1974 AMC 12 Solutions

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1. If $x \neq 0$ or 4 and $y \neq 0$ or 6, then $\frac{2}{x} + \frac{3}{y} = \frac{1}{2}$ is equivalent to

A $4x + 3y = xy$

B $y = \frac{4x}{6-y}$

C $\frac{x}{2} + \frac{y}{3} = 2$

D $\frac{4y}{y-6} = x$

E none of these

Solution:

Multiplying by $2xy$ gives $4y + 6x = xy$, so $4y = x(y-6)$. Since $y \neq 6$,

$$x = \frac{4y}{y-6}.$$

Therefore, the correct answer is **D**.

2. Let x_1 and x_2 be such that $x_1 \neq x_2$ and $3x_i^2 - hx_i = b, i = 1, 2.$ Then $x_1 + x_2$ equals

A $-\frac{h}{3}$

B $\frac{h}{3}$

C $\frac{b}{3}$

D $2b$

E $-\frac{b}{3}$

Solution:

The distinct numbers x_1, x_2 are the two roots of

$$3x^2 - hx - b = 0.$$

By Vieta's formulas, their sum is $-\frac{(-h)}{3} = \frac{h}{3}$.

Therefore, the correct answer is **B**.

3. The coefficient of x^7 in the polynomial expansion of

$$(1+2x-x^2)^4$$

is

A -8

B 12

C 6

D -12

E none of these

Solution:

The only way four selected terms can have total degree 7 is to choose $2x$ once and $-x^2$ three times. There are 4 choices for the first factor, so the coefficient is

$$4(2)(-1)^3 = -8.$$

Therefore, the correct answer is **A**.

4. What is the remainder when $x^{51} + 51$ is divided by $x + 1$?

A 0

B 1

C 49

D 50

E 51

Solution:

By the polynomial remainder theorem, the remainder is

$$(-1)^{51} + 51 = -1 + 51 = 50.$$

Therefore, the correct answer is **D**.

5. Given a quadrilateral $ABCD$ inscribed in a circle with side AB extended beyond B to point E, if $\angle BAD = 92^\circ$ and $\angle ADC = 68^\circ$, find $\angle EBC$.

A 66°

B 68°

C 70°

D 88°

E 92°

Solution:

Because $ABCD$ is cyclic, $\angle ABC + \angle ADC = 180^\circ$. Since BE extends BA, $\angle EBC + \angle ABC = 180^\circ$ as well. Hence

$$\angle EBC = \angle ADC = 68^\circ.$$

The given 92° angle is unnecessary.

Therefore, the correct answer is **B**.

6. For positive real numbers x and y define $x * y = \frac{x \cdot y}{x+y}$; then

A “$*$” is commutative but not associative

B “$*$” is associative but not commutative

C “$*$” is neither commutative nor associative

D “$*$” is commutative and associative

E none of these

Solution:

Symmetry in x, y makes the operation commutative. Also,

$$\frac{1}{x * y} = \frac{1}{x} + \frac{1}{y}.$$

Thus both $(x * y) * z$ and $x * (y * z)$ have reciprocal $\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$, so they are equal. The operation is associative too.

Therefore, the correct answer is **D**.

7. A town's population increased by $1{,}200$ people, and then this new population decreased by 11%. The town now had 32 less people than it did before the $1{,}200$ increase. What is the original population?

- A $1{,}200$
- B $11{,}200$
- C $9{,}968$
- D $10{,}000$
- E none of these

Solution:

Let the original population be P. Then

$$0.89(P + 1200) = P - 32.$$

Therefore $0.11P = 1100$, giving $P = 10{,}000$.

Therefore, the correct answer is **D**.

8. What is the smallest prime number dividing the sum $3^{11} + 5^{13}$?

A 2

B 3

C 5

D $3^{11} + 5^{13}$

E none of these

Solution:

Both 3^{11} and 5^{13} are odd, so their sum is even. Hence it is divisible by $2,$ the smallest prime.

Therefore, the correct answer is **A**.

9. The integers greater than one are arranged in five columns as follows:

$$\begin{array}{ccccc} & 2 & 3 & 4 & 5 \\ 9 & 8 & 7 & 6 & \\ & 10 & 11 & 12 & 13 \\ 17 & 16 & 15 & 14 & \\ & \vdots & \vdots & \vdots & \vdots \end{array}$$

(Four consecutive integers appear in each row; in the first, third and other odd numbered rows, the integers appear in the last four columns and increase from left to right; in the second, fourth and other even numbered rows, the integers appear in the first four columns and increase from right to left.)

In which column will the number $1{,}000$ fall?

A first

B second

C third

D fourth

E fifth

Solution:

Each pair of rows is a repeating block of eight integers. In every such block, its multiple of 8 lies in the second column, as do 8 and $16.$ Since $1000 = 125 \cdot 8,$ it also lies in the second column.

