1980 AMC 12 Solutions

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1. The largest whole number such that seven times the number is less than 100 is

A 12

B 13

C 14

D 15

E 16

Solution:

The number must be less than $\frac{100}{7} = 14 + \frac{2}{7}$. The largest whole number below this value is 14.

Therefore, the correct answer is **C**.

2. The degree of $(x^2+1)^4(x^3+1)^3$ as a polynomial in x is

- A 5
- B 7
- C 12
- D 17
- E 72

Solution:

The first factor has degree $2 \cdot 4 = 8$, and the second has degree $3 \cdot 3 = 9$. Their product therefore has degree $8 + 9 = 17$.

Therefore, the correct answer is **D**.

3. If the ratio of $2x - y$ to $x + y$ is $\frac{2}{3}$, what is the ratio of x to y?

- A $\frac{1}{5}$
- B $\frac{4}{5}$
- C 1
- D $\frac{6}{5}$
- E $\frac{5}{4}$

Solution:

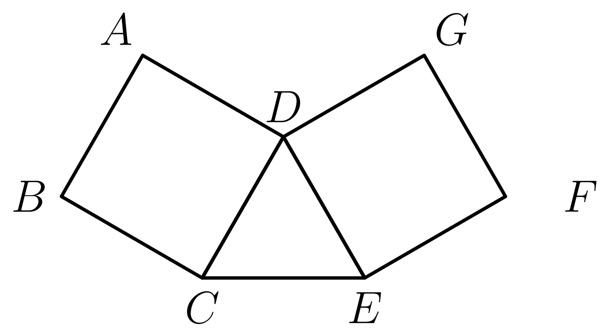
The condition gives

$$\frac{2x - y}{x + y} = \frac{2}{3}.$$

Thus $6x - 3y = 2x + 2y$, so $4x = 5y$ and $\frac{x}{y} = \frac{5}{4}$.

Therefore, the correct answer is **E**.

4. In the adjoining figure, CDE is an equilateral triangle and $ABCD$ and $DEFG$ are squares. The measure of $\angle GDA$ is



A 90°

B 105°

C 120°

D 135°

E 150°

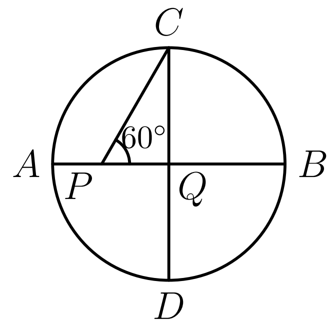
Solution:

At D, the two square angles are 90° each and the equilateral-triangle angle is 60°. Hence

$$\begin{aligned}\angle GDA &= 360^\circ - 90^\circ \\ &\quad - 60^\circ - 90^\circ \\ &= 120^\circ.\end{aligned}$$

Therefore, the correct answer is **C**.

5. If AB and CD are perpendicular diameters of circle Q, and $\angle QPC = 60^\circ$, then the length of PQ divided by the length of AQ is



A $\frac{\sqrt{3}}{2}$

B $\frac{\sqrt{3}}{3}$

C $\frac{\sqrt{2}}{2}$

D $\frac{1}{2}$

E $\frac{2}{3}$

Solution:

Triangle PQC is right at Q and has $\angle P = 60^\circ$, so it is a 30°-60°-90° triangle. Since $QC = AQ$,

$$\frac{PQ}{AQ} = \frac{PQ}{QC} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}.$$

Therefore, the correct answer is **B**.

6. A positive number x satisfies the inequality $\sqrt{x} < 2x$ if and only if

A $x > \frac{1}{4}$

B $x > 2$

C $x > 4$

D $x < \frac{1}{4}$

E $x < 4$

Solution:

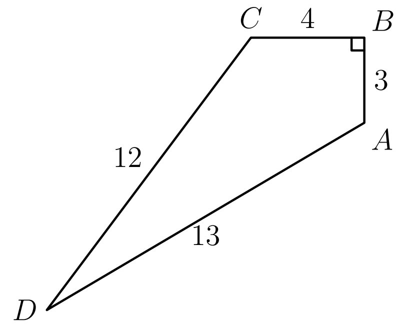
Because $x > 0$, division by $\sqrt{x}$ preserves the inequality:

$$1 < 2\sqrt{x}.$$

Thus $\sqrt{x} > \frac{1}{2}$, which is equivalent to $x > \frac{1}{4}$.

