1975 AMC 12 Solutions

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1. The value of

$$\cfrac{1}{2-\cfrac{1}{2-\cfrac{1}{2-\frac{1}{2}}}}$$

is

A $\frac{3}{4}$

B $\frac{4}{5}$

C $\frac{5}{6}$

D $\frac{6}{7}$

E $\frac{6}{5}$

Solution:

Working outward,

$$\begin{aligned} 2-\frac{1}{2} &= \frac{3}{2}, \\ 2-\frac{1}{\frac{3}{2}} &= \frac{4}{3}, \\ 2-\frac{1}{\frac{4}{3}} &= \frac{5}{4}. \end{aligned}$$

The given value is therefore $\frac{1}{\frac{5}{4}} = \frac{4}{5}$.

Therefore, the correct answer is **B**.

2. For which real values of m are the simultaneous equations

$$y = mx + 3,$$
$$y = (2m - 1)x + 4$$

satisfied by at least one pair of real numbers (x, y)?

A all m

B all $m \neq 0$

C all $m \neq \frac{1}{2}$

D all $m \neq 1$

E no values of m

Solution:

The lines fail to intersect only if $m = 2m - 1$, which gives $m = 1$. Their y-intercepts are 3 and 4, so at that value they are distinct parallel lines. Every other m gives one intersection.

Therefore, the correct answer is **D**.

3. Which of the following inequalities are satisfied for all real numbers a, b, c, x, y, z which satisfy $x < a, y < b$, and $z < c$?

I. $xy + yz + zx < ab + bc + ca$

II. $x^2 + y^2 + z^2 < a^2 + b^2 + c^2$

III. $xyz < abc$

A None are satisfied.

B I only

C II only

D III only

E All are satisfied.

Solution:

Take $a = b = 1,\ c = -1$, and $x = y = 0,\ z = -10$. The required three coordinate inequalities hold. But I becomes $0 < -1$, II becomes $100 < 3$, and III becomes $0 < -1$, all false. Thus none is universally true.

Therefore, the correct answer is **A**.

4. If the side of one square is the diagonal of a second square, what is the ratio of the area of the first square to the area of the second?

A 2

B $\sqrt{2}$

C $\frac{1}{2}$

D $2\sqrt{2}$

E 4

Solution:

Let the first square have side s. The second has diagonal s, so its side is $\frac{s}{\sqrt{2}}$ and its area is $\frac{s^2}{2}$. The ratio is $\frac{s^2}{\frac{s^2}{2}} = 2.$

Therefore, the correct answer is **A**.

5. The polynomial $(x+y)^9$ is expanded in decreasing powers of x. The second and third terms have equal values when evaluated at $x=p$ and $y=q$, where p and q are positive numbers whose sum is one. What is the value of p?

- A $\frac{1}{5}$
- **B** $\frac{4}{5}$
- C $\frac{1}{4}$
- D $\frac{3}{4}$
- E $\frac{8}{9}$

Solution:

The terms are $9p^8q$ and $36p^7q^2$. Their equality, with $p, q > 0$, gives $p = 4q$. Since $p + q = 1$, we get $5q = 1$ and $p = \frac{4}{5}$.

Therefore, the correct answer is **B**.

6. The sum of the first eighty positive odd integers subtracted from the sum of the first eighty positive even integers is

- A 0
- B 20
- C 40
- D 60
- E 80

Solution:

The desired difference is

$$(2-1)+(4-3)+\cdots \\ +(160-159).$$

It is the sum of 80 ones, hence 80.

Therefore, the correct answer is **E**.

7. For which nonzero real numbers x is $\frac{|x-|x||}{x}$ a positive integer?

A for negative x only

B for positive x only

C only for x an even integer

D for all nonzero real numbers x

E for no nonzero real numbers x

Solution:

If $x > 0,$ the quotient is $0.$ If $x < 0,$ then $x - |x| = 2x < 0,$ so $|x - |x|| = -2x$ and the quotient is $-2.$ Neither value is a positive integer.

Therefore, the correct answer is **E**.

8. If the statement "All shirts in this store are on sale." is false, then which of the following statements must be true?

I. All shirts in this store are at non-sale prices.

II. There is some shirt in this store not on sale.

III. No shirt in this store is on sale.

IV. Not all shirts in this store are on sale.

Note: Originally, statement I read: All shirts in this store are not on sale.

A II only

B IV only

C I and III only

D II and IV only

E I, II and IV only

Solution:

The negation of "Every shirt is on sale" is "Some shirt is not on sale," which is II and is equivalently phrased by IV. Statements I and III make the stronger claim that no shirt is on sale, which need not follow.