Therefore, the correct answer is **B**.

10. What is the smallest integral value of k such that

$$2x(kx-4)-x^2+6=0$$

has no real roots?

A -1

B 2

C 3

D 4

E 5

Solution:

The equation is $(2k-1)x^2-8x+6=0$. Its discriminant is

$$64-24(2k-1)=88-48k.$$

This is negative exactly when $k>\frac{11}{6}$, so the least integer possible is 2.

Therefore, the correct answer is **B**.

11. If (a, b) and (c, d) are two points on the line whose equation is $y = mx + k$, then the distance between (a, b) and (c, d), in terms of a, c, and m, is

A $|a-c|\sqrt{1+m^2}$

B $|a+c|\sqrt{1+m^2}$

C $\frac{|a-c|}{\sqrt{1+m^2}}$

D $|a-c|(1+m^2)$

E $|a-c||m|$

Solution:

Since both points are on the line, $b-d=m(a-c)$. Their distance is therefore

$$\begin{aligned}\ell &= \sqrt{(a-c)^2+(b-d)^2} \\ &= \sqrt{(a-c)^2(1+m^2)} \\ &= |a-c|\sqrt{1+m^2}.\end{aligned}$$

Therefore, the correct answer is **A**.

12. If $g(x) = 1 - x^2$ and $f(g(x)) = \frac{1-x^2}{x^2}$ when $x \neq 0,$ then $f(\frac{1}{2})$ equals

A $\frac{3}{4}$

B 1

C 3

D $\frac{\sqrt{2}}{2}$

E $\sqrt{2}$

Solution:

If $g(x) = \frac{1}{2}$, then $x^2 = \frac{1}{2}$, and either corresponding x is nonzero. Hence

$$f(\frac{1}{2}) = \frac{1-x^2}{x^2} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1.$$

Therefore, the correct answer is **B**.

13. Which of the following is equivalent to "If P is true then Q is false"?

A "P is true or Q is false."

B "If Q is false then P is true."

C "If P is false then Q is true."

D "If Q is true then P is false."

E "If Q is true then P is true."

Solution:

The contrapositive of "P true implies Q false" is "Q not false implies P not true." This is exactly "If Q is true then P is false."

Therefore, the correct answer is **D**.

14. Which statement is correct?

A If $x < 0$, then $x^2 > x$.

B If $x^2 > 0$, then $x > 0$.

C If $x^2 > x$, then $x > 0$.

D If $x^2 > x$, then $x < 0$.

E If $x < 1$, then $x^2 < x$.

Solution:

If $x < 0$, then $x^2 > 0 > x$, so choice A always holds. The values $x = -1, x = -1$, $x = 2$, and $x = -1$ respectively disprove B, C, D, and E.

Therefore, the correct answer is **A**.

15. If $x < -2$ then $\left|1 - |1 + x|\right|$ equals

A $2 + x$

B $-2 - x$

C x

D $-x$

E -2

Solution:

Since $x < -2, 1 + x < 0$, so $|1 + x| = -1 - x$. Thus $\left|1 - |1 + x|\right|$ becomes $|1 - (-1 - x)|$, which is $|2 + x| = -2 - x$ because $2 + x < 0$.

Therefore, the correct answer is **B**.

16. A circle of radius r is inscribed in a right isosceles triangle, and a circle of radius R is circumscribed about the triangle. Then $\frac{R}{r}$ equals

A $1+\sqrt{2}$

B $\frac{2+\sqrt{2}}{2}$

C $\frac{\sqrt{2}-1}{2}$

D $\frac{1+\sqrt{2}}{2}$

E $2(2-\sqrt{2})$

Solution:

With legs a, a and hypotenuse $a\sqrt{2}$,

$$\begin{aligned} R &= \frac{a\sqrt{2}}{2}, \\ r &= \frac{a+a-a\sqrt{2}}{2}. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{R}{r} &= \frac{\sqrt{2}}{2-\sqrt{2}} \\ &= \frac{1}{\sqrt{2}-1} = 1+\sqrt{2}. \end{aligned}$$

Therefore, the correct answer is **A**.

17. If $i^2 = -1$, then $(1+i)^{20} - (1-i)^{20}$ equals

A -1024

B $-1024i$

C 0

D 1024

E $1024i$

Solution:

Since $(1+i)^2 = 2i$, we have $(1+i)^4 = -4$. Likewise $(1-i)^2 = -2i$ and $(1-i)^4 = -4$. Hence each twentieth power equals $(-4)^5 = -1024$, and their difference is 0.

Therefore, the correct answer is **C**.