Therefore, the correct answer is **A**.

7. Sides $AB, BC, CD,$ and DA of convex quadrilateral $ABCD$ have lengths $3, 4, 12,$ and $13,$ respectively; and $\angle CBA$ is a right angle. The area of the quadrilateral is



A 32

B 36

C 39

D 42

E 48

Solution:

Triangle ABC is a 3-4-5 right triangle, so $AC = 5$ and its area is $6.$ Since $5^2 + 12^2 = 13^2,$ triangle ACD is right at C and has area $(\frac{1}{2})(5)(12) = 30.$ The quadrilateral's area is $6 + 30 = 36.$

Therefore, the correct answer is **B**.

8. How many pairs (a, b) of nonzero real numbers satisfy the equation

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{a+b}?$$

A none

B 1

C 2

D one pair for each $b \neq 0$

E two pairs for each $b \neq 0$

Solution:

Clearing denominators gives

$$(a+b)^2 = ab,$$

or $a^2 + ab + b^2 = 0$. Since $b \neq 0$, let $t = \frac{a}{b}$. Then $t^2 + t + 1 = 0$, whose discriminant is $1 - 4 = -3$. Thus there are no real pairs.

Therefore, the correct answer is **A**.

9. A man walks x miles due west, turns 150° to his left and walks 3 miles in the new direction. If he finishes at a point $\sqrt{3}$ miles from his starting point, then x is

- A $\sqrt{3}$
- B $2\sqrt{3}$
- C $\frac{3}{2}$
- D 3
- E not uniquely determined by the given information

Solution:

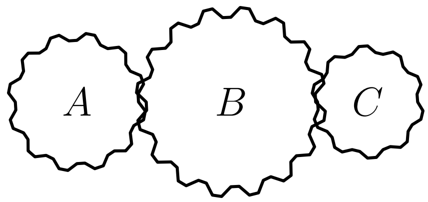
Take east as the positive horizontal direction. After the turn, the 3-mile displacement has components $(\frac{3\sqrt{3}}{2}, -\frac{3}{2})$. Therefore

$$\left(\frac{3\sqrt{3}}{2} - x\right)^2 + \left(\frac{3}{2}\right)^2 = 3.$$

Hence $\frac{3\sqrt{3}}{2} - x = \pm\frac{\sqrt{3}}{2}$, giving $x = \sqrt{3}$ or $x = 2\sqrt{3}$. The value is not unique.

Therefore, the correct answer is **E**.

10. The number of teeth in three meshed circular gears A, B, C are x, y, z, respectively. (The teeth on all gears are the same size and regularly spaced as in the figure.) The angular speeds, in revolutions per minute, of A, B, C are in the proportion



A $x : y : z$

B $z : y : x$

C $y : z : x$

D $yz : xz : xy$

E $xz : yx : zy$

Solution:

The gears' circumferences are proportional to x, y, z, while their tangential speeds have equal magnitude. Their angular speeds are therefore proportional to

$$\frac{1}{x} : \frac{1}{y} : \frac{1}{z}.$$

Multiplying all three terms by xyz gives $yz : xz : xy$.

Therefore, the correct answer is **D**.

11. If the sum of the first 10 terms and the sum of the first 100 terms of a given arithmetic progression are 100 and 10, respectively, then the sum of the first 110 terms is

A 90

B -90

C 110

D -110

E -100

Solution:

The two given sums yield

$$\begin{aligned} 2a + 9d &= 20, \\ 2a + 99d &= \frac{1}{5}. \end{aligned}$$

Subtracting gives $90d = -\frac{99}{5}$, so $d = -\frac{11}{50}$ and $2a = \frac{1099}{50}$. Therefore

$$\begin{aligned} S_{110} &= 55(2a + 109d) \\ &= 55(-2) \\ &= -110. \end{aligned}$$

Therefore, the correct answer is **D**.

