1977 AMC 12 Solutions

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1. If $y = 2x$ and $z = 2y$, then $x + y + z$ equals

A x

B $3x$

C $5x$

D $7x$

E $9x$

Solution:

Since $y = 2x$, we have $z = 2y = 4x$. Therefore $x + y + z = x + 2x + 4x = 7x$.

Therefore, the correct answer is **D**.

2. Which one of the following statements is false? All equilateral triangles are

- A equiangular
- B isosceles
- C regular polygons
- D congruent to each other
- E similar to each other

Solution:

Every equilateral triangle is equiangular, isosceles, regular, and similar to every other equilateral triangle. Two equilateral triangles with different side lengths are not congruent, however.

Therefore, the correct answer is **D**.

3. A man has $\$2.73$ in pennies, nickels, dimes, quarters and half dollars. If he has an equal number of coins of each kind, then the total number of coins he has is

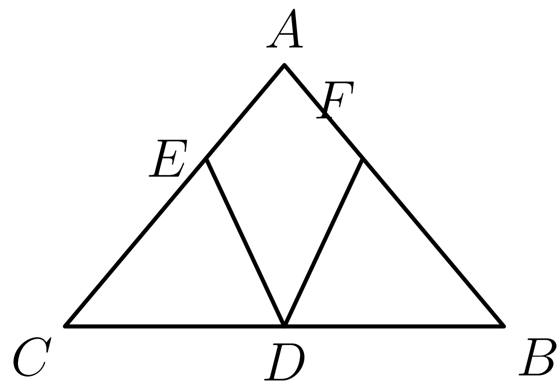
- A 3
- B 5
- C 9
- D 10
- E 15

Solution:

One coin of each kind is worth $1+5+10+25+50=91$ cents. Since $\frac{273}{91}=3$, there are three coins of each of the five kinds, or $5 \cdot 3=15$ coins.

Therefore, the correct answer is **E**.

4. In triangle $ABC, AB = AC$ and $\angle A = 80°$. If points D, E and F lie on sides BC, AC and AB, respectively, and $CE = CD$ and $BF = BD$, then $\angle EDF$ equals



A $30°$

B $40°$

C $50°$

D $65°$

E none of these

Solution:

The base angles of $\triangle ABC$ are each $50°$. Since $CE = CD$, triangle CED has vertex angle $50°$, so $\angle EDC = 65°$. Similarly, $BF = BD$ gives $\angle BDF = 65°$. The three angles above the straight line CB sum to $180°$, so

$$\begin{aligned}\angle EDF &= 180° - 65° - 65° \\ &= 50°.\end{aligned}$$

Therefore, the correct answer is **C**.

5. The set of all points P such that the sum of the (undirected) distances from P to two fixed points A and B equals the distance between A and B is

- A. the line segment from A to B
- B. the line passing through A and B
- C. the perpendicular bisector of the line segment from A to B
- D. an ellipse having positive area
- E. a parabola

Solution:

The triangle inequality gives $AP + PB \geq AB$. Equality holds exactly when A, P, B are collinear with P between A and B, including the endpoints. Thus the locus is the segment from A to B.

Therefore, the correct answer is **A**.

6. If x, y and $2x + \frac{y}{2}$ are not zero, then

$$\left(2x + \frac{y}{2}\right)^{-1}\left[(2x)^{-1} + \left(\frac{y}{2}\right)^{-1}\right]$$

equals

- A 1
- B xy^{-1}
- C $x^{-1}y$
- D $(xy)^{-1}$
- E none of these

Solution:

We have

$$\begin{aligned}\left(2x + \frac{y}{2}\right)^{-1} &= \frac{2}{4x+y},\\ (2x)^{-1} + \left(\frac{y}{2}\right)^{-1} &= \frac{1}{2x} + \frac{2}{y}\\ &= \frac{y+4x}{2xy}.\end{aligned}$$

Their product is $\frac{1}{xy} = (xy)^{-1}$.

Therefore, the correct answer is **D**.

