1979 AMC 12 Solutions

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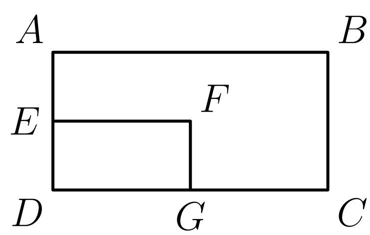
1. If rectangle $ABCD$ has area 72 square meters and E and G are the midpoints of sides AD and CD, respectively, then the area of rectangle $DEFG$ in square meters is



A 8

B 9

C 12

D 18

E 24

Solution:

The side lengths of $DEFG$ are one half the corresponding side lengths of $ABCD$. Its area is therefore $(\frac{1}{2})(\frac{1}{2}) = \frac{1}{4}$ of 72, which is 18.

Therefore, the correct answer is **D**.

2. For all nonzero real numbers x and y such that $x - y = xy$, $\frac{1}{x} - \frac{1}{y}$ equals

A $\frac{1}{xy}$

B $\frac{1}{x-y}$

C 0

D -1

E $y - x$

Solution:

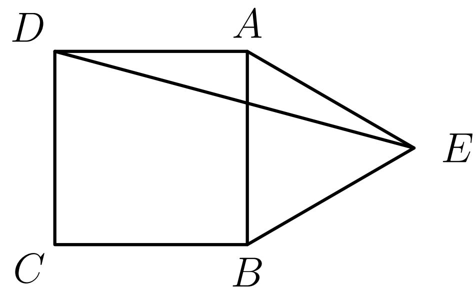
We have

$$\begin{aligned} \frac{1}{x} - \frac{1}{y} &= \frac{y-x}{xy} \\ &= -\frac{x-y}{xy} \\ &= -1, \end{aligned}$$

because $x - y = xy$.

Therefore, the correct answer is **D**.

3. In the adjoining figure, $ABCD$ is a square, ABE is an equilateral triangle and point E is outside square $ABCD$. What is the measure of $\angle AED$ in degrees?



A 10

B 12.5

C 15

D 20

E 25

Solution:

The square and equilateral triangle give $DA = AB = AE$. Also

$$\begin{aligned}\angle DAE &= \angle DAB + \angle BAE \\ &= 90^\circ + 60^\circ \\ &= 150^\circ.\end{aligned}$$

Thus $\triangle DAE$ is isosceles, and each base angle is $\frac{180^\circ - 150^\circ}{2} = 15^\circ$.

Therefore, the correct answer is **C**.

4. For all real numbers $x, x[x\{x(2-x)-4\}+10]+1$ equals

A $-x^4+2x^3+4x^2+10x+1$

B $-x^4-2x^3+4x^2+10x+1$

C $-x^4-2x^3-4x^2+10x+1$

D $-x^4-2x^3-4x^2-10x+1$

E $-x^4+2x^3-4x^2+10x+1$

Solution:

Expanding from the inside gives

$$\begin{aligned} & x[x\{x(2-x)-4\}+10]+1 \\ & \quad = x[x(2x-x^2-4)+10]+1 \\ & \quad = -x^4+2x^3-4x^2 \\ & \qquad +10x+1. \end{aligned}$$

Therefore, the correct answer is **E**.

5. Find the sum of the digits of the largest even three digit number (in base ten representation) which is not changed when its units and hundreds digits are interchanged.

A 22

B 23

C 24

D 25

E 26

Solution:

The equal hundreds and units digit must be even. The largest possible nonzero such digit is 8, and the tens digit can be 9. Thus the largest number is 898, whose digit sum is $8+9+8=25$.

Therefore, the correct answer is **D**.

6. $\frac{3}{2}+\frac{5}{4}+\frac{9}{8}+\frac{17}{16}+\frac{33}{32}+\frac{65}{64}-7=$

A $-\frac{1}{64}$

B $-\frac{1}{16}$

C 0

D $\frac{1}{16}$

E $\frac{1}{64}$

Solution:

Each numerator is one more than its denominator, so the expression is

$$6+\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}\right)-7.$$

The parenthesized geometric sum is $1-\frac{1}{64}=\frac{63}{64}$. Therefore the value is $6+\frac{63}{64}-7=-\frac{1}{64}$.