Therefore, the correct answer is **D**.

9. Let $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ be arithmetic progressions such that $a_1 = 25, b_1 = 75,$ and $a_{100} + b_{100} = 100$. Find the sum of the first one hundred terms of the progression $a_1 + b_1, a_2 + b_2, \ldots$.

- A 0
- B 100
- C $10{,}000$
- D $505{,}000$
- E not enough information given to solve the problem

Solution:

Let $c_i = a_i + b_i$. This is an arithmetic progression with $c_1 = 25 + 75 = 100$ and $c_{100} = 100$. Equal endpoint terms force its common difference to be 0, so all 100 terms equal 100. Their sum is $10{,}000$.

Therefore, the correct answer is **C**.

10. The sum of the digits in base ten of $\left(10^{4n^2+8}+1\right)^2$, where n is a positive integer, is

A 4

B $4n$

C $2+2n$

D $4n^2$

E n^2+n+2

Solution:

For $k = 4n^2 + 8$,

$$(10^k + 1)^2 = 10^{2k} + 2 \cdot 10^k + 1.$$

Its only nonzero digits are $1, 2, 1$, so their sum is 4.

Therefore, the correct answer is **A**.

11. Let P be an interior point of circle K other than the center of K. Form all chords of K which pass through P, and determine their midpoints. The locus of these midpoints is

A a circle with one point deleted

B a circle if the distance from P to the center of K is less than one half the radius of K; otherwise a circular arc of less than 360°

C a semicircle with one point deleted

D a semicircle

E a circle

Solution:

If M is a chord midpoint and O is the center, then OM is perpendicular to the chord, so $\angle OMP = 90^\circ$. Thus M lies on the circle with diameter OP. Conversely, for every M on that circle, the line PM is perpendicular to OM and cuts a chord of K whose midpoint is M. The endpoint case $M = P$ also gives the chord through P perpendicular to OP.

Therefore, the correct answer is **E**.

12. If $a \neq b, a^3 - b^3 = 19x^3$, and $a - b = x$, which of the following conclusions is correct?

- A $a = 3x$
- B $a = 3x$ or $a = -2x$
- C $a = -3x$ or $a = 2x$
- D $a = 3x$ or $a = 2x$
- E $a = 2x$

Solution:

Factoring and dividing by $x \neq 0$ gives $a^2 + ab + b^2 = 19x^2$. With $b = a - x$,

$$3a^2 - 3ax + x^2 = 19x^2,$$

so

$$a^2 - ax - 6x^2 = 0,$$
$$(a - 3x)(a + 2x) = 0.$$

Hence $a = 3x$ or $a = -2x$.

Therefore, the correct answer is **B**.

13. The equation $x^6 - 3x^5 - 6x^3 - x + 8 = 0$ has

- A no real roots
- B exactly two distinct negative roots
- C exactly one negative root
- D no negative roots, but at least one positive root
- E none of these

Solution:

For $x < 0$, each of $x^6, -3x^5, -6x^3, -x,$ and 8 is positive, so there is no negative root. At $x = 0$ the polynomial is $8,$ while at $x = 1$ it is $-1.$ Continuity therefore gives a positive root between 0 and $1.$

Therefore, the correct answer is **D**.

14. If the whatsis is so when the whosis is is and the so and so is is · so, what is the whosis · whatsis when the whosis is so, the so and so is so · so, and the is is two (whatsis, whosis, is and so are variables taking positive values)?

A whosis · is · so

B whosis

C is

D so

E so and so

Solution:

The first condition says that $H = I$ and $2S = IS$ imply $W = S$. Since $S > 0$, the latter equation gives $I = 2$. In the requested case, $H = S,\ 2S = S^2$, and $I = 2$, so positivity gives $H = I = S = 2$. Hence $W = S = 2$ and $HW = 4 = S + S$.

Therefore, the correct answer is **E**.

15. In the sequence of numbers $1, 3, 2, \ldots,$ each term after the first two is equal to the term preceding it minus the term preceding that. The sum of the first one hundred terms of the sequence is

- A 5
- B 4
- C 2
- D 1
- E -1

Solution:

The sequence begins

$$1, 3, 2, -1, -3, -2, 1, 3, \ldots$$

and repeats every 6 terms, with period sum 0. The first 96 terms sum to 0, and the last four sum to $1 + 3 + 2 - 1 = 5$.

Therefore, the correct answer is **A**.