12. The equations of L_1 and L_2 are $y = mx$ and $y = nx$, respectively. Suppose L_1 makes twice as large an angle with the horizontal (measured counterclockwise from the positive x-axis) as does L_2, and that L_1 has 4 times the slope of L_2. If L_1 is not horizontal, then mn is

A $\frac{\sqrt{2}}{2}$

B $-\frac{\sqrt{2}}{2}$

C 2

D -2

E not uniquely determined by the given information

Solution:

Let $\tan\theta = n$. Then $m = \tan(2\theta)$ and also $m = 4n$. Hence

$$4n = \frac{2n}{1-n^2}.$$

The nonhorizontal condition gives $n \neq 0$, so $2(1-n^2) = 1$, or $n^2 = \frac{1}{2}$. Thus $mn = 4n^2 = 2$.

Therefore, the correct answer is **C**.

13. A bug (of negligible size) starts at the origin on the coordinate plane. First it moves 1 unit right to $(1,0)$. Then it makes a 90° turn counterclockwise and travels $\frac{1}{2}$ a unit to $(1, \frac{1}{2})$. If it continues in this fashion, each time making a 90° turn counterclockwise and traveling half as far as in the previous move, to which of the following points will it come closest?

A $(\frac{2}{3}, \frac{2}{3})$

B $(\frac{4}{5}, \frac{2}{5})$

C $(\frac{2}{3}, \frac{4}{5})$

D $(\frac{2}{3}, \frac{1}{3})$

E $(\frac{2}{5}, \frac{4}{5})$

Solution:

As complex numbers, the successive displacement vectors are

$$1, \frac{i}{2}, \left(\frac{i}{2}\right)^2, \ldots.$$

Their sum is

$$\frac{1}{1-\frac{i}{2}} = \frac{1+\frac{i}{2}}{1+\frac{1}{4}} = \frac{4}{5} + \frac{2}{5}i.$$

Thus the bug approaches $(\frac{4}{5}, \frac{2}{5})$.

Therefore, the correct answer is **B**.

14. If the function f defined by

$$f(x) = \frac{cx}{2x+3},$$
$$x \neq -\frac{3}{2},$$
$$c \text{ a constant},$$

satisfies $f(f(x)) = x$ for all real numbers x except $-\frac{3}{2}$, then c is

- A -3
- B $-\frac{3}{2}$
- C $\frac{3}{2}$
- D 3
- E not uniquely determined by the given information

Solution:

Direct composition gives

$$f(f(x)) = \frac{c^2 x}{(2c+6)x+9}.$$

For this to equal x identically, the denominator must be the constant c^2. Thus $2c+6=0$ and $c^2=9$, both of which give $c=-3$.

Therefore, the correct answer is **A**.

15. A store prices an item in dollars and cents so that when 4% sales tax is added no rounding is necessary because the result is exactly n dollars, where n is a positive integer. The smallest value of n is

A 1

B 13

C 25

D 26

E 100

Solution:

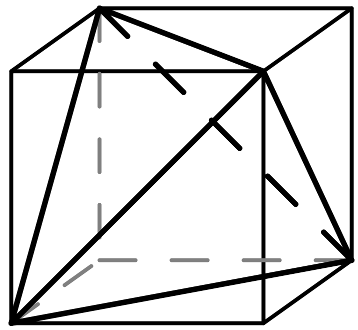
Let the price be p cents. Then

$$\frac{104}{100}p = 100n,$$

so $13p = 1250n$. Since 13 and 1250 are relatively prime, 13 must divide n. The smallest positive possibility is $n = 13$.

Therefore, the correct answer is **B**.

16. Four of the eight vertices of a cube are vertices of a regular tetrahedron. Find the ratio of the surface area of the cube to the surface area of the tetrahedron.



A $\sqrt{2}$

B $\sqrt{3}$

C $\sqrt{\frac{3}{2}}$

D $\frac{2}{\sqrt{3}}$

E 2

Solution:

Let the cube edge be s. Each tetrahedron edge is a cube face diagonal, so it has length $s\sqrt{2}$. The tetrahedron's surface area is

$$4\left(\frac{\sqrt{3}}{4}(s\sqrt{2})^2\right) = 2\sqrt{3}\,s^2.$$

The cube's surface area is $6s^2$, so the ratio is $\frac{6}{2\sqrt{3}} = \sqrt{3}.$

Therefore, the correct answer is **B**.