7. If $t = \dfrac{1}{1-\sqrt[4]{2}}$, then t equals

A $(1-\sqrt[4]{2})(2-\sqrt{2})$

B $(1-\sqrt[4]{2})(1+\sqrt{2})$

C $(1+\sqrt[4]{2})(1-\sqrt{2})$

D $(1+\sqrt[4]{2})(1+\sqrt{2})$

E $-(1+\sqrt[4]{2})(1+\sqrt{2})$

Solution:

Let $u = \sqrt[4]{2}$. Since

$$\begin{aligned}(1-u)\big(-(1+u)(1+u^2)\big) \\ &= -(1-u^2)(1+u^2) \\ &= u^4 - 1 = 1,\end{aligned}$$

it follows that

$$t = -(1+\sqrt[4]{2})(1+\sqrt{2}).$$

Therefore, the correct answer is **E**.

8. For every triple (a, b, c) of nonzero real numbers, form the number

$$\frac{a}{|a|} + \frac{b}{|b|} + \frac{c}{|c|} + \frac{abc}{|abc|}.$$

The set of all numbers formed is

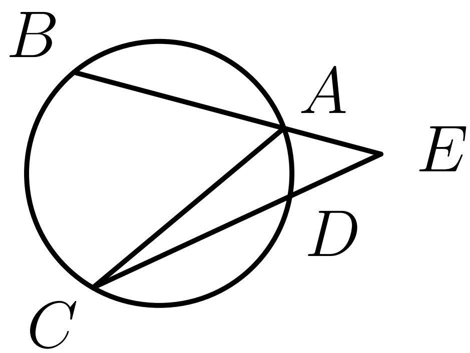
- A $\{0\}$
- B $\{-4, 0, 4\}$
- C $\{-4, -2, 0, 2, 4\}$
- D $\{-4, -2, 2, 4\}$
- E none of these

Solution:

Let $r, s, t \in \{-1, 1\}$ be the signs of a, b, c. The expression is $r + s + t + rst$. If all three signs are positive it is 4, and if all are negative it is -4. If the signs are mixed, direct cancellation gives 0. Hence the set is $\{-4, 0, 4\}$.

Therefore, the correct answer is **B**.

9. In the adjoining figure $\angle E = 40^\circ$ and arc AB, arc BC and arc CD all have equal length. Find the measure of $\angle ACD$.



A 10°

B 15°

C 20°

D $\left(\frac{45}{2}\right)^\circ$

E 30°

Solution:

Let each of arcs AB, BC, CD measure x° and let arc AD measure y°. Then

$$\begin{aligned} 3x + y &= 360, \\ \frac{x-y}{2} &= 40. \end{aligned}$$

Solving gives $y = 30$. The inscribed angle $\angle ACD$ subtends arc AD, so it measures $\frac{y}{2} = 15^\circ$.

Therefore, the correct answer is **B**.

10. If the polynomial $(3x-1)^7$ is written as $a_7x^7 + a_6x^6 + \cdots + a_0$, then $a_7 + a_6 + \cdots + a_0$ equals

A 0

B 1

C 64

D -64

E 128

Solution:

Setting $x = 1$ makes the right side the desired coefficient sum. Thus

$$\begin{aligned} a_7 + a_6 + \cdots + a_0 &= (3 \cdot 1 - 1)^7 \\ &= 2^7 = 128. \end{aligned}$$

Therefore, the correct answer is **E**.

11. For each real number x, let $[x]$ be the largest integer not exceeding x (i.e., the integer n such that $n \leq x < n+1$). Which of the following statements is (are) true?

I. $[x+1] = [x] + 1$ for all x

II. $[x+y] = [x] + [y]$ for all x and y

III. $[xy] = [x][y]$ for all x and y

A none

B I only

C I and II only

D III only

E all

Solution:

If $n \leq x < n+1$, then $n+1 \leq x+1 < n+2$, so I is true. Taking $x = y = \frac{3}{2}$ gives $[x+y] = 3 \neq 2 = [x] + [y]$ and $[xy] = 2 \neq 1 = [x][y]$. Thus II and III are false.

Therefore, the correct answer is **B**.

12. Al's age is 16 more than the sum of Bob's age and Carl's age, and the square of Al's age is 1632 more than the square of the sum of Bob's age and Carl's age. The sum of the ages of Al, Bob and Carl is

A 64

B 94

C 96

D 102

E 140

Solution:

Let a be Al's age and s the sum of the other two ages. Then $a - s = 16$ and

$$\begin{aligned} 1632 &= a^2 - s^2 \\ &= (a - s)(a + s) \\ &= 16(a + s). \end{aligned}$$

Hence the requested total $a + s$ is $\frac{1632}{16} = 102.$

Therefore, the correct answer is **D**.