18. If $\log_8 3 = p$ and $\log_3 5 = q$, then, in terms of p and q, $\log_{10} 5$ equals

A pq

B $\frac{3p+q}{5}$

C $\frac{1+3pq}{p+q}$

D $\frac{3pq}{1+3pq}$

E $p^2 + q^2$

Solution:

From $p = \log_8 3, \log_2 3 = 3p$. Therefore

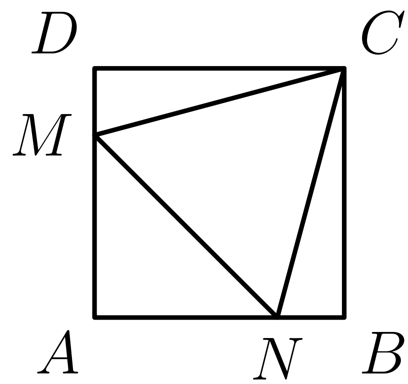
$$\log_2 5 = (\log_2 3)(\log_3 5) = 3pq.$$

Since $\log_2 10 = 1 + \log_2 5 = 1 + 3pq$,

$$\log_{10} 5 = \frac{\log_2 5}{\log_2 10} = \frac{3pq}{1 + 3pq}.$$

Therefore, the correct answer is **D**.

19. In the adjoining figure $ABCD$ is a square and CMN is an equilateral triangle. If the area of $ABCD$ is one square inch, then the area of CMN in square inches is



A $2\sqrt{3}-3$

B $1-\frac{\sqrt{3}}{3}$

C $\frac{\sqrt{3}}{4}$

D $\frac{\sqrt{2}}{3}$

E $4-2\sqrt{3}$

Solution:

The square has side 1. Since $CM = CN$, symmetry of their squared lengths gives $DM = NB = x$. Also,

$$CM^2 = 1 + x^2,$$
$$MN^2 = 2(1-x)^2.$$

Equating these because CMN is equilateral gives $x^2 - 4x + 1 = 0$. The root in $(0,1)$ is $x = 2 - \sqrt{3}$.

The triangle's side has square $1+x^2$, so its area is

$$\begin{aligned} A_{\triangle} &= \frac{\sqrt{3}}{4}(1+x^2) \\ &= \frac{\sqrt{3}}{4}\big(1+(2-\sqrt{3})^2\big) \\ &= 2\sqrt{3}-3. \end{aligned}$$

Therefore, the correct answer is **A**.

20. Let

$$T = \frac{1}{3-\sqrt{8}} - \frac{1}{\sqrt{8}-\sqrt{7}} + \frac{1}{\sqrt{7}-\sqrt{6}} - \frac{1}{\sqrt{6}-\sqrt{5}} + \frac{1}{\sqrt{5}-2};$$

then

A $T < 1$

B $T = 1$

C $1 < T < 2$

D $T > 2$

E $T = \dfrac{1}{(3-\sqrt{8})(\sqrt{8}-\sqrt{7})(\sqrt{7}-\sqrt{6}) \cdot (\sqrt{6}-\sqrt{5})(\sqrt{5}-2)}$

Solution:

Rationalizing the five terms gives

$$T = (3+\sqrt{8}) - (\sqrt{8}+\sqrt{7}) + (\sqrt{7}+\sqrt{6}) - (\sqrt{6}+\sqrt{5}) + (\sqrt{5}+2) = 5.$$

Thus $T > 2.$

Therefore, the correct answer is **D**.

21. In a geometric series of positive terms the difference between the fifth and fourth terms is 576, and the difference between the second and first terms is 9. What is the sum of the first five terms of this series?

A 1061

B 1023

C 1024

D 768

E none of these

Solution:

Let the first term be a and the ratio be r. The conditions give

$$\begin{aligned} ar^3(r-1) &= 576, \\ a(r-1) &= 9. \end{aligned}$$

Their quotient gives $r^3 = 64$, so $r = 4$. Then $3a = 9$, hence $a = 3$. The sum is

$$\begin{aligned} S &= 3(1+4+16+64+256) \\ &= 3 \cdot 341 \\ &= 1023. \end{aligned}$$

Therefore, the correct answer is **B**.

22. The minimum value of $\sin \frac{A}{2} - \sqrt{3} \cos \frac{A}{2}$ is attained when A is

- A -180°
- B 60°
- C 120°
- D 0°
- E none of these

Solution:

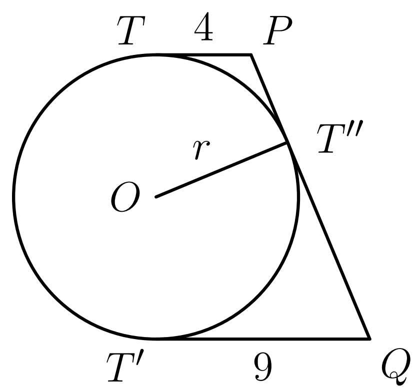
Let $u = \frac{A}{2}$. Then $\sin u - \sqrt{3} \cos u$ equals $2 \sin(u - 60^\circ)$. Its minimum occurs when $u - 60^\circ = 270^\circ + 360^\circ n$, so

$$A = 660^\circ + 720^\circ n$$

for an integer n. None of the first four choices has this form.

