1978 AMC 12 Solutions

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1. If $1 - \frac{4}{x} + \frac{4}{x^2} = 0$, then $\frac{2}{x}$ equals

A -1

B 1

C 2

D -1 or 2

E -1 or -2

Solution:

With $u = \frac{2}{x}$, the equation becomes $1 - 2u + u^2 = 0$, or $(u-1)^2 = 0$. Therefore $\frac{2}{x} = u = 1$.

Therefore, the correct answer is **B**.

2. If four times the reciprocal of the circumference of a circle equals the diameter of the circle, then the area of the circle is

A $\frac{1}{\pi^2}$

B $\frac{1}{\pi}$

C 1

D π

E π^2

Solution:

The condition is

$$4\left(\frac{1}{2\pi r}\right) = 2r.$$

Multiplying by πr gives $2 = 2\pi r^2$, so the area πr^2 equals 1.

Therefore, the correct answer is **C**.

3. For all nonzero numbers x and y such that $x = \frac{1}{y}$,

$$\left(x - \frac{1}{x}\right)\left(y + \frac{1}{y}\right)$$

equals

- A $2x^2$
- B $2y^2$
- C $x^2 + y^2$
- D $x^2 - y^2$
- E $y^2 - x^2$

Solution:

Since $x = \frac{1}{y}$, we also have $y = \frac{1}{x}$. Thus the product is

$$(x - y)(y + x) = x^2 - y^2.$$

Therefore, the correct answer is **D**.

4. If $a = 1, b = 10, c = 100$ and $d = 1000$, then

$$\begin{aligned}&(a+b+c-d)+(a+b-c+d)\\&\quad+(a-b+c+d)\\&\quad+(-a+b+c+d)\end{aligned}$$

is equal to

A 1111

B 2222

C 3333

D 1212

E 4242

Solution:

Each variable appears positively three times and negatively once, so the expression is

$$\begin{aligned}&2(a+b+c+d)\\&=2a+2b+2c+2d\\&=2+20+200+2000\\&=2222.\end{aligned}$$

Therefore, the correct answer is **B**.

5. Four boys bought a boat for $\$60$. The first boy paid one half of the sum of the amounts paid by the other boys; the second boy paid one third of the sum of the amounts paid by the other boys; and the third boy paid one fourth of the sum of the amounts paid by the other boys. How much did the fourth boy pay?

A $\$10$

B $\$12$

C $\$13$

D $\$14$

E $\$15$

Solution:

Let the first three payments be w, x, y. Then

$$\begin{aligned} w &= \frac{1}{2}(60 - w), \\ x &= \frac{1}{3}(60 - x), \\ y &= \frac{1}{4}(60 - y). \end{aligned}$$

Hence $w = 20, x = 15, y = 12$. The fourth payment is $60 - 20 - 15 - 12 = 13$ dollars.

Therefore, the correct answer is **C**.

6. The number of distinct pairs (x, y) of real numbers satisfying both of the following equations:

$$x = x^2 + y^2,$$
$$y = 2xy.$$

is

A 0

B 1

C 2

D 3

E 4

Solution:

If $y = 0$, the first equation gives $x = x^2$, so $(x, y) = (0, 0)$ or $(1, 0)$. If $y \neq 0$, the second equation gives $x = \frac{1}{2}$. The first then gives $y^2 = \frac{1}{4}$, producing $(\frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{2}, -\frac{1}{2})$. There are four pairs.

Therefore, the correct answer is **E**.

7. Opposite sides of a regular hexagon are 12 inches apart. The length of each side, in inches, is

A 7.5

B $6\sqrt{2}$

C $5\sqrt{2}$

D $\frac{9}{2}\sqrt{3}$

E $4\sqrt{3}$

Solution:

The distance between opposite sides is twice the apothem, hence $s\sqrt{3}$. Therefore $s\sqrt{3} = 12$, so $s = \frac{12}{\sqrt{3}} = 4\sqrt{3}$.

Therefore, the correct answer is **E**.