13. If $a_1, a_2, a_3, \ldots$ is a sequence of positive numbers such that $a_{n+2} = a_n a_{n+1}$ for all positive integers n, then the sequence $a_1, a_2, a_3, \ldots$ is a geometric progression

A. for all positive values of a_1 and a_2

B. if and only if $a_1 = a_2$

C. if and only if $a_1 = 1$

D. if and only if $a_2 = 1$

E. if and only if $a_1 = a_2 = 1$

Solution:

The next terms are $a_3 = a_1 a_2, a_4 = a_1 a_2^2$, and $a_5 = a_1^2 a_2^3$. If the sequence is geometric, then

$$\frac{a_2}{a_1} = \frac{a_3}{a_2} = a_1,$$
$$\frac{a_4}{a_3} = a_2 = a_1.$$

Thus $a_2 = a_1^2$ and $a_2 = a_1$. Positivity forces $a_1 = a_2 = 1$. Conversely, these initial values produce the constant geometric sequence $1, 1, 1, \ldots$.

Therefore, the correct answer is **E**.

14. How many pairs (m, n) of integers satisfy the equation $m + n = mn$?

A 1

B 2

C 3

D 4

E more than 4

Solution:

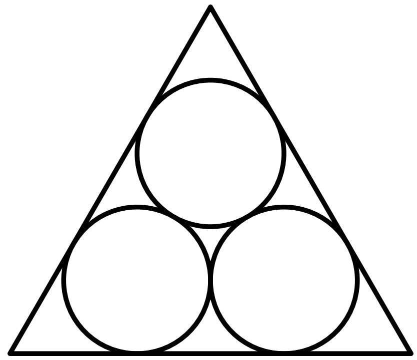
Rearranging and completing the product gives

$$mn - m - n = 0,$$
$$(m-1)(n-1) = 1.$$

The integer factor pairs of 1 are $(1, 1)$ and $(-1, -1)$, producing $(m, n) = (2, 2)$ and $(0, 0)$. Thus there are two ordered pairs.

Therefore, the correct answer is **B**.

15. Each of the three circles in the adjoining figure is externally tangent to the other two, and each side of the triangle is tangent to two of the circles. If each circle has radius three, then the perimeter of the triangle is



A $36+9\sqrt{2}$

B $36+6\sqrt{3}$

C $36+9\sqrt{3}$

D $18+18\sqrt{3}$

E 45

Solution:

The circle centers form an equilateral triangle of side 6, whose inradius is $\frac{6\sqrt{3}}{6}=\sqrt{3}$. Each side of the outer triangle is a parallel tangent line three units farther out, so the outer triangle has inradius $3+\sqrt{3}$. An equilateral triangle of inradius r has side $2\sqrt{3}r$, hence each outer side is

$$2\sqrt{3}(3+\sqrt{3})=6+6\sqrt{3}.$$

The perimeter is $3(6+6\sqrt{3})=18+18\sqrt{3}$.

Therefore, the correct answer is **D**.

16. If $i^2 = -1$, then the sum

$$\cos 45^\circ + i \cos 135^\circ + \cdots \\ + i^n \cos(45 + 90n)^\circ + \cdots \\ + i^{40} \cos 3645^\circ$$

equals

A $\frac{\sqrt{2}}{2}$

B $-10i\sqrt{2}$

C $\frac{21\sqrt{2}}{2}$

D $\frac{\sqrt{2}}{2}(21 - 20i)$

E $\frac{\sqrt{2}}{2}(21 + 20i)$

Solution:

Let $u_n = i^n \cos(45 + 90n)^\circ$. Then

$$\begin{aligned} u_{n+2} &= i^{n+2} \cos(225 + 90n)^\circ \\ &= (-i^n)(-\cos(45 + 90n)^\circ) \\ &= u_n. \end{aligned}$$

Every even-indexed term equals $\frac{\sqrt{2}}{2}$, and every odd-indexed term equals $-\frac{i\sqrt{2}}{2}$. There are 21 even indices and 20 odd indices, so the sum is $\frac{\sqrt{2}}{2}(21 - 20i)$.