Therefore, the correct answer is **A**.

7. The square of an integer is called a *perfect square*. If x is a perfect square, the next larger perfect square is

A $x+1$

B x^2+1

C x^2+2x+1

D x^2+x

E $x+2\sqrt{x}+1$

Solution:

Let $x=n^2$ with $n\geq 0$. The next larger perfect square is

$$\begin{aligned}(n+1)^2 &= n^2+2n+1\\ &= x+2\sqrt{x}+1.\end{aligned}$$

Therefore, the correct answer is **E**.

8. Find the area of the smallest region bounded by the graphs of $y = |x|$ and $x^2 + y^2 = 4$.

A $\frac{\pi}{4}$

B $\frac{3\pi}{4}$

C π

D $\frac{3\pi}{2}$

E 2π

Solution:

The two rays meet the circle at angles 45° and 135°, so the smallest bounded region is a 90° sector of the radius-2 circle. Its area is

$$\frac{90^\circ}{360^\circ}\pi(2)^2 = \pi.$$

Therefore, the correct answer is **C**.

9. The product of $\sqrt[3]{4}$ and $\sqrt[4]{8}$ equals

A $\sqrt[7]{12}$

B $2\sqrt[7]{12}$

C $\sqrt[7]{32}$

D $\sqrt[12]{32}$

E $2\sqrt[12]{32}$

Solution:

Writing both radicals as powers of 2 gives

$$\begin{aligned} 4^{\frac{1}{3}}8^{\frac{1}{4}} &= 2^{\frac{2}{3}}2^{\frac{3}{4}} \\ &= 2^{\frac{17}{12}} \\ &= 2\sqrt[12]{2^5} \\ &= 2\sqrt[12]{32}. \end{aligned}$$

Therefore, the correct answer is **E**.

10. If $P_1P_2P_3P_4P_5P_6$ is a regular hexagon whose apothem (distance from the center to the midpoint of a side) is 2, and Q_i is the midpoint of side P_iP_{i+1} for $i = 1, 2, 3, 4$, then the area of quadrilateral $Q_1Q_2Q_3Q_4$ is

- A 6
- B $2\sqrt{6}$
- C $\frac{8\sqrt{3}}{3}$
- D $3\sqrt{3}$
- E $4\sqrt{3}$

Solution:

Let O be the center. Consecutive apothems OQ_i and OQ_{i+1} have length 2 and form a 60° angle, so each $\triangle OQ_iQ_{i+1}$ is equilateral of side 2. The quadrilateral is the union of three such triangles, and hence has area

$$3\left(\frac{\sqrt{3}}{4}\cdot 2^2\right) = 3\sqrt{3}.$$

Therefore, the correct answer is **D**.

11. Find a positive integral solution to the equation

$$\frac{\substack{1+3+5+\cdots\\+(2n-1)}}{\substack{2+4+6+\cdots\\+2n}} = \frac{115}{116}.$$

A 110

B 115

C 116

D 231

E The equation has no positive integral solutions.

Solution:

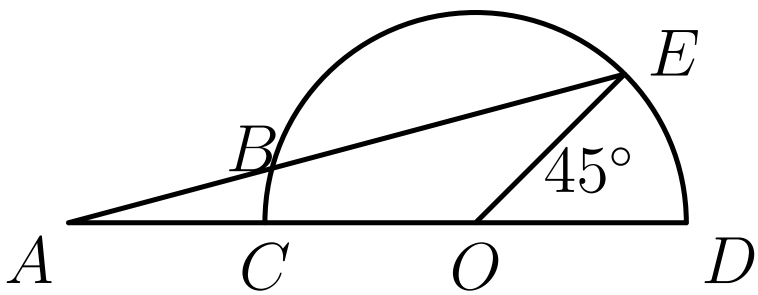
The numerator is n^2, and the denominator is $n(n+1)$. Thus

$$\frac{n}{n+1} = \frac{115}{116},$$

which gives $116n = 115n + 115$, so $n = 115$.