8. If $x \neq y$ and the sequences x, a_1, a_2, y and x, b_1, b_2, b_3, y each are in arithmetic progression, then $\frac{a_2 - a_1}{b_2 - b_1}$ equals

A $\frac{2}{3}$

B $\frac{3}{4}$

C 1

D $\frac{4}{3}$

E $\frac{3}{2}$

Solution:

The first common difference is $\frac{y-x}{3}$, and the second is $\frac{y-x}{4}$. Since $x \neq y$, their ratio is

$$\frac{\frac{y-x}{3}}{\frac{y-x}{4}} = \frac{4}{3}.$$

Therefore, the correct answer is **D**.

9. If $x < 0$, then $\left|x - \sqrt{(x-1)^2}\right|$ equals

A 1

B $1 - 2x$

C $-2x - 1$

D $1 + 2x$

E $2x - 1$

Solution:

Because $x - 1 < 0$, $\sqrt{(x-1)^2} = |x-1| = 1 - x$. Therefore

$$\begin{aligned} |x - (1 - x)| &= |2x - 1| \\ &= 1 - 2x, \end{aligned}$$

since $2x - 1 < 0$.

Therefore, the correct answer is **B**.

10. If B is a point on circle C with center $P,$ then the set of all points A in the plane of circle C such that the distance between A and B is less than or equal to the distance between A and any other point on circle C is

- A the line segment from P to B
- B the ray beginning at P and passing through B
- C a ray beginning at B
- D a circle whose center is P
- E a circle whose center is B

Solution:

For any $A \neq P,$ the closest point of the circle to A is where the ray from P through A meets the circle. This point is B exactly when A lies on the ray from P through $B.$ The center P also qualifies because every point on the circle is equally distant from it. Thus the locus is that ray.

Therefore, the correct answer is **B**.

11. If r is positive and the line whose equation is $x+y=r$ is tangent to the circle whose equation is $x^2+y^2=r$, then r equals

A $\frac{1}{2}$

B 1

C 2

D $\sqrt{2}$

E $2\sqrt{2}$

Solution:

The distance from the origin to $x+y-r=0$ is $\frac{r}{\sqrt{2}}$. Tangency requires this to equal the radius $\sqrt{r}$, so

$$\frac{r}{\sqrt{2}}=\sqrt{r}.$$

Since $r>0$, squaring and dividing by r gives $r=2$.

Therefore, the correct answer is **C**.

12. In $\triangle ADE, \angle ADE = 140^\circ$, points B and C lie on sides AD and AE, respectively, and points A, B, C, D, E are distinct. If lengths AB, BC, CD, and DE are all equal, then the measure of $\angle EAD$ is

- A 5°
- B 6°
- C 7.5°
- D 8°
- E 10°

Solution:

Let $x = \angle BAC = \angle BCA, y = \angle CBD = \angle CDB$, and $z = \angle DCE = \angle DEC$. The exterior-angle theorem applied successively gives

$$y = 2x, \qquad z = x + y = 3x.$$

The angles of $\triangle ADE$ are $x, 140^\circ, z$, so $x + 140^\circ + 3x = 180^\circ$. Hence $x = 10^\circ$.

Therefore, the correct answer is **E**.

13. If $a, b, c,$ and d are nonzero numbers such that c and d are the solutions of $x^2 + ax + b = 0$ and a and b are the solutions of $x^2 + cx + d = 0,$ then $a + b + c + d$ equals

A 0

B -2

C 2

D 4

E $\frac{-1+\sqrt{5}}{2}$

Solution:

Vieta's formulas give

$$\begin{aligned} c + d &= -a, \quad cd = b, \\ a + b &= -c, \quad ab = d. \end{aligned}$$

The two sum equations imply $b = d$. Since $b = d \neq 0,$ the product equations give $a = c = 1$. Then $a + b = -c$ gives $b = d = -2,$ and therefore $a + b + c + d = -2.$

Therefore, the correct answer is **B**.