Therefore, the correct answer is **D**.

17. Three fair dice are tossed (all faces have the same probability of coming up). What is the probability that the three numbers turned up can be arranged to form an arithmetic progression with common difference one?

A $\frac{1}{6}$

B $\frac{1}{9}$

C $\frac{1}{27}$

D $\frac{1}{54}$

E $\frac{7}{36}$

Solution:

The possible sets are $\{1, 2, 3\}$, $\{2, 3, 4\}$, $\{3, 4, 5\}$ and $\{4, 5, 6\}$. Each has $3! = 6$ orderings, so 24 of the $6^3 = 216$ outcomes work. The probability is $\frac{24}{216} = \frac{1}{9}$.

Therefore, the correct answer is **B**.

18. If

$$
\begin{aligned}
y = {} & (\log_2 3)(\log_3 4)\cdots \\
& (\log_n[n+1])\cdots(\log_{31} 32),
\end{aligned}
$$

then

A $4 < y < 5$

B $y = 5$

C $5 < y < 6$

D $y = 6$

E $6 < y < 7$

Solution:

By change of base,

$$
\begin{aligned}
y &= \frac{\log 3}{\log 2} \cdot \frac{\log 4}{\log 3} \cdots \frac{\log 32}{\log 31} \\
&= \frac{\log 32}{\log 2} \\
&= \log_2 32 = 5.
\end{aligned}
$$

Therefore, the correct answer is **B**.

19. Let E be the point of intersection of the diagonals of convex quadrilateral $ABCD$, and let P, Q, R and S be the centers of the circles circumscribing triangles ABE, BCE, CDE and ADE, respectively. Then

A $PQRS$ is a parallelogram

B $PQRS$ is a parallelogram if and only if $ABCD$ is a rhombus

C $PQRS$ is a parallelogram if and only if $ABCD$ is a rectangle

D $PQRS$ is a parallelogram if and only if $ABCD$ is a parallelogram

E none of the above are true

Solution:

Both P and Q lie on the perpendicular bisector of BE, while both R and S lie on the perpendicular bisector of DE. Since B, E, D are collinear, these bisectors are parallel, so $PQ \parallel RS$. Likewise, Q, R lie on the perpendicular bisector of CE, and S, P lie on that of AE. Since A, E, C are collinear, $QR \parallel SP$. Therefore $PQRS$ is always a parallelogram.

Therefore, the correct answer is **A**.

20. For how many paths consisting of a sequence of horizontal and/or vertical line segments, with each segment connecting a pair of adjacent letters in the diagram below, is the word CONTEST spelled out as the path is traversed from beginning to end?

$$\begin{array}{ccccccccccccc}
 & & & & & & C & & & & & & \\
 & & & & & C & O & C & & & & & \\
 & & & & C & O & N & O & C & & & & \\
 & & & C & O & N & T & N & O & C & & & \\
 & & C & O & N & T & E & T & N & O & C & & \\
 & C & O & N & T & E & S & E & T & N & O & C & \\
C & O & N & T & E & S & T & S & E & T & N & O & C
\end{array}$$

A 63

B 128

C 129

D 255

E none of these

Solution:

Reverse the paths and spell TSETNOC from the central bottom T. In the family whose horizontal moves go left, each of the six steps has two choices: up or left. This gives $2^6 = 64$ paths. By symmetry, 64 paths have horizontal moves going right. The all-vertical central path belongs to both families, so the total is

$$64 + 64 - 1 = 127.$$

This number is not among the first four choices.

Therefore, the correct answer is **E**.

21. For how many values of the coefficient a do the equations

$$x^2 + ax + 1 = 0,$$
$$x^2 - x - a = 0$$

have a common real solution?

A 0

B 1

C 2

D 3

E infinitely many

Solution:

Subtracting the second equation from the first gives

$$(a+1)x + (a+1)$$
$$= (a+1)(x+1) = 0.$$

If $a = -1$, the common equation is $x^2 - x + 1 = 0$, which has no real roots. Otherwise $x = -1$, and substitution into $x^2 - x - a = 0$ gives $a = 2$. This value works, so exactly one value of a qualifies.