16. If the first term of an infinite geometric series is a positive integer, the common ratio is the reciprocal of a positive integer, and the sum of the series is 3, then the sum of the first two terms of the series is

A $\frac{1}{3}$

B $\frac{2}{3}$

C $\frac{8}{3}$

D 2

E $\frac{9}{2}$

Solution:

Let the first term be a and the ratio $\frac{1}{n}$. Convergence gives $n > 1$, and

$$\frac{a}{1-\frac{1}{n}} = 3, \qquad a = 3 - \frac{3}{n}.$$

Thus n divides 3, so $n = 3$ and $a = 2$. The first two terms sum to $2 + \frac{2}{3} = \frac{8}{3}$.

Therefore, the correct answer is **C**.

17. A man can commute either by train or by bus. If he goes to work on the train in the morning, he comes home on the bus in the afternoon; and if he comes home in the afternoon on the train, he took the bus in the morning. During a total of x working days, the man took the bus to work in the morning 8 times, came home by bus in the afternoon 15 times, and commuted by train (either morning or afternoon) 9 times. Find x.

A 19

B 18

C 17

D 16

E not enough information given to solve the problem

Solution:

There were $8 + 15 = 23$ bus trips and 9 train trips, hence 32 one-way trips in all. Since each working day contributes two trips, $2x = 32$ and $x = 16$. The conditional statements are consistent but not needed for this count.

Therefore, the correct answer is **D**.

18. A positive integer N with three digits in its base ten representation is chosen at random, with each three-digit number having an equal chance of being chosen. The probability that $\log_2 N$ is an integer is

A 0

B $\frac{3}{899}$

C $\frac{1}{225}$

D $\frac{1}{300}$

E $\frac{1}{450}$

Solution:

There are $999 - 100 + 1 = 900$ three-digit integers. The three powers of 2 among them are $128, 256,$ and $512.$ Thus the probability is $\frac{3}{900} = \frac{1}{300}.$

Therefore, the correct answer is **D**.

19. Which positive numbers x satisfy the equation $(\log_3 x)(\log_x 5) = \log_3 5$?

A 3 and 5 only

B $3, 5$ and 15 only

C only numbers of the form $5^n \cdot 3^m$, where n and m are positive integers

D all positive $x \neq 1$

E none of these

Solution:

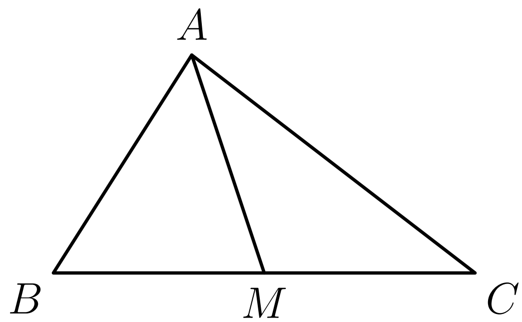
For $x > 0,\ x \neq 1,$ change of base makes the left side

$$\frac{\log x}{\log 3} \cdot \frac{\log 5}{\log x}.$$

Canceling $\log x$ leaves $\frac{\log 5}{\log 3} = \log_3 5.$ The expression is undefined at $x = 1.$

Therefore, the correct answer is **D**.

20. In the adjoining figure triangle ABC is such that $AB = 4$ and $AC = 8$. If M is the midpoint of BC and $AM = 3$, what is the length of BC?



A $2\sqrt{26}$

B $2\sqrt{31}$

C 9

D $4 + 2\sqrt{13}$

E not enough information given to solve the problem

Solution:

Let $BM = CM = x$, so $BC = 2x$. By Apollonius' theorem,

$$4^2 + 8^2 = 2(3^2 + x^2).$$

Hence $80 = 18 + 2x^2,\ x^2 = 31$, and $BC = 2\sqrt{31}$.

Therefore, the correct answer is **B**.

21. Suppose $f(x)$ is defined for all real numbers x; $f(x) > 0$ for all x; and $f(a)f(b) = f(a+b)$ for all a and b. Which of the following statements are true?

I. $f(0) = 1$

II. $f(-a) = \frac{1}{f(a)}$ for all a

III. $f(a) = \sqrt[3]{f(3a)}$ for all a

IV. $f(b) > f(a)$ if $b > a$

A $\quad$ III and IV only

B $\quad$ I, III and IV only

C $\quad$ I, II and IV only

D $\quad$ I, II and III only

E $\quad$ All are true.

Solution:

Taking $a = 0$ and using positivity gives $f(0) = 1$. Taking $b = -a$ gives $f(a)f(-a) = 1$. Also $f(3a) = f(a)^3$, so positivity permits the positive cube root in III. But $f(x) = 1$ satisfies the functional equation and is not strictly increasing, so IV need not hold.

