\sqrt{a+x}=x^2$. Rearranging and using $a+x=(\sqrt{a+x})^2$ yields

$$1=\sqrt{a+x}-x.$$

Hence $\sqrt{a+x}=x+1,$ so $a+x=x^2+2x+1$ and

$$x^2+x+1-a=0.$$

The only nonnegative root is $x=\frac{\sqrt{4a-3}-1}{2}$, and it satisfies the original equation. It is therefore also the sum of all real solutions.

Therefore, the correct answer is **E**.

30. If a, b, c, d are the solutions of the equation $x^4 - bx - 3 = 0$, then an equation whose solutions are

$$\frac{a+b+c}{d^2}, \quad \frac{a+b+d}{c^2}, \quad \frac{a+c+d}{b^2}, \quad \frac{b+c+d}{a^2}$$

is

A $3x^4 + bx + 1 = 0$

B $3x^4 - bx + 1 = 0$

C $3x^4 + bx^3 - 1 = 0$

D $3x^4 - bx^3 - 1 = 0$

E none of these

Solution:

Vieta's formulas give $a + b + c + d = 0$. Thus, for example,

$$\frac{a+b+c}{d^2} = -\frac{1}{d},$$

and similarly the new roots are $-\frac{1}{a}, -\frac{1}{b}, -\frac{1}{c}, -\frac{1}{d}$. Put $x = -\frac{1}{r}$ in $r^4 - br - 3 = 0$. Multiplying by x^4 gives $1 + bx^3 - 3x^4 = 0$, or

$$3x^4 - bx^3 - 1 = 0.$$

Therefore, the correct answer is **D**.

Problems: https://live.poshenloh.com/past-contests/amc12/1981