1976 AMC 12 Solutions

Typeset by: LIVE by Po-Shen Loh
https://live.poshenloh.com/past-contests/amc12/1976/solutions

1. If one minus the reciprocal of $(1-x)$ equals the reciprocal of $(1-x)$, then x equals

A -2

B -1

C $\frac{1}{2}$

D 2

E 3

Solution:

The equation is

$$1-\frac{1}{1-x}=\frac{1}{1-x}.$$

Multiplying by $1-x \neq 0$ gives $1-x-1=1$, so $x=-1$.

Therefore, the correct answer is **B**.

2. For how many real numbers x is $\sqrt{-(x+1)^2}$ a real number?

A none

B one

C two

D a finite number greater than two

E infinitely many

Solution:

Since $(x+1)^2 \geq 0$, its negative is at most 0. It is nonnegative only when $(x+1)^2 = 0$, which occurs for the single value $x = -1$.

Therefore, the correct answer is **B**.

3. The sum of the distances from one vertex of a square with sides of length two to the midpoints of each of the sides of the square is

A $2\sqrt{5}$

B $2+\sqrt{3}$

C $2+2\sqrt{3}$

D $2+\sqrt{5}$

E $2+2\sqrt{5}$

Solution:

The two midpoints on sides meeting the chosen vertex are each at distance $1.$ The other two midpoints have coordinate differences 2 and $1,$ so each is at distance $\sqrt{2^2+1^2}=\sqrt{5}.$ The sum is $2+2\sqrt{5}.$

Therefore, the correct answer is **E**.

4. Let a geometric progression with n terms have first term one, common ratio r and sum s, where r and s are not zero. The sum of the geometric progression formed by replacing each term of the original progression by its reciprocal is

A $\frac{1}{s}$

B $\frac{1}{r^n s}$

C $\frac{s}{r^{n-1}}$

D $\frac{r^n}{s}$

E $\frac{r^{n-1}}{s}$

Solution:

Let S' be the reciprocal sum. Multiplying it by r^{n-1} reverses the original terms:

$$\begin{aligned} r^{n-1}S' &= r^{n-1} + r^{n-2} + \cdots + 1 \\ &= s. \end{aligned}$$

Hence $S' = \frac{s}{r^{n-1}}$. This also holds when $r = 1$.

Therefore, the correct answer is **C**.

5. How many integers greater than ten and less than one hundred, written in base ten notation, are increased by nine when their digits are reversed?

- A 0
- B 1
- C 8
- D 9
- E 10

Solution:

The condition is $9(u-t)=9$, so $u=t+1$. The possibilities are $12, 23, 34, 45, 56, 67, 78, 89$, giving 8 integers.

Therefore, the correct answer is **C**.

6. If c is a real number and the negative of one of the solutions of $x^2 - 3x + c = 0$ is a solution of $x^2 + 3x - c = 0$, then the solutions of $x^2 - 3x + c = 0$ are

- A $1, 2$
- B $-1, -2$
- C $0, 3$
- D $0, -3$
- E $\frac{3}{2}, \frac{3}{2}$

Solution:

For such a root r,

$$r^2 - 3r + c = 0,$$
$$(-r)^2 + 3(-r) - c = 0.$$

Subtracting gives $2c = 0$. Thus the first equation is $x(x-3) = 0$, with roots 0 and 3.

Therefore, the correct answer is **C**.

7. If x is a real number, then the quantity $(1-|x|)(1+x)$ is positive if and only if

A $|x| < 1$

B $x < 1$

C $|x| > 1$

D $x < -1$

E $x < -1$ or $-1 < x < 1$

Solution:

Both factors are positive exactly when $|x| < 1$, giving $-1 < x < 1$. Both are negative when $|x| > 1$ and $x < -1$, which reduces to $x < -1$. The endpoints make the product zero.

Therefore, the correct answer is **E**.

8. A point in the plane, both of whose rectangular coordinates are integers with absolute value less than or equal to four, is chosen at random, with all such points having an equal probability of being chosen. What is the probability that the distance from the point to the origin is at most two units?