Therefore, the correct answer is **B**.

12. In the adjoining figure, CD is the diameter of a semicircle with center O. Point A lies on the extension of DC past C; point E lies on the semicircle, and B is the point of intersection (distinct from E) of line segment AE with the semicircle. If length AB equals length OD, and the measure of $\angle EOD$ is 45°, then the measure of $\angle BAO$ is



A 10°

B 15°

C 20°

D 25°

E 30°

Solution:

Let $y = \angle BAO$. Since $AB = OB$, $\angle AOB = y$, so $\angle EBO = 2y$. Also $OB = OE$, hence $\angle BEO = 2y$ and $\angle BOE = 180^\circ - 4y$. On the other hand,

$$\angle BOD = 180^\circ - y,$$

so $\angle BOE = 135^\circ - y$. Therefore $180^\circ - 4y = 135^\circ - y$, giving $y = 15^\circ$.

Therefore, the correct answer is **B**.

13. The inequality $y - x < \sqrt{x^2}$ is satisfied if and only if

A $y < 0$ or $y < 2x$ (or both inequalities hold)

B $y > 0$ or $y < 2x$ (or both inequalities hold)

C $y^2 < 2xy$

D $y < 0$

E $x > 0$ and $y < 2x$

Solution:

If $x \geq 0$, the inequality becomes $y - x < x$, or $y < 2x$. If $x < 0$, it becomes $y - x < -x$, or $y < 0$. Combining the two cases gives exactly $y < 0$ or $y < 2x$, with overlap allowed.

Therefore, the correct answer is **A**.

14. In a certain sequence of numbers, the first number is 1, and, for all $n \geq 2$, the product of the first n numbers in the sequence is n^2. The sum of the third and the fifth numbers in the sequence is

A $\frac{25}{9}$

B $\frac{31}{15}$

C $\frac{61}{16}$

D $\frac{576}{225}$

E 34

Solution:

For $n \geq 2$, the nth term is

$$a_n = \frac{n^2}{(n-1)^2}.$$

Therefore $a_3 = \frac{9}{4}$ and $a_5 = \frac{25}{16}$, whose sum is $\frac{36}{16} + \frac{25}{16} = \frac{61}{16}$.

Therefore, the correct answer is **C**.

15. Two identical jars are filled with alcohol solutions, the ratio of the volume of alcohol to the volume of water being $p : 1$ in one jar and $q : 1$ in the other jar. If the entire contents of the two jars are mixed together, the ratio of the volume of alcohol to the volume of water in the mixture is

A $\frac{p+q}{2}$

B $\frac{p^2+q^2}{p+q}$

C $\frac{2pq}{p+q}$

D $\frac{2(p^2+pq+q^2)}{3(p+q)}$

E $\frac{p+q+2pq}{p+q+2}$

Solution:

For equal total volumes, the alcohol fractions are $\frac{p}{p+1}$ and $\frac{q}{q+1}$, while the water fractions are $\frac{1}{p+1}$ and $\frac{1}{q+1}$. Their ratio is

$$\frac{\frac{p}{p+1}+\frac{q}{q+1}}{\frac{1}{p+1}+\frac{1}{q+1}}=\frac{p+q+2pq}{p+q+2}.$$

Therefore, the correct answer is **E**.

16. A circle with area A_1 is contained in the interior of a larger circle with area $A_1 + A_2$. If the radius of the larger circle is 3, and if $A_1, A_2, A_1 + A_2$ is an arithmetic progression, then the radius of the smaller circle is

A $\frac{\sqrt{3}}{2}$

B 1

C $\frac{2}{\sqrt{3}}$

D $\frac{3}{2}$

E $\sqrt{3}$

Solution:

The progression condition gives

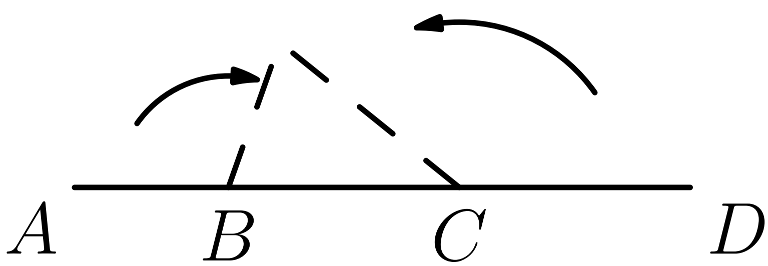
$$2A_2 = A_1 + (A_1 + A_2),$$

so $A_2 = 2A_1$. Hence the larger area is $3A_1 = 9\pi$, and $A_1 = 3\pi$. If the smaller radius is r, then $\pi r^2 = 3\pi$, so $r = \sqrt{3}$.

Therefore, the correct answer is **E**.

17. Points $A, B, C,$ and D are distinct and lie, in the given order, on a straight line. Line segments $AB, AC,$ and AD have lengths $x, y,$ and z, respectively. If line segments AB and CD may be rotated about points B and C, respectively, so that points A and D coincide, to form a triangle with positive area, then which of the following three inequalities must be satisfied?

$$\text{I. } x < \frac{z}{2}$$
$$\text{II. } y < x + \frac{z}{2}$$
$$\text{III. } y < \frac{z}{2}$$



A | I only
B | II only
C | I and II only
D | II and III only
E | I, II and III

Solution:

The new triangle has side lengths $x, y-x, z-y$. Its triangle inequalities give

$$\begin{aligned}
x &< (y-x)+(z-y), \\
&\implies x < \frac{z}{2}; \\
y-x &< x+(z-y), \\
&\implies y < x+\frac{z}{2}; \\
z-y &< x+(y-x), \\
&\implies y > \frac{z}{2}.
\end{aligned}$$

Thus I and II must hold, while III must fail.

Therefore, the correct answer is **C**.

18. To the nearest thousandth, $\log_{10} 2$ is 0.301 and $\log_{10} 3$ is 0.477. Which of the following is the best approximation of $\log_5 10$?

A $\frac{8}{7}$

B $\frac{9}{7}$

C $\frac{10}{7}$

D $\frac{11}{7}$

E $\frac{12}{7}$

Solution:

We have $\log_{10} 5 \approx 1 - 0.301 = 0.699$. Therefore

$$\begin{aligned}\log_5 10 &= \frac{1}{\log_{10} 5} \\ &\approx \frac{1}{0.699} \\ &\approx 1.431.\end{aligned}$$

Of the choices, $\frac{10}{7} \approx 1.429$ is closest.

Therefore, the correct answer is **C**.

19. Find the sum of the squares of all real numbers satisfying the equation $x^{256} - 256^{32} = 0$.

A 8

B 128

C 512

D $65{,}536$

E $2(256^{32})$

Solution:

Since $256 = 2^8$,

$$256^{32} = 2^{256}.$$

Thus $(\frac{x}{2})^{256} = 1$. The real solutions are $x = 2$ and $x = -2$, and the sum of their squares is $4 + 4 = 8$.

Therefore, the correct answer is **A**.

20. If $a = \frac{1}{2}$ and $(a+1)(b+1) = 2$, then the radian measure of $\mathrm{Arctan}\, a + \mathrm{Arctan}\, b$ equals

A $\frac{\pi}{2}$

B $\frac{\pi}{3}$

C $\frac{\pi}{4}$

D $\frac{\pi}{5}$

E $\frac{\pi}{6}$

Solution:

Substituting $a = \frac{1}{2}$ gives $b = \frac{1}{3}$. If $u = \mathrm{Arctan}\, a$ and $v = \mathrm{Arctan}\, b$, then

$$\begin{aligned} \tan(u+v) &= \frac{a+b}{1-ab} \\ &= \frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{6}} \\ &= 1. \end{aligned}$$

Both angles are positive and their sum is less than $\frac{\pi}{2}$, so $u + v = \frac{\pi}{4}$.