17. Given that $i^2 = -1$, for how many integers n is $(n+i)^4$ an integer?

A none

B 1

C 2

D 3

E 4

Solution:

Expansion gives

$$(n+i)^4 = n^4 - 6n^2 + 1 \\ + 4n(n^2 - 1)i.$$

The imaginary part vanishes exactly when $n = 0, 1, -1$. For each of these three integers the real part is an integer, so there are 3 values.

Therefore, the correct answer is **D**.

18. If $b > 1, \sin x > 0, \cos x > 0$ and $\log_b \sin x = a$, then $\log_b \cos x$ equals

A $2\log_b(1 - b^{\frac{a}{2}})$

B $\sqrt{1 - a^2}$

C b^{a^2}

D $\frac{1}{2}\log_b(1 - b^{2a})$

E none of these

Solution:

We have $\sin x = b^a$, so

$$\cos x = \sqrt{1 - b^{2a}}.$$

Therefore

$$\log_b \cos x = \frac{1}{2}\log_b(1 - b^{2a}).$$

Therefore, the correct answer is **D**.

19. Let C_1, C_2, and C_3 be three parallel chords of a circle on the same side of the center. The distance between C_1 and C_2 is the same as the distance between C_2 and C_3. The lengths of the chords are $20, 16$, and 8. The radius of the circle is

A 12

B $4\sqrt{7}$

C $\frac{5\sqrt{65}}{3}$

D $\frac{5\sqrt{22}}{2}$

E not uniquely determined by the given information

Solution:

Let r be the radius, u the distance to the 20-unit chord, and v the common spacing. Then

$$r^2 = u^2 + 10^2,$$
$$r^2 = (u+v)^2 + 8^2,$$
$$r^2 = (u+2v)^2 + 4^2.$$

Consecutive subtraction gives $2uv + v^2 = 36$ and $2uv + 3v^2 = 48$, so $v^2 = 6$ and $u = \frac{15}{\sqrt{6}}$. Hence

$$r^2 = u^2 + 100 = \frac{275}{2},$$

and $r = \frac{5\sqrt{22}}{2}$.

Therefore, the correct answer is **D**.

20. A box contains 2 pennies, 4 nickels and 6 dimes. Six coins are drawn without replacement, with each coin having an equal probability of being chosen. What is the probability that the value of the coins drawn is at least 50 cents?

A $\frac{37}{924}$

B $\frac{91}{924}$

C $\frac{127}{924}$

D $\frac{132}{924}$

E none of these

Solution:

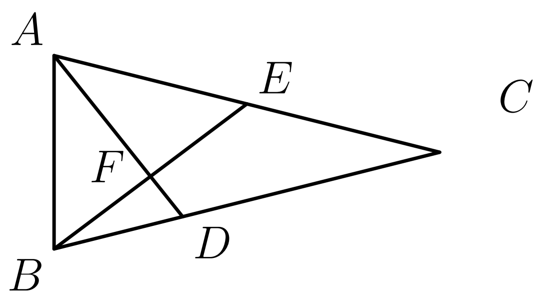
There are $\binom{12}{6} = 924$ selections. A value of at least 50 cents occurs with six dimes; with five dimes and any one other coin; or with four dimes and two nickels. The favorable count is

$$\begin{aligned}&\binom{6}{6}\\ &\quad+\binom{6}{5}\binom{6}{1}\\ &\quad+\binom{6}{4}\binom{4}{2}\\ &=1+36+90\\ &=127.\end{aligned}$$

Thus the probability is $\frac{127}{924}$.

Therefore, the correct answer is **C**.

21. In triangle $ABC, \angle CBA = 72^\circ, E$ is the midpoint of side AC, and D is a point on side BC such that $2BD = DC$; AD and BE intersect at F. The ratio of the area of $\triangle BDF$ to the area of quadrilateral $FDCE$ is



A $\frac{1}{5}$

B $\frac{1}{4}$

C $\frac{1}{3}$

D $\frac{2}{5}$

E none of these

Solution:

Scale the total area of $\triangle ABC$ to 1. Since $BD : DC = 1 : 2$, we have $[ABD] = \frac{1}{3}$ and $[ACD] = \frac{2}{3}$. The midpoint and trisection conditions give $AF : FD = 3 : 1$. Hence

$$[BDF] = \frac{1}{4}[ABD] = \frac{1}{12}.$$

Also

$$\begin{aligned}[AEF] &= \frac{AE}{AC}\frac{AF}{AD}[ACD] \\ &= \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{2}{3} \\ &= \frac{1}{4}.\end{aligned}$$

Therefore $[FDCE] = \frac{2}{3} - \frac{1}{4} = \frac{5}{12}$, and the requested ratio is $\frac{\frac{1}{12}}{\frac{5}{12}} = \frac{1}{5}$.