A $\frac{13}{81}$

B $\frac{15}{81}$

C $\frac{13}{64}$

D $\frac{\pi}{16}$

E the square of a rational number

Solution:

There are $9^2 = 81$ possible points. Within distance 2 are the origin; the four points $(\pm 1, 0), (0, \pm 1)$; the four points $(\pm 1, \pm 1)$; and the four points $(\pm 2, 0), (0, \pm 2)$. Thus 13 points qualify, for probability $\frac{13}{81}$.

Therefore, the correct answer is **A**.

9. In triangle ABC, D is the midpoint of AB; E is the midpoint of DB; and F is the midpoint of BC. If the area of $\triangle ABC$ is 96, then the area of $\triangle AEF$ is

A 16

B 24

C 32

D 36

E 48

Solution:

Since D is the midpoint of AB and E is the midpoint of DB, $AE = \frac{3}{4}AB$. Since F is the midpoint of BC, its distance from line AB is half that of C. Therefore

$$\begin{aligned}[AEF] &= \frac{3}{4} \cdot \frac{1}{2}[ABC] \\ &= \frac{3}{8}(96) = 36.\end{aligned}$$

Therefore, the correct answer is **D**.

10. If m, n, p and q are real numbers and $f(x) = mx + n$ and $g(x) = px + q$, then the equation $f(g(x)) = g(f(x))$ has a solution

- A for all choices of m, n, p and q
- B if and only if $m = p$ and $n = q$
- C if and only if $mq - np = 0$
- D if and only if $n(1-p) - q(1-m) = 0$
- E if and only if $(1-n)(1-p) - (1-q)(1-m) = 0$

Solution:

We have

$$f(g(x)) = mpx + mq + n,$$
$$g(f(x)) = pmx + pn + q.$$

The x-terms cancel for every x, so a solution exists exactly when $mq + n = pn + q$. Rearranging gives $n(1-p) - q(1-m) = 0$.

Therefore, the correct answer is **D**.

11. Which of the following statements is (are) equivalent to the statement "If the pink elephant on planet alpha has purple eyes, then the wild pig on planet beta does not have a long nose"?

I. "If the wild pig on planet beta has a long nose, then the pink elephant on planet alpha has purple eyes."

II. "If the pink elephant on planet alpha does not have purple eyes, then the wild pig on planet beta does not have a long nose."

III. "If the wild pig on planet beta has a long nose, then the pink elephant on planet alpha does not have purple eyes."

IV. "The pink elephant on planet alpha does not have purple eyes, or the wild pig on planet beta does not have a long nose."

The word "or" is used here in the inclusive sense (as is customary in mathematical writing).

A. I and III only

B. III and IV only

C. II and IV only

D. II and III only

E. III only

Solution:

Let P mean the elephant has purple eyes and Q mean the pig has a long nose. The original statement $P \Rightarrow \neg Q$ is equivalent to its contrapositive $Q \Rightarrow \neg P$, which is III, and to $\neg P \vee \neg Q$, which is IV. Statements I and II are converses and need not follow.

Therefore, the correct answer is **B**.

12. A supermarket has 128 crates of apples. Each crate contains at least 120 apples and at most 144 apples. What is the largest integer n such that there must be at least n crates containing the same number of apples?

- A 4
- B 5
- C 6
- D 24
- E 25

Solution:

There are $144 - 120 + 1 = 25$ possible apple counts. Since $128 > 25 \cdot 5$, some count occurs at least 6 times. This is sharp: distribute 128 crates so that three counts occur 6 times and the other twenty-two occur 5 times. Thus the largest guaranteed n is 6.

Therefore, the correct answer is **C**.

13. If x cows give $x+1$ cans of milk in $x+2$ days, how many days will it take $x+3$ cows to give $x+5$ cans of milk?

A $\frac{x(x+2)(x+5)}{(x+1)(x+3)}$

B $\frac{x(x+1)(x+5)}{(x+2)(x+3)}$

C $\frac{(x+1)(x+3)(x+5)}{x(x+2)}$

D $\frac{(x+1)(x+3)}{x(x+2)(x+5)}$

E none of these

Solution:

The production rate is $\frac{x+1}{x(x+2)}$ cans per cow-day. If the new time is T, then

$$\frac{x+5}{(x+3)T} = \frac{x+1}{x(x+2)}.$$

Hence $T = \frac{x(x+2)(x+5)}{(x+1)(x+3)}$.