Therefore, the correct answer is **D**.

22. If p and q are primes and $x^2 - px + q = 0$ has distinct positive integral roots, then which of the following statements are true?

I. The difference of the roots is odd.

II. At least one root is prime.

III. $p^2 - q$ is prime.

IV. $p + q$ is prime.

A $\quad \mathrm{I}$ only

B $\quad \mathrm{II}$ only

C $\quad \mathrm{II}$ and III only

D $\quad \mathrm{I}, \mathrm{II}$ and IV only

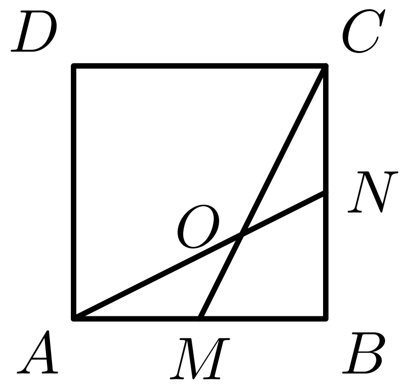
E $\quad$ All are true.

Solution:

The positive integer roots have product q, so they are 1 and q. Their sum is $p = q + 1$, which is prime only when $q = 2$, giving $p = 3$ and roots $1, 2$. Their difference is 1, one root is prime, $p^2 - q = 7$, and $p + q = 5$; therefore all four statements hold.

Therefore, the correct answer is **E**.

23. In the adjoining figure AB and BC are adjacent sides of square $ABCD$; M is the midpoint of AB; N is the midpoint of BC; and AN and CM intersect at O. The ratio of the area of $AOCD$ to the area of $ABCD$ is



A $\frac{5}{6}$

B $\frac{3}{4}$

C $\frac{2}{3}$

D $\frac{\sqrt{3}}{2}$

E $\frac{\sqrt{3}-1}{2}$

Solution:

Take $A=(0,0), B=(1,0), C=(1,1), D=(0,1)$. Since O is the centroid of triangle ABC, $O=(\frac{2}{3},\frac{1}{3})$. Triangles AOB and COB each have area $\frac{1}{6}$. Removing them from the unit square leaves

$$[AOCD]=1-\frac{1}{6}-\frac{1}{6}=\frac{2}{3}.$$

Therefore, the correct answer is **C**.

24. In triangle $ABC, \angle C = \theta$ and $\angle B = 2\theta$, where $0^\circ < \theta < 60^\circ$. The circle with center A and radius AB intersects AC at D and intersects BC, extended if necessary, at B and at E (E may coincide with B). Then $EC = AD$

A for no values of θ

B only if $\theta = 45^\circ$

C only if $0^\circ < \theta \leq 45^\circ$

D only if $45^\circ \leq \theta < 60^\circ$

E for all θ such that $0^\circ < \theta < 60^\circ$

Solution:

Since $AB = AE$, triangle ABE is isosceles. For $0^\circ < \theta < 45^\circ, E$ lies between B, C; the exterior-angle theorem in triangle AEC gives $2\theta = \angle EAC + \theta$, so $\angle EAC = \theta$. For $45^\circ < \theta < 60^\circ, E$ lies beyond B; then $\angle AEC = 180^\circ - 2\theta$, hence

$$\begin{aligned}\angle EAC &= 180^\circ \\ &\quad - (180^\circ - 2\theta) - \theta \\ &= \theta.\end{aligned}$$

At $\theta = 45^\circ,\ E = B$ and the conclusion is immediate. Thus triangle AEC is always isosceles, so $EC = EA = AD$.

Therefore, the correct answer is **E**.

25. A woman, her brother, her son and her daughter are chess players (all relations by birth). The worst player's twin (who is one of the four players) and the best player are of opposite sex. The worst player and the best player are the same age. Who is the worst player?

- A the woman
- B her son
- C her brother
- D her daughter
- E No solution is consistent with the given information.

Solution:

If the son is worst, the daughter can be his twin and the brother can be best; the son and brother may have the same age, so all conditions can hold. If the woman is worst, the brother is her twin and the daughter must be best, but mother and daughter cannot be the same age. If the brother is worst, the woman is his twin and the son must be best, again impossible in age. If the daughter is worst, her twin is the son and the woman must be best, also impossible in age. Thus only the son works.

Therefore, the correct answer is **B**.