14. If an integer n, greater than 8, is a solution of the equation $x^2 - ax + b = 0$ and the representation of a in the base n numeration system is 18, then the base n representation of b is

A 18

B 28

C 80

D 81

E 280

Solution:

In ordinary notation, $a = (18)_n = n + 8$. Since one root is n, the other root must be 8. Their product is $b = 8n$, whose base-n representation is 80.

Therefore, the correct answer is **C**.

15. If $\sin x + \cos x = \frac{1}{5}$ and $0 \leq x < \pi$, then $\tan x$ is

A $-\frac{4}{3}$

B $-\frac{3}{4}$

C $\frac{3}{4}$

D $\frac{4}{3}$

E not completely determined by the given information

Solution:

Squaring gives $1 + 2\sin x \cos x = \frac{1}{25}$, so $\sin x \cos x = -\frac{12}{25}$. Thus $\sin x$ and $\cos x$ are the roots of

$$t^2 - \frac{1}{5}t - \frac{12}{25} = 0,$$

namely $\frac{4}{5}$ and $-\frac{3}{5}$. Because $\sin x \geq 0$ on the given interval, $\sin x = \frac{4}{5}$ and $\cos x = -\frac{3}{5}$. Hence $\tan x = -\frac{4}{3}$.

Therefore, the correct answer is **A**.

16. In a room containing N people, $N > 3,$ at least one person has not shaken hands with everyone else in the room. What is the maximum number of people in the room that could have shaken hands with everyone else?

- A 0
- B 1
- C $N - 1$
- D N
- E none of these

Solution:

If one person has missed a handshake, the other person in that missed pair also has not shaken hands with everyone. Thus at most $N - 2$ people can have shaken hands with everyone. This is attainable when exactly two people fail to shake hands with each other and every other handshake occurs. Since $N - 2$ is not listed, the answer is "none of these."

Therefore, the correct answer is **E**.

17. If k is a positive number and f is a function such that, for every positive number x,

$$\left[f\left(x^2+1\right)\right]^{\sqrt{x}}=k,$$

then, for every positive number y,

$$\left[f\left(\frac{9+y^2}{y^2}\right)\right]^{\sqrt{\frac{12}{y}}}$$

is equal to

- A $\sqrt{k}$
- B $2k$
- C $k\sqrt{k}$
- D k^2
- E $y\sqrt{k}$

Solution:

Set $x=\frac{3}{y}$, which is positive. Then

$$x^2+1=\frac{9+y^2}{y^2},$$
$$\sqrt{\frac{12}{y}}=2\sqrt{x}.$$

Therefore the requested expression is

$$\left(\left[f\left(x^2+1\right)\right]^{\sqrt{x}}\right)^2=k^2.$$

Therefore, the correct answer is **D**.

18. What is the smallest positive integer n such that $\sqrt{n} - \sqrt{n-1} < 0.01$?

A 2499

B 2500

C 2501

D $10{,}000$

E There is no such integer.

Solution:

Rationalizing gives

$$\sqrt{n} - \sqrt{n-1} = \frac{1}{\sqrt{n} + \sqrt{n-1}}.$$

For $n = 2500$, the denominator is $50 + \sqrt{2499} < 100$, so the difference exceeds 0.01. For $n = 2501$, the denominator is $\sqrt{2501} + 50 > 100$, so the difference is less than 0.01. The denominator increases with n, making 2501 the least such integer.

Therefore, the correct answer is **C**.

19. A positive integer n not exceeding 100 is chosen in such a way that if $n \leq 50$, then the probability of choosing n is p, and if $n > 50$, then the probability of choosing n is $3p$. The probability that a perfect square is chosen is

- A 0.05
- B 0.065
- C 0.08
- D 0.09
- E 0.1

Solution:

The total probability is $50p + 50(3p) = 200p = 1$, so $p = 0.005$. There are seven perfect squares at most 50 and three more, $64, 81, 100$, above 50. Hence the desired probability is

$$7p + 3(3p) = 16p = 0.08.$$

Therefore, the correct answer is **C**.