Therefore, the correct answer is **C**.

21. The length of the hypotenuse of a right triangle is h, and the radius of the inscribed circle is r. The ratio of the area of the circle to the area of the triangle is

A $\frac{\pi r}{h+2r}$

B $\frac{\pi r}{h+r}$

C $\frac{\pi r}{2h+r}$

D $\frac{\pi r^2}{h^2+r^2}$

E none of these

Solution:

If the legs are x, y, then the right-triangle inradius identity is $x + y = h + 2r$. Hence the semiperimeter is

$$s = \frac{x+y+h}{2} = h + r.$$

The triangle's area is $rs = r(h+r)$, while the circle's area is πr^2. Their ratio is $\frac{\pi r}{h+r}$.

Therefore, the correct answer is **B**.

22. Find the number of pairs (m, n) of integers which satisfy the equation

$$m^3 + 6m^2 + 5m = 27n^3 + 9n^2 + 9n + 1.$$

A 0

B 1

C 3

D 9

E infinitely many

Solution:

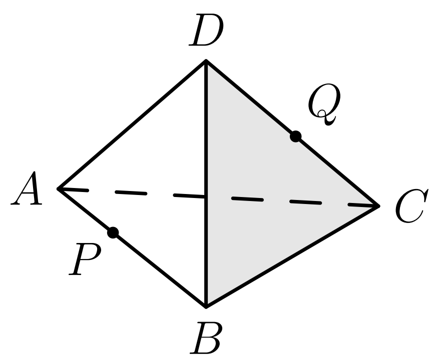
The left side is

$$\begin{aligned} &m(m+1)(m+5) \\ &= m(m+1)(m+2) \\ &\quad +3m(m+1), \end{aligned}$$

which is divisible by 3 for every integer m. The right side is $3(9n^3 + 3n^2 + 3n) + 1$, which is congruent to $1 \pmod 3$. Therefore no integer pair satisfies the equation.

Therefore, the correct answer is **A**.

23. The edges of a regular tetrahedron with vertices $A, B, C,$ and D each have length one. Find the least possible distance between a pair of points P and Q, where P is on edge AB and Q is on edge CD.



A $\frac{1}{2}$

B $\frac{3}{4}$

C $\frac{\sqrt{2}}{2}$

D $\frac{\sqrt{3}}{2}$

E $\frac{\sqrt{3}}{3}$

Solution:

Before scaling, take

$$\begin{aligned} A &= (1, 1, 1), \\ B &= (1, -1, -1), \\ C &= (-1, 1, -1), \\ D &= (-1, -1, 1). \end{aligned}$$

These edges have length $2\sqrt{2}$, so scale by $\frac{1}{2\sqrt{2}}$. Points on AB and CD have forms $P = (1, t, t)$ and $Q = (-1, u, -u)$. Before scaling,

$$\begin{aligned} PQ^2 &= 4 + (t-u)^2 + (t+u)^2 \\ &= 4 + 2t^2 + 2u^2, \end{aligned}$$

minimized at $t = u = 0$. After scaling, the minimum squared distance is $\frac{4}{8} = \frac{1}{2}$, so the distance is $\frac{\sqrt{2}}{2}$.

Therefore, the correct answer is **C**.

24. Sides $AB, BC,$ and CD of (simple) quadrilateral $ABCD$ have lengths $4, 5,$ and $20,$ respectively. If vertex angles B and C are obtuse and $\sin C = -\cos B = \frac{3}{5}$, then side AD has length. (A polygon is called "simple" if it is not self-intersecting.)

A 24

B 24.5

C 24.6

D 24.8

E 25

Solution:

Put $B = (0,0)$ and $C = (5,0)$. Since B is obtuse and $-\cos B = \frac{3}{5}$, we may take

$$\begin{aligned} A &= 4(-\frac{3}{5}, \frac{4}{5}) \\ &= (-\frac{12}{5}, \frac{16}{5}). \end{aligned}$$

The simple configuration and the obtuse angle at C give

$$\begin{aligned} D &= C + 20(\frac{4}{5}, \frac{3}{5}) \\ &= (21, 12). \end{aligned}$$

Thus

$$\begin{aligned} AD &= \sqrt{\begin{matrix} (21 + \frac{12}{5})^2 \\ + (12 - \frac{16}{5})^2 \end{matrix}} \\ &= \sqrt{(\frac{117}{5})^2 + (\frac{44}{5})^2} \\ &= 25. \end{aligned}$$

Therefore, the correct answer is **E**.