26. In acute triangle ABC the bisector of $\angle A$ meets side BC at D. The circle with center B and radius BD intersects side AB at M; and the circle with center C and radius CD intersects side AC at N. Then it is always true that

A $\angle CND + \angle BMD - \angle DAC = 120^\circ$

B $AMDN$ is a trapezoid

C BC is parallel to MN

D $AM - AN = \frac{3(DB-DC)}{2}$

E $AB - AC = \frac{3(DB-DC)}{2}$

Solution:

The angle bisector theorem gives

$$\frac{BD}{CD} = \frac{AB}{AC}.$$

Since $BM = BD$ and $CN = CD$, we have $\frac{BM}{CN} = \frac{AB}{AC}$. Therefore M and N divide AB and AC proportionally from B and C, so the converse of the side-splitter theorem gives $MN \parallel BC$.

Therefore, the correct answer is **C**.

27. If $p, q,$ and r are distinct roots of $x^3 - x^2 + x - 2 = 0,$ then $p^3 + q^3 + r^3$ equals

A -1

B 1

C 3

D 5

E none of these

Solution:

Vieta gives $p + q + r = 1$ and $pq + pr + qr = 1.$ Thus

$$\begin{aligned} p^2 + q^2 + r^2 &= (p + q + r)^2 \\ &\quad - 2(pq + pr + qr) \\ &= -1. \end{aligned}$$

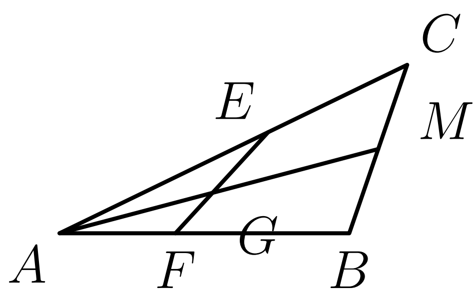
Each root t satisfies $t^3 = t^2 - t + 2.$ Summing over the three roots yields

$$p^3 + q^3 + r^3 = -1 - 1 + 6 = 4.$$

This is not among choices A–D.

Therefore, the correct answer is **E**.

28. In triangle ABC shown in the adjoining figure, M is the midpoint of side BC, $AB = 12$, and $AC = 16$. Points E and F are taken on AC and AB, respectively, and lines EF and AM intersect at G. If $AE = 2AF$, then $\frac{EG}{GF}$ equals



A $\frac{3}{2}$

B $\frac{4}{3}$

C $\frac{5}{4}$

D $\frac{6}{5}$

E not enough information given to solve the problem

Solution:

Put A at the origin and write the position vectors of B, C as $\mathbf{b}, \mathbf{c}$. If $F = t\mathbf{b}$, then $AF = 12t$ and $AE = 24t$, so $E = (\frac{3t}{2})\mathbf{c}$. Write $G = (1-u)F + uE$. Because G also lies on the median AM, its $\mathbf{b}$ and $\mathbf{c}$ coefficients are equal:

$$(1-u)t = \frac{3}{2}ut.$$

Hence $u = \frac{2}{5}$. Along FE, $\frac{FG}{FE} = \frac{2}{5}$ and $\frac{EG}{FE} = \frac{3}{5}$, so $\frac{EG}{GF} = \frac{3}{2}$.

Therefore, the correct answer is **A**.

29. What is the smallest integer larger than $(\sqrt{3}+\sqrt{2})^6$?

A 972

B 971

C 970

D 969

E 968

Solution:

Let $\alpha = \sqrt{3}+\sqrt{2}$ and $\beta = \sqrt{3}-\sqrt{2}$. Adding their sixth powers cancels all odd radical terms:

$$\begin{aligned}\alpha^6+\beta^6 &= 2\big(3^3+15(3^2)(2)\\ &\qquad +15(3)(2^2)+2^3\big)\\ &= 970.\end{aligned}$$

Since $0<\beta<1$, we have $969<\alpha^6 = 970-\beta^6<970$. Thus the smallest larger integer is 970.

Therefore, the correct answer is **C**.

30. Let $x = \cos 36^\circ - \cos 72^\circ$. Then x equals

- A $\frac{1}{3}$
- B $\frac{1}{2}$
- C $3 - \sqrt{6}$
- D $2\sqrt{3} - 3$
- E none of these

Solution:

Set $w = \cos 36^\circ$ and $y = \cos 72^\circ$. The double-angle identities give

$$y = 2w^2 - 1, \qquad w = 1 - 2y^2.$$

Adding yields

$$\begin{aligned} w + y &= 2(w^2 - y^2) \\ &= 2(w - y)(w + y). \end{aligned}$$

Since $w + y \neq 0$, division gives $w - y = \frac{1}{2}$. Thus $x = \frac{1}{2}$.

Therefore, the correct answer is **B**.

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