20. If a, b, c are nonzero real numbers such that

$$\frac{a+b-c}{c} = \frac{a-b+c}{b} = \frac{-a+b+c}{a},$$

and

$$x = \frac{(a+b)(b+c)(c+a)}{abc},$$

and $x < 0$, then x equals

A -1

B -2

C -4

D -6

E -8

Solution:

Let the common value be t. The first two resulting equations imply

$$(b-c)(t+2) = 0,$$

and cyclic comparisons give the analogous relations. Thus either $a = b = c$, which gives $x = 8$, or $t = -2$. In the latter case $a + b = -c, b + c = -a$, and $c + a = -b$, so

$$x = \frac{(-c)(-a)(-b)}{abc} = -1.$$

The condition $x < 0$ selects the latter value.

Therefore, the correct answer is **A**.

21. For all positive numbers x distinct from 1,

$$\frac{1}{\log_3 x} + \frac{1}{\log_4 x} + \frac{1}{\log_5 x}$$

equals

A $\frac{1}{\log_{60} x}$

B $\frac{1}{\log_x 60}$

C $\frac{1}{(\log_3 x)(\log_4 x)(\log_5 x)}$

D $\frac{12}{(\log_3 x)+(\log_4 x)+(\log_5 x)}$

E $\frac{\log_2 x}{(\log_3 x)(\log_5 x)} + \frac{\log_3 x}{(\log_2 x)(\log_5 x)} + \frac{\log_5 x}{(\log_2 x)(\log_3 x)}$

Solution:

Let S denote the given sum. Changing bases and combining logarithms gives

$$\begin{aligned} S &= \frac{1}{\log_3 x} + \frac{1}{\log_4 x} \\ &\quad + \frac{1}{\log_5 x} \\ &= \log_x(3 \cdot 4 \cdot 5) \\ &= \log_x 60 \\ &= \frac{1}{\log_{60} x}. \end{aligned}$$

Therefore, the correct answer is **A**.

22. The following four statements, and only these, are found on a card:

On this card exactly one statement is false.
On this card exactly two statements are false.
On this card exactly three statements are false.
On this card exactly four statements are false.

(Assume each statement on the card is either true or false.) Among them the number of false statements is exactly

A 0

B 1

C 2

D 3

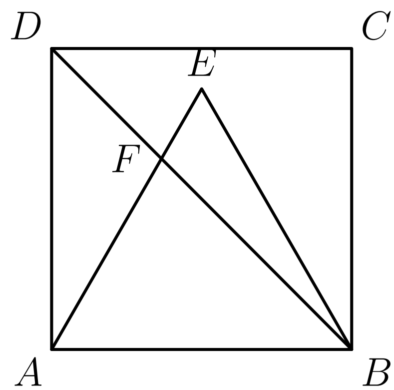
E 4

Solution:

If the actual number m of false statements is one of $1, 2, 3, 4,$ exactly one displayed statement—the one naming m—is true. Therefore exactly three statements are false, forcing $m = 3.$ This is consistent: the third statement is true and the other three are false.

Therefore, the correct answer is **D**.

23. Vertex E of equilateral triangle ABE is in the interior of square $ABCD$, and F is the point of intersection of diagonal BD and line segment AE. If length AB is $\sqrt{1+\sqrt{3}}$, then the area of $\triangle ABF$ is



A 1

B $\frac{\sqrt{2}}{2}$

C $\frac{\sqrt{3}}{2}$

D $4-2\sqrt{3}$

E $\frac{1}{2}+\frac{\sqrt{3}}{4}$

Solution:

Let $A=(0,0)$, $B=(s,0)$, and $D=(0,s)$, where $s=\sqrt{1+\sqrt{3}}$. The two lines containing F have equations

$$\begin{aligned} AE: &\quad y=\sqrt{3}x, \\ BD: &\quad y=s-x. \end{aligned}$$

Hence the altitude of F above AB is $\frac{s\sqrt{3}}{1+\sqrt{3}}$. Therefore

$$[\triangle ABF]=\frac{s^2\sqrt{3}}{2(1+\sqrt{3})}=\frac{\sqrt{3}}{2}.$$

Therefore, the correct answer is **C**.