25. If $q_1(x)$ and r_1 are the quotient and remainder, respectively, when the polynomial x^8 is divided by $x + \frac{1}{2}$, and if $q_2(x)$ and r_2 are the quotient and remainder, respectively, when $q_1(x)$ is divided by $x + \frac{1}{2}$, then r_2 equals

A $\frac{1}{256}$

B $-\frac{1}{16}$

C 1

D -16

E 256

Solution:

Let $a = -\frac{1}{2}$. Since the first remainder is a^8,

$$\begin{aligned} q_1(x) &= \frac{x^8 - a^8}{x - a} \\ &= x^7 + ax^6 + \cdots + a^7. \end{aligned}$$

Therefore the second remainder is

$$\begin{aligned} q_1(a) &= 8a^7 \\ &= 8(-\frac{1}{2})^7 \\ &= -\frac{1}{16}. \end{aligned}$$

Therefore, the correct answer is **B**.

26. The function f satisfies the functional equation

$$f(x) + f(y) = f(x + y) - xy - 1$$

for every pair x, y of real numbers. If $f(1) = 1$, then the number of integers $n \neq 1$ for which $f(n) = n$ is

A 0

B 1

C 2

D 3

E infinite

Solution:

Setting $x = 1$ gives $f(y + 1) - f(y) = y + 2$. Setting $x = y = 0$ gives $f(0) = -1$. Extending the recurrence in both directions yields, for every integer n,

$$f(n) = \frac{n^2 + 3n - 2}{2}.$$

Thus $f(n) = n$ is equivalent to

$$\begin{aligned} n^2 + n - 2 &= (n + 2)(n - 1) \\ &= 0. \end{aligned}$$

The integer solutions are $n = -2, 1$, and after excluding 1 exactly one remains.

Therefore, the correct answer is **B**.

27. An ordered pair (b, c) of integers, each of which has absolute value less than or equal to five, is chosen at random, with each such ordered pair having an equal likelihood of being chosen. What is the probability that the equation $x^2 + bx + c = 0$ will *not* have distinct positive real roots?

A $\frac{106}{121}$

B $\frac{108}{121}$

C $\frac{110}{121}$

D $\frac{112}{121}$

E none of these

Solution:

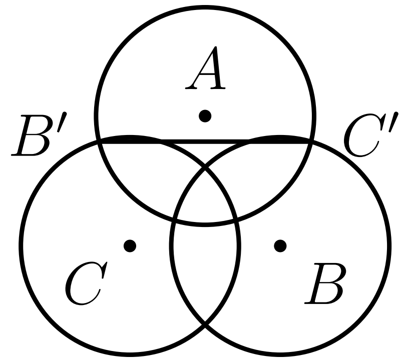
There are $11^2 = 121$ ordered pairs. Distinct positive roots require $b < 0$, $c > 0$, and $b^2 > 4c$. For $b = -1, -2$ there are no choices; for $b = -3, -4, -5$ there are respectively $2, 3, 5$ choices of c. Thus only 10 pairs produce distinct positive roots, so the requested probability is

$$1 - \frac{10}{121} = \frac{111}{121},$$

which is not listed.

Therefore, the correct answer is **E**.