Therefore, the correct answer is **A**.

14. The measures of the interior angles of a convex polygon are in arithmetic progression. If the smallest angle is 100° and the largest angle is 140°, then the number of sides the polygon has is

A 6

B 8

C 10

D 11

E 12

Solution:

The average angle is $\frac{100+140}{2} = 120^\circ$, so the angle sum is $120n$. A convex n-gon has angle sum $180(n-2)$, hence $120n = 180(n-2)$ and $n = 6$.

Therefore, the correct answer is **A**.

15. If r is the remainder when each of the numbers $1059, 1417$ and 2312 is divided by d, where d is an integer greater than one, then $d - r$ equals

A 1

B 15

C 179

D $d - 15$

E $d - 1$

Solution:

The divisor d divides both $358 = 2 \cdot 179$ and $895 = 5 \cdot 179$. Their greatest common divisor is 179, a prime, so $d = 179$. Since $1059 = 5 \cdot 179 + 164$, we have $r = 164$ and $d - r = 15$.

Therefore, the correct answer is **B**.

16. In triangles ABC and DEF, lengths AC, BC, DF and EF are all equal. Length AB is twice the length of the altitude of $\triangle DEF$ from F to DE. Which of the following statements is (are) true?

I. $\angle ACB$ and $\angle DFE$ must be complementary.

II. $\angle ACB$ and $\angle DFE$ must be supplementary.

III. The area of $\triangle ABC$ must equal the area of $\triangle DEF$.

IV. The area of $\triangle ABC$ must equal twice the area of $\triangle DEF$.

A II only

B III only

C IV only

D I and III only

E II and III only

Solution:

Let the equal sides have length s, let $AB = 2a$, and let the altitude from F be a. In the isosceles triangles, the altitude from C bisects AB, while the altitude from F bisects DE. A half of ABC has hypotenuse s and leg a; a half of DEF has the same hypotenuse and altitude leg a. The two right triangles are congruent with their legs interchanged. Thus their relevant acute angles are complementary, so the full vertex angles $\angle ACB$ and $\angle DFE$ sum to $180°$. The corresponding half-areas are equal as well, hence the full triangle areas are equal. Statements II and III hold.

Therefore, the correct answer is **E**.

17. If θ is an acute angle and $\sin 2\theta = a$, then $\sin\theta + \cos\theta$ equals

A $\sqrt{a+1}$

B $(\sqrt{2}-1)a+1$

C $\sqrt{a+1} - \sqrt{a^2 - a}$

D $\sqrt{a+1} + \sqrt{a^2 - a}$

E $\sqrt{a+1} + a^2 - a$

Solution:

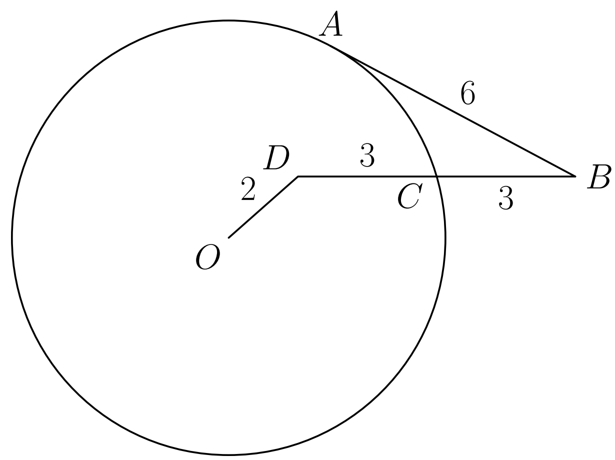
We have

$$\begin{aligned}(\sin\theta + \cos\theta)^2 &= \sin^2\theta + \cos^2\theta \\ &\quad + 2\sin\theta\cos\theta \\ &= 1 + a.\end{aligned}$$

Both sine and cosine are positive for acute θ, so the sum is $\sqrt{a+1}$.

Therefore, the correct answer is **A**.