24. If the distinct nonzero numbers $x(y-z), y(z-x), z(x-y)$ form a geometric progression with common ratio r, then r satisfies the equation

A $r^2+r+1=0$

B $r^2-r+1=0$

C $r^4+r^2-1=0$

D $(r+1)^4+r=0$

E $(r-1)^4+r=0$

Solution:

The three expressions have sum

$$x(y-z)+y(z-x)$$
$$+z(x-y)=0.$$

Writing the nonzero geometric progression as u, ur, ur^2 gives $u(1+r+r^2)=0$. Since $u \neq 0, r^2+r+1=0$.

Therefore, the correct answer is **A**.

25. Let a be a positive number. Consider the set S of all points whose rectangular coordinates (x, y) satisfy all of the following conditions:

$$\text{(i) } \frac{a}{2} \leq x \leq 2a$$
$$\text{(ii) } \frac{a}{2} \leq y \leq 2a$$
$$\text{(iii) } x + y \geq a$$
$$\text{(iv) } x + a \geq y$$
$$\text{(v) } y + a \geq x.$$

The boundary of set S is a polygon with

A 3 sides

B 4 sides

C 5 sides

D 6 sides

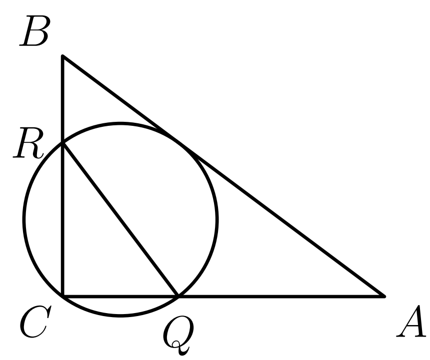
E 7 sides

Solution:

The first two conditions form the square $[\frac{a}{2}, 2a] \times [\frac{a}{2}, 2a]$. Within this square, $x + y \geq a$ is automatic. The last two conditions are equivalent to $|x - y| \leq a$; their boundary lines cut off the corners $(2a, \frac{a}{2})$ and $(\frac{a}{2}, 2a)$. Cutting two opposite corners from a square produces a hexagon, so the boundary has six sides.

Therefore, the correct answer is **D**.

26. In $\triangle ABC, AB = 10, AC = 8,$ and $BC = 6.$ Circle P is the circle with smallest radius which passes through C and is tangent to $AB.$ Let Q and R be the points of intersection, distinct from $C,$ of circle P with sides AC and $BC,$ respectively. The length of segment QR is



A 4.75

B 4.8

C 5

D $4\sqrt{2}$

E $3\sqrt{5}$

Solution:

Let H be the foot of the altitude from C to $AB.$ Among circles through C tangent to $AB,$ the least radius occurs when the tangency point is $H,$ so CH is a diameter. The area of the right triangle gives

$$CH = \frac{AC \cdot BC}{AB} = \frac{8 \cdot 6}{10} = 4.8.$$

Also $\angle QCR = 90^\circ,$ so QR is a diameter of circle $P.$ Therefore $QR = CH = 4.8.$

Therefore, the correct answer is **B**.

27. There is more than one integer greater than 1 which, when divided by any integer k such that $2 \leq k \leq 11,$ has a remainder of $1.$ What is the difference between the two smallest such integers?

A 2310

B 2311

C $27{,}720$

D $27{,}721$

E none of these

Solution:

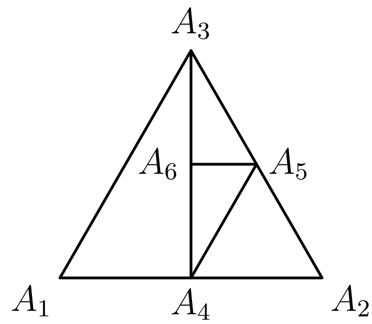
A qualifying integer is congruent to 1 modulo every integer from 2 through $11,$ hence modulo

$$\begin{aligned} L &= \operatorname{lcm}(2, \ldots, 11) \\ &= 2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \\ &= 27720. \end{aligned}$$

The two smallest qualifying integers greater than 1 are $L+1$ and $2L+1,$ whose difference is $L = 27720.$

Therefore, the correct answer is **C**.