Therefore, the correct answer is **A**.

22. For each real number x, let $f(x)$ be the minimum of the numbers $4x+1, x+2,$ and $-2x+4.$ Then the maximum value of $f(x)$ is

A $\frac{1}{3}$

B $\frac{1}{2}$

C $\frac{2}{3}$

D $\frac{5}{2}$

E $\frac{8}{3}$

Solution:

The lower envelope is $4x+1$ until it meets $x+2$ at $x=\frac{1}{3}$, then is $x+2$ until that line meets $-2x+4$ at $x=\frac{2}{3}$, and thereafter is $-2x+4.$ Thus its maximum occurs at $x=\frac{2}{3}$ and equals

$$\frac{2}{3}+2=\frac{8}{3}.$$

Therefore, the correct answer is **E**.

23. Line segments drawn from the vertex opposite the hypotenuse of a right triangle to the points trisecting the hypotenuse have lengths $\sin x$ and $\cos x$, where x is a real number such that $0 < x < \frac{\pi}{2}$. The length of the hypotenuse is

- A $\frac{4}{3}$
- B $\frac{3}{2}$
- C $\frac{3\sqrt{5}}{5}$
- D $\frac{2\sqrt{5}}{3}$
- E not uniquely determined by the given information

Solution:

Let the hypotenuse have endpoints $(0,0)$, $(h,0)$ and let the right-angle vertex be (u,v). The right-angle condition gives $u^2+v^2=hu$. The squared distances to $(\frac{h}{3},0)$ and $(\frac{2h}{3},0)$ sum to

$$(u-\frac{h}{3})^2+v^2$$
$$+(u-\frac{2h}{3})^2+v^2=\frac{5h^2}{9}.$$

This also equals $\sin^2 x+\cos^2 x=1$. Hence $h=\frac{3}{\sqrt{5}}=\frac{3\sqrt{5}}{5}$.

Therefore, the correct answer is **C**.

24. For some real number r, the polynomial $8x^3 - 4x^2 - 42x + 45$ is divisible by $(x - r)^2$. Which of the following numbers is closest to r?

- A 1.22
- B 1.32
- C 1.42
- D 1.52
- E 1.62

Solution:

The polynomial factors as

$$\begin{aligned} &8x^3 - 4x^2 - 42x + 45 \\ &\quad = (2x - 3)^2(2x + 5). \end{aligned}$$

Thus the repeated root is $r = \frac{3}{2} = 1.5$, and the nearest listed number is 1.52.

Therefore, the correct answer is **D**.

25. In the nondecreasing sequence of odd integers

$$(a_1, a_2, a_3, \ldots) = (1, 3, 3, 3, \\ 5, 5, 5, 5, 5, \ldots)$$

each positive odd integer k appears k times. It is a fact that there are integers b, c, and d such that, for all positive integers n,

$$a_n = b \left\lfloor \sqrt{n + c} \right\rfloor + d,$$

where $\lfloor x \rfloor$ denotes the largest integer not exceeding x. The sum $b + c + d$ equals

- A 0
- B 1
- C 2
- D 3
- E 4

Solution:

The final occurrence of $2m - 1$ is in position

$$1 + 3 + \cdots + (2m - 1) = m^2.$$

Therefore

$$\begin{aligned} a_n &= 2 \left\lceil \sqrt{n} \right\rceil - 1 \\ &= 2 \left\lfloor \sqrt{n - 1} \right\rfloor + 1. \end{aligned}$$

Thus $b = 2, c = -1, d = 1$, and $b + c + d = 2$.

Therefore, the correct answer is **C**.