18. In the adjoining figure, AB is tangent at A to the circle with center O; point D is interior to the circle; and DB intersects the circle at C. If $BC = DC = 3, OD = 2$ and $AB = 6$, then the radius of the circle is



A $3+\sqrt{3}$

B $\frac{15}{\pi}$

C $\frac{9}{2}$

D $2\sqrt{6}$

E $\sqrt{22}$

Solution:

Extend BD through D to meet the circle at E. Since $BC = 3$ and $BD = 6$, the tangent-secant theorem gives

$$3 \cdot BE = AB^2 = 36,$$

so $BE = 12$ and $DE = 6$. The power of D gives

$$\begin{aligned} DC \cdot DE &= (R - OD)(R + OD) \\ &= R^2 - OD^2, \end{aligned}$$

hence $3 \cdot 6 = (R-2)(R+2) = R^2 - 4$. Thus $R^2 = 22$ and $R = \sqrt{22}$.

Therefore, the correct answer is **E**.

19. A polynomial $p(x)$ has remainder three when divided by $x-1$ and remainder five when divided by $x-3$. The remainder when $p(x)$ is divided by $(x-1)(x-3)$ is

A $x-2$

B $x+2$

C 2

D 8

E 15

Solution:

Let the remainder be $r(x)=ax+b$. Since $p(1)=3$ and $p(3)=5$,

$$a+b=3, \qquad 3a+b=5.$$

Thus $a=1,\ b=2$, and the remainder is $x+2$.

Therefore, the correct answer is **B**.

20. Let a, b and x be positive real numbers distinct from one. Then

$$\begin{aligned} 4(\log_a x)^2 + 3(\log_b x)^2 \\ = 8(\log_a x)(\log_b x) \end{aligned}$$

A for all values of a, b and x

B if and only if $a = b^2$

C if and only if $b = a^2$

D if and only if $x = ab$

E none of these

Solution:

Factoring gives

$$\begin{aligned} &(2\log_a x - \log_b x) \\ &\cdot (2\log_a x - 3\log_b x) = 0. \end{aligned}$$

Since $x \neq 1$, change of base shows that the first factor vanishes when $a = b^2$, while the second vanishes when $a^3 = b^2$. Thus the equation holds under either of two conditions, and no listed "if and only if" statement describes their union.

Therefore, the correct answer is **E**.

21. What is the smallest positive odd integer n such that the product

$$2^{\frac{1}{7}}2^{\frac{3}{7}}\cdots 2^{\frac{2n+1}{7}}$$

is greater than 1000? (In the product the denominators of the exponents are all sevens, and the numerators are the successive odd integers from 1 to $2n+1$.)

- A 7
- B 9
- C 11
- D 17
- E 19

Solution:

The product is

$$2^{\frac{1+3+\cdots+(2n+1)}{7}} = 2^{\frac{(n+1)^2}{7}}.$$

For $n=7$, the exponent $\frac{64}{7}$ is less than $\frac{19}{2}$, so the product is less than $2^{\frac{19}{2}} = 512\sqrt{2} < 768 < 1000$. All smaller positive odd n also fail. For $n=9$, the exponent is $\frac{100}{7} > 10$, so the product exceeds $2^{10} = 1024$. Thus the first allowable odd n is 9.

Therefore, the correct answer is **B**.

22. Given an equilateral triangle with side of length s, consider the locus of all points P in the plane of the triangle such that the sum of the squares of the distances from P to the vertices of the triangle is a fixed number a. This locus

A is a circle if $a > s^2$

B contains only three points if $a = 2s^2$ and is a circle if $a > 2s^2$

C is a circle with positive radius only if $s^2 < a < 2s^2$

D contains only a finite number of points for any value of a

E is none of these

Solution:

For vertices A, B, C with centroid G, put $T = GA^2 + GB^2 + GC^2$. The standard centroid identity gives

$$PA^2 + PB^2 + PC^2 = 3PG^2 + T.$$

In an equilateral triangle, each $GA = \frac{s}{\sqrt{3}}$, so $T = s^2$. Hence $3PG^2 = a - s^2$. The locus is empty for $a < s^2$, one point for $a = s^2$, and a circle of positive radius for $a > s^2$.

Therefore, the correct answer is **A**.