Therefore, the correct answer is **B**.

22. If $f(x)$ is a real-valued function of the real variable x, and $f(x)$ is not identically zero, and for all a and b

$$\begin{aligned} &f(a+b)+f(a-b) \\ &= 2f(a)+2f(b), \end{aligned}$$

then for all x and y

A $f(0)=1$

B $f(-x)=-f(x)$

C $f(-x)=f(x)$

D $f(x+y)=f(x)+f(y)$

E there is a positive number T such that $f(x+T)=f(x)$

Solution:

Setting $a=b=0$ gives $2f(0)=4f(0)$, so $f(0)=0$. Now set $a=0$ and $b=x$. The equation becomes

$$\begin{aligned} f(x)+f(-x) &= 2f(0)+2f(x) \\ &= 2f(x), \end{aligned}$$

and hence $f(-x)=f(x)$ for every real x.

Therefore, the correct answer is **C**.

23. If the solutions of the equation $x^2 + px + q = 0$ are the cubes of the solutions of the equation $x^2 + mx + n = 0$, then

A $p = m^3 + 3mn$

B $p = m^3 - 3mn$

C $p + q = m^3$

D $\left(\frac{m}{n}\right)^3 = \frac{p}{q}$

E none of these

Solution:

Let the roots of $x^2 + mx + n$ be u, v. Then $u + v = -m$ and $uv = n$, while the roots of the first equation are u^3, v^3. Thus

$$\begin{aligned} -p &= u^3 + v^3 \\ &= (u+v)^3 - 3uv(u+v) \\ &= -m^3 + 3mn. \end{aligned}$$

Therefore $p = m^3 - 3mn.$

Therefore, the correct answer is **B**.

24. Find the sum

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\cdots + \frac{1}{(2n-1)(2n+1)}+\cdots + \frac{1}{255\cdot 257}.$$

A $\frac{127}{255}$

B $\frac{128}{255}$

C $\frac{1}{2}$

D $\frac{128}{257}$

E $\frac{129}{257}$

Solution:

For $1 \leq n \leq 128$,

$$\frac{1}{(2n-1)(2n+1)} = \frac{1}{2(2n-1)} - \frac{1}{2(2n+1)}.$$

Hence the sum telescopes to

$$\frac{1}{2}\left(1-\frac{1}{257}\right) = \frac{128}{257}.$$

Therefore, the correct answer is **D**.

25. Determine the largest positive integer n such that $1005!$ is divisible by 10^n.

A 102

B 112

C 249

D 502

E none of these

Solution:

There are more factors of 2 than of 5, so the exponent of 10 is

$$\begin{aligned} n = &\left\lfloor \frac{1005}{5} \right\rfloor + \left\lfloor \frac{1005}{25} \right\rfloor \\ &+ \left\lfloor \frac{1005}{125} \right\rfloor + \left\lfloor \frac{1005}{625} \right\rfloor \\ &= 201 + 40 + 8 + 1 = 250. \end{aligned}$$

Since 250 is not listed, the correct choice is “none of these.”

Therefore, the correct answer is **E**.

26. Let a, b, c and d be the lengths of sides MN, NP, PQ and QM, respectively, of quadrilateral $MNPQ$. If A is the area of $MNPQ$, then

A $A = \left(\frac{a+c}{2}\right)\left(\frac{b+d}{2}\right)$ if and only if $MNPQ$ is convex

B $A = \left(\frac{a+c}{2}\right)\left(\frac{b+d}{2}\right)$ if and only if $MNPQ$ is a rectangle

C $A \leq \left(\frac{a+c}{2}\right)\left(\frac{b+d}{2}\right)$ if and only if $MNPQ$ is a rectangle

D $A \leq \left(\frac{a+c}{2}\right)\left(\frac{b+d}{2}\right)$ if and only if $MNPQ$ is a parallelogram

E $A \geq \left(\frac{a+c}{2}\right)\left(\frac{b+d}{2}\right)$ if and only if $MNPQ$ is a parallelogram

Solution:

Splitting along diagonal MP and using $\sin\theta \leq 1$ gives $A \leq \frac{ab+cd}{2}$. Splitting along NQ similarly gives $A \leq \frac{ad+bc}{2}$. These bounds remain valid for a nonconvex quadrilateral by taking the appropriate difference of triangle areas. Adding them yields

$$2A \leq \frac{ab+ad+bc+cd}{2} = \frac{(a+c)(b+d)}{2},$$

or $A \leq \left(\frac{a+c}{2}\right)\left(\frac{b+d}{2}\right)$. Equality requires equality in all four sine bounds, so all four angles are right angles; conversely a rectangle gives equality. Thus the equality holds exactly for rectangles.