28. If $\triangle A_1A_2A_3$ is equilateral and A_{n+3} is the midpoint of line segment A_nA_{n+1} for all positive integers n, then the measure of $\angle A_{44}A_{45}A_{43}$ equals



A 30°

B 45°

C 60°

D 90°

E 120°

Solution:

Let $\mathbf{d}_n = \overrightarrow{A_nA_{n+1}}$. The midpoint rule gives $\mathbf{d}_{n+3} = \frac{1}{2}(\mathbf{d}_n + \mathbf{d}_{n+1})$ and also $\mathbf{d}_n + \mathbf{d}_{n+1} + \mathbf{d}_{n+2} = \frac{1}{2}\mathbf{d}_n$. Consequently,

$$\begin{aligned}\mathbf{d}_{n+4} &= \frac{1}{2}(\mathbf{d}_{n+1} + \mathbf{d}_{n+2}) \\ &= -\frac{1}{4}\mathbf{d}_n.\end{aligned}$$

Thus $\mathbf{d}_{43}$ and $\mathbf{d}_{44}$ are the same positive scalar multiple of $\mathbf{d}_3$ and $\mathbf{d}_4$, respectively, so $\angle A_{44}A_{45}A_{43} = \angle A_4A_5A_3$. Since A_4 and A_5 are the midpoints of A_1A_2 and A_2A_3, $A_4A_5 \parallel A_1A_3$. The equilateral-triangle angles then give $\angle A_4A_5A_3 = 120^\circ$.

Therefore, the correct answer is **E**.

29. Sides $AB, BC, CD,$ and $DA,$ respectively, of convex quadrilateral $ABCD$ are extended past $B, C, D,$ and A to points $B', C', D',$ and A'. Also, $AB = BB' = 6,$ $BC = CC' = 7, CD = DD' = 8,$ and $DA = AA' = 9;$ and the area of $ABCD$ is 10. The area of $A'B'C'D'$ is

- A 20
- B 40
- C 45
- D 50
- E 60

Solution:

Using position vectors, the equal extensions give

$$\begin{aligned} A' &= 2A - D, \quad B' = 2B - A, \\ C' &= 2C - B, \quad D' = 2D - C. \end{aligned}$$

Substitution into the oriented polygon-area sum shows that the diagonal cross terms cancel:

$$\begin{aligned} &A' \times B' + B' \times C' \\ &\quad + C' \times D' + D' \times A' \\ &= 5A \times B + 5B \times C \\ &\quad + 5C \times D + 5D \times A. \end{aligned}$$

Hence the outer area is five times the original area, or $5 \cdot 10 = 50.$

Therefore, the correct answer is **D**.

30. In a tennis tournament, n women and $2n$ men play, and each player plays exactly one match with every other player. If there are no ties and the ratio of the number of matches won by women to the number of matches won by men is $\frac{7}{5}$, then n equals

A 2

B 4

C 6

D 7

E none of these

Solution:

There are $\frac{3n(3n-1)}{2}$ matches, so women must win $\frac{7n(3n-1)}{8}$ matches. If women win k of the $2n^2$ mixed matches, then

$$\frac{n(n-1)}{2}+k=\frac{7n(3n-1)}{8},$$

giving $k=\frac{n(17n-3)}{8}$. The bound $k \leq 2n^2$ forces $n \leq 3.$ For $n=1,2,$ this formula is not an integer, while $n=3$ gives $k=18,$ which is possible. Thus $n=3,$ which is not among the listed numerical choices.

Therefore, the correct answer is **E**.

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