23. For integers k and n such that $1 \leq k < n$, let

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Then $\left(\frac{n-2k-1}{k+1}\right)\binom{n}{k}$ is an integer

A for all k and n

B for all even values of k and n, but not for all k and n

C for all odd values of k and n, but not for all k and n

D if $k = 1$ or $n-1$, but not for all odd values of k and n

E if n is divisible by k, but not for all even values of k and n

Solution:

Using

$$\binom{n}{k+1} = \frac{n-k}{k+1}\binom{n}{k},$$

and writing

$$\frac{n-2k-1}{k+1} = \frac{n-k}{k+1} - 1,$$

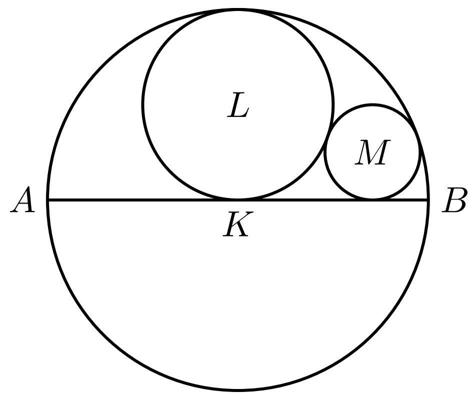
the given expression becomes

$$\binom{n}{k+1} - \binom{n}{k}.$$

This is an integer for every permitted k, n.

Therefore, the correct answer is **A**.

24. In the adjoining figure, circle K has diameter AB; circle L is tangent to circle K and to AB at the center of circle K; and circle M is tangent to circle K, to circle L and to AB. The ratio of the area of circle K to the area of circle M is



A 12

B 14

C 16

D 18

E not an integer

Solution:

Take K to have center $(0,0)$ and radius 2, with AB the x-axis. Then L has center $(0,1)$ and radius 1. If M has center (x,t) and radius t, internal tangency to K and external tangency to L give

$$x^2+t^2=(2-t)^2,$$
$$x^2+(t-1)^2=(1+t)^2.$$

These simplify to $x^2=4-4t$ and $x^2=4t$, so $t=\frac{1}{2}$. The radius ratio is $2:\left(\frac{1}{2}\right)=4:1$, hence the area ratio is 16.

Therefore, the correct answer is **C**.

25. For a sequence $u_1, u_2, \ldots,$ define $\Delta^1(u_n) = u_{n+1} - u_n$ and, for all integers $k > 1$, $\Delta^k(u_n) = \Delta^1(\Delta^{k-1}(u_n))$. If $u_n = n^3 + n$, then $\Delta^k(u_n) = 0$ for all n

A if $k = 1$

B if $k = 2$, but not if $k = 1$

C if $k = 3$, but not if $k = 2$

D if $k = 4$, but not if $k = 3$

E for no value of k

Solution:

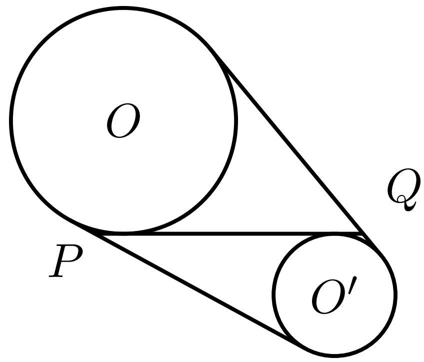
Directly,

$$\begin{aligned} \Delta u_n &= 3n^2 + 3n + 2, \\ \Delta^2 u_n &= 6n + 6, \\ \Delta^3 u_n &= 6, \\ \Delta^4 u_n &= 0. \end{aligned}$$

Thus the fourth differences vanish for all n, but the third differences do not.

Therefore, the correct answer is **D**.

26. In the adjoining figure, every point of circle O' is exterior to circle O. Let P and Q be the points of intersection of an internal common tangent with the two external common tangents. Then the length of PQ is



A the average of the lengths of the internal and external common tangents

B equal to the length of an external common tangent if and only if circles O and O' have equal radii

C always equal to the length of an external common tangent

D greater than the length of an external common tangent

E the geometric mean of the lengths of the internal and external common tangents

Solution:

Let the internal tangent touch O and O' at R and S. Let the external tangent through P touch them at X, Y, and let the one through Q touch them at V, W. Equal tangent segments from a point give

$$\begin{aligned} PR = PX, \quad PS = PY, \\ QR = QV, \quad QS = QW. \end{aligned}$$

Along the internal tangent, $PR + QR = PQ$ and $PS + QS = PQ$. Adding,

$$PR + QR + PS + QS = 2PQ.$$

Along the external tangents, $PX + PY = XY$ and $QV + QW = VW$. Therefore

$2PQ = XY + VW$. The two external common tangent segments have equal length, so $XY = VW$ and hence $PQ = XY = VW$.