26. Four balls of radius 1 are mutually tangent, three resting on the floor and the fourth resting on the others. A tetrahedron, each of whose edges has length s, is circumscribed around the balls. Then s equals

A $4\sqrt{2}$

B $4\sqrt{3}$

C $2\sqrt{6}$

D $1+2\sqrt{6}$

E $2+2\sqrt{6}$

Solution:

The ball centers form a regular tetrahedron of edge 2. Its inradius is

$$d=\frac{2\sqrt{6}}{12}=\frac{\sqrt{6}}{6}.$$

Each face of the circumscribed tetrahedron is parallel to the corresponding face of this center tetrahedron and lies one unit farther from the common center. By similarity,

$$\frac{s}{2}=\frac{d+1}{d}.$$

Hence $s=2+\frac{2}{d}=2+2\sqrt{6}$.

Therefore, the correct answer is **E**.

27. The sum

$$\sqrt[3]{5+2\sqrt{13}}+\sqrt[3]{5-2\sqrt{13}}$$

equals

A $\frac{3}{2}$

B $\frac{\sqrt[3]{65}}{4}$

C $\frac{1+\sqrt[6]{13}}{2}$

D $\sqrt[3]{2}$

E none of these

Solution:

Let the two real cube roots be u, v. Then

$$\begin{aligned} uv &= \sqrt[3]{25-52} = -3, \\ u^3+v^3 &= 10. \end{aligned}$$

If $t = u+v$, then $t^3 = 10 - 9t$, so

$$\begin{aligned} & t^3+9t-10 \\ &= (t-1)(t^2+t+10) \\ &= 0. \end{aligned}$$

The only real solution is $t = 1$, which is not listed.

Therefore, the correct answer is **E**.

28. The polynomial $x^{2n} + 1 + (x+1)^{2n}$ is not divisible by $x^2 + x + 1$ if n equals

A 17

B 20

C 21

D 64

E 65

Solution:

Let $\omega^2 + \omega + 1 = 0$. Since $1 + \omega = -\omega^2$, the polynomial evaluated at ω is

$$\omega^{2n} + 1 + \omega^{4n}.$$

This is 0 unless $\omega^{2n} = 1$, in which case it is 3. The latter occurs exactly when $3 \mid n$. Among the choices only 21 is divisible by 3, so that is the value for which divisibility fails.

Therefore, the correct answer is **C**.

29. How many ordered triples (x, y, z) of integers satisfy the system of equations below?

$$\begin{aligned} x^2 - 3xy + 2y^2 - z^2 &= 31, \\ -x^2 + 6yz + 2z^2 &= 44, \\ x^2 + xy + 8z^2 &= 100. \end{aligned}$$

A 0

B 1

C 2

D a finite number greater than two

E infinitely many

Solution:

Adding the three equations gives

$$\begin{aligned} x^2 - 2xy + 2y^2 + 6yz + 9z^2 \\ = 175, \end{aligned}$$

or

$$(x - y)^2 + (y + 3z)^2 = 175.$$

A square is congruent to 0 or $1 \pmod 4$, so a sum of two squares cannot be congruent to $3 \pmod 4$. But $175 \equiv 3 \pmod 4$, a contradiction. Thus there are no integer triples.

Therefore, the correct answer is **A**.

30. A six digit number (base 10) is *squarish* if it satisfies the following conditions:

(i) none of its digits is zero;

(ii) it is a perfect square; and

(iii) the first two digits, the middle two digits and the last two digits of the number are all perfect squares when considered as two digit numbers.

How many squarish numbers are there?

A 0

B 2

C 3

D 8

E 9

Solution:

Each two-digit block must belong to

$$\{16, 25, 36, 49, 64, 81\}.$$

Restricting the square root by the first block and retaining only squares with an allowed final block leaves the following possible middle blocks:

first block	possible middle blocks
16	$32, 40, 48, 56, 64, 72$
25	$40, 50, 60, 70, 80, 90$
36	$48, 60, 72, 84, 96$
49	$56, 70, 84, 98$
64	$64, 80, 96$
81	$72, 90$

Only the middle block 64 is allowed, producing

$$166464 = 408^2,$$
$$646416 = 804^2.$$

Thus there are 2 squarish numbers.

Therefore, the correct answer is **B**.

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