Therefore, the correct answer is **B**.

27. There are two spherical balls of different sizes lying in two corners of a rectangular room, each touching two walls and the floor. If there is a point on each ball which is 5 inches from each wall which that ball touches and 10 inches from the floor, then the sum of the diameters of the balls is

- A 20 inches
- B 30 inches
- C 40 inches
- D 60 inches
- E not determined by the given information

Solution:

For radius r, the center is (r, r, r) and the given point is $(5, 5, 10)$. Thus

$$2(5-r)^2 + (10-r)^2 = r^2,$$

which simplifies to $r^2 - 20r + 75 = 0$, or $(r-5)(r-15) = 0$. The two radii are 5 and 15, so the sum of the diameters is $2(5+15) = 40$ inches.

Therefore, the correct answer is **C**.

28. Let $g(x) = x^5 + x^4 + x^3 + x^2 + x + 1$. What is the remainder when the polynomial $g(x^{12})$ is divided by the polynomial $g(x)$?

A 6

B $5 - x$

C $4 - x + x^2$

D $3 - x + x^2 - x^3$

E $2 - x + x^2 - x^3 + x^4$

Solution:

Let $R(x)$ be the remainder, so $\deg R \leq 4$. The five roots α of g satisfy $\alpha^6 = 1$ and $\alpha \neq 1$. Hence $\alpha^{12} = 1$ and

$$R(\alpha) = g(\alpha^{12}) = g(1) = 6.$$

Therefore $R(x) - 6$, a polynomial of degree at most 4, has five distinct roots. It must be identically zero, so $R(x) = 6$.

Therefore, the correct answer is **A**.

29. Find the smallest integer n such that

$$(x^2 + y^2 + z^2)^2 \leq n(x^4 + y^4 + z^4)$$

for all real numbers x, y and z.

A 2

B 3

C 4

D 6

E There is no such integer n.

Solution:

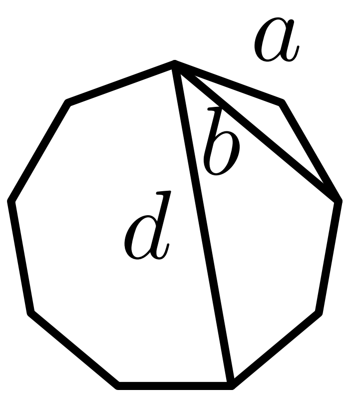
By Cauchy-Schwarz,

$$(x^2 + y^2 + z^2)^2 \leq 3Q, \quad Q = x^4 + y^4 + z^4.$$

Equality occurs when $x^2 = y^2 = z^2 \neq 0$, so no smaller value can work. Thus the smallest integer is 3.

Therefore, the correct answer is **B**.

30. If a, b and d are the lengths of a side, a shortest diagonal and a longest diagonal, respectively, of a regular nonagon (see adjoining figure), then



A $d = a + b$

B $d^2 = a^2 + b^2$

C $d^2 = a^2 + ab + b^2$

D $b = \frac{a+d}{2}$

E $b^2 = ad$

Solution:

If the circumradius is r, the three chords subtend central angles $40^\circ, 80^\circ, 160^\circ$, respectively. Thus

$$\begin{aligned} a &= 2r \sin 20^\circ, \\ b &= 2r \sin 40^\circ, \\ d &= 2r \sin 80^\circ. \end{aligned}$$

The sum-to-product identity gives

$$\begin{aligned} &\sin 20^\circ + \sin 40^\circ \\ &= 2 \sin 30^\circ \cos 10^\circ \\ &= \cos 10^\circ = \sin 80^\circ. \end{aligned}$$

Multiplying by $2r$ yields $a + b = d$.

Therefore, the correct answer is **A**.

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