Therefore, the correct answer is **C**.

27. If

$$N = \frac{\sqrt{\sqrt{5}+2} + \sqrt{\sqrt{5}-2}}{\sqrt{\sqrt{5}+1}} - \sqrt{3-2\sqrt{2}},$$

then N equals

A 1

B $2\sqrt{2}-1$

C $\frac{\sqrt{5}}{2}$

D $\sqrt{\frac{5}{2}}$

E none of these

Solution:

Let T be the first fraction. Its numerator squared is

$$\begin{aligned}&(\sqrt{5}+2)+(\sqrt{5}-2)\\&\quad+2\sqrt{(\sqrt{5}+2)(\sqrt{5}-2)}\\&=2\sqrt{5}+2.\end{aligned}$$

Thus $T^2 = \frac{2\sqrt{5}+2}{\sqrt{5}+1} = 2$, and $T = \sqrt{2}$. Also $\sqrt{3-2\sqrt{2}} = \sqrt{2}-1$. Hence $N = \sqrt{2} - (\sqrt{2}-1) = 1$.

Therefore, the correct answer is **A**.

28. Lines $L_1, L_2, \ldots, L_{100}$ are distinct. All lines L_{4n}, n a positive integer, are parallel to each other. All lines L_{4n-3}, n a positive integer, pass through a given point A. The maximum number of points of intersection of pairs of lines from the complete set $\{L_1, L_2, \ldots, L_{100}\}$ is

A 4350

B 4351

C 4900

D 4901

E 9851

Solution:

There are 25 indices divisible by 4 and 25 congruent to $1 \pmod 4$. Starting from $\binom{100}{2} = 4950$, the parallel group contributes no intersections, removing $\binom{25}{2} = 300$. The concurrent group's 300 pairs all give one point rather than 300 distinct points, removing another 299. All remaining intersections can be chosen distinct, so the maximum is

$$4950 - 300 - 299 = 4351.$$

Therefore, the correct answer is **B**.

29. Ann and Barbara were comparing their ages and found that Barbara is as old as Ann was when Barbara was as old as Ann had been when Barbara was half as old as Ann is. If the sum of their present ages is 44 years, then Ann's age is

A 22

B 24

C 25

D 26

E 28

Solution:

Let Ann's and Barbara's present ages be x and y, and let their constant age difference be $d = x - y$. At the first referenced time Ann was y, so Barbara was $y - d$. At the earlier referenced time Ann was that age $y - d$, so Barbara was $y - 2d$. The final clause says this last age was $\frac{x}{2}$. Hence

$$y - 2(x - y) = \frac{x}{2},$$

which gives $6y = 5x$. Together with $x + y = 44$, this yields $x = 24$ and $y = 20$.

Therefore, the correct answer is **B**.

30. How many distinct ordered triples (x, y, z) satisfy the equations

$$\begin{aligned} x + 2y + 4z &= 12, \\ xy + 4yz + 2xz &= 22, \\ xyz &= 6? \end{aligned}$$

A none

B 1

C 2

D 4

E 6

Solution:

Set $x = 2u,\ y = v,\ z = \frac{w}{2}$. Dividing the first two equations by 2 gives

$$\begin{aligned} u + v + w &= 6, \\ uv + vw + uw &= 11, \end{aligned}$$

while $uvw = 6$. Thus u, v, w are the three roots of $p(t) = t^3 - 6t^2 + 11t - 6$. This polynomial factors as

$$p(t) = (t - 1)(t - 2)(t - 3),$$

so they are $1, 2, 3$ in any order. Their $3! = 6$ permutations produce 6 distinct ordered triples (x, y, z).

Therefore, the correct answer is **E**.

Problems: https://live.poshenloh.com/past-contests/amc12/1976