28. Circles with centers A, B, and C each have radius r, where $1 < r < 2$. The distance between each pair of centers is 2. If B' is the point of intersection of circle A and circle C which is outside circle B, and if C' is the point of intersection of circle A and circle B which is outside circle C, then length $B'C'$ equals



A $3r - 2$

B r^2

C $r + \sqrt{3(r-1)}$

D $1 + \sqrt{3(r^2-1)}$

E none of these

Solution:

Let A' be the analogous outer intersection of the circles centered at B, C. Then $A'B'C'$ and ABC are concentric equilateral triangles. If K is their common centroid and M is the midpoint of BC, then

$$\begin{aligned} A'M &= \sqrt{r^2-1}, \\ MK &= \frac{\sqrt{3}}{3}, \\ AK &= \frac{2\sqrt{3}}{3}. \end{aligned}$$

Similarity gives

$$\begin{aligned}\frac{B'C'}{2} &= \frac{A'K}{AK} \\ &= \frac{\sqrt{r^2-1}+\frac{\sqrt{3}}{3}}{\frac{2\sqrt{3}}{3}}.\end{aligned}$$

Hence $B'C' = 1 + \sqrt{3(r^2-1)}$.

Therefore, the correct answer is **D**.

29. For each positive number x, let

$$f(x)=\frac{\left(x+\frac{1}{x}\right)^{6}-\left(x^{6}+\frac{1}{x^{6}}\right)-2}{\left(x+\frac{1}{x}\right)^{3}\left(x^{3}+\frac{1}{x^{3}}\right)}.$$

The minimum value of $f(x)$ is

A 1

B 2

C 3

D 4

E 6

Solution:

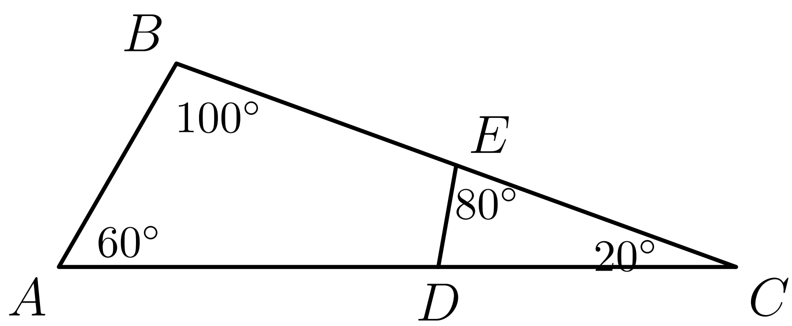
Let $t=x+\frac{1}{x}$ and $u=x^3+x^{-3}=t^3-3t$. Since $x^6+x^{-6}=u^2-2$, the numerator is

$$\begin{aligned} t^6-u^2 &= (t^3-u)(t^3+u) \\ &= 3t(t^3+u). \end{aligned}$$

The denominator is t^3+u, so $f(x)=3t=3(x+\frac{1}{x})$. For $x>0, x+\frac{1}{x}\geq 2,$ with equality at $x=1$. Thus the minimum is 6.

Therefore, the correct answer is **E**.

30. In $\triangle ABC$, E is the midpoint of side BC and D is on side AC. If the length of AC is 1 and $\angle BAC = 60^\circ$, $\angle ABC = 100^\circ$, $\angle ACB = 20^\circ$, and $\angle DEC = 80^\circ$, then the area of $\triangle ABC$ plus twice the area of $\triangle CDE$ equals



A $\frac{1}{4}\cos 10^\circ$

B $\frac{\sqrt{3}}{8}$

C $\frac{1}{4}\cos 40^\circ$

D $\frac{1}{4}\cos 50^\circ$

E $\frac{1}{8}$

Solution:

Extend AB through B to F with $AF = AC = 1$. Then $\triangle ACF$ is equilateral. Choose G on BF so that $\angle BCG = 20^\circ$. The angle conditions give $\triangle FGC \cong \triangle ABC$, while $\triangle BCG \sim \triangle DCE$. Since E is the midpoint of BC, the similarity scale is 2, so

$$[BCG] = 4[CDE].$$

Decomposing the equilateral triangle gives

$$\frac{\sqrt{3}}{4} = 2[ABC] + 4[CDE].$$

Dividing by 2 yields $[ABC] + 2[CDE] = \frac{\sqrt{3}}{8}$.

Therefore, the correct answer is **B**.

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