Therefore, the correct answer is **E**.

23. In the adjoining figure TP and $T'Q$ are parallel tangents to a circle of radius r, with T and T' the points of tangency. $PT''Q$ is a third tangent with T'' as point of tangency. If $TP = 4$ and $T'Q = 9$ then r is



A $\frac{25}{6}$

B 6

C $\frac{25}{4}$

D a number other than $\frac{25}{6}$, 6, or $\frac{25}{4}$

E not determinable from the given information

Solution:

The radii to the tangency points make congruent right triangles from each external point, so OP and OQ bisect the two tangent angles. Because the other two tangents are parallel, these half-angles add to 90°; hence $\angle POQ = 90^\circ$.

Thus $OT'' = r$ is the altitude to the hypotenuse PQ. Equal tangent segments give $PT'' = PT = 4$ and $QT'' = QT' = 9$. The altitude theorem yields

$$r^2 = (PT'')(QT'') = 4 \cdot 9 = 36,$$

so $r = 6$.

Therefore, the correct answer is **B**.

24. A fair die is rolled six times. The probability of rolling at least a five at least five times is

A $\frac{13}{729}$

B $\frac{12}{729}$

C $\frac{2}{729}$

D $\frac{3}{729}$

E none of these

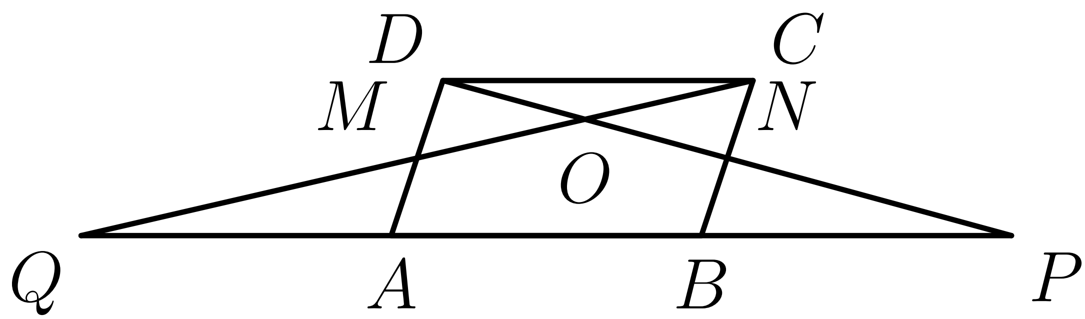
Solution:

Let a success be a 5 or $6,$ so its probability is $\frac{1}{3}$. The desired probability is

$$\begin{aligned} & \binom{6}{5}\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right)+\left(\frac{1}{3}\right)^6 \\ & =\frac{12}{729}+\frac{1}{729} \\ & =\frac{13}{729}. \end{aligned}$$

Therefore, the correct answer is **A**.

25. In parallelogram $ABCD$ of the accompanying diagram, line DP is drawn bisecting BC at N and meeting AB (extended) at P. From vertex C, line CQ is drawn bisecting side AD at M and meeting AB (extended) at Q. Lines DP and CQ meet at O. If the area of parallelogram $ABCD$ is k, then the area of triangle QPO is equal to



A k

B $\frac{6k}{5}$

C $\frac{9k}{8}$

D $\frac{5k}{4}$

E $2k$

Solution:

Choose coordinates $A = (0, 0), B = (1, 0), D = (u, v)$, and $C = (u + 1, v)$, so $k = v$. The midpoints are

$$M = (\frac{u}{2}, \frac{v}{2}),$$
$$N = (1 + \frac{u}{2}, \frac{v}{2}).$$

Extending CM and DN to $y = 0$ gives $Q = (-1, 0)$ and $P = (2, 0)$.

Parameterizing DP and CQ shows their intersection is $O = (\frac{3u}{4} + \frac{1}{2}, \frac{3v}{4})$. Thus $QP = 3$ and the height from O is $\frac{3v}{4}$, so

$$
\begin{aligned}
[QPO] &= \frac{1}{2}(3)\left(\frac{3v}{4}\right) \\
&= \frac{9v}{8} = \frac{9k}{8}.
\end{aligned}
$$

Therefore, the correct answer is **C**.

26. The number of distinct positive integral divisors of $(30)^4$ excluding 1 and $(30)^4$ is

A 100

B 125

C 123

D 30

E none of these

Solution:

Because

$$30^4 = 2^4 \cdot 3^4 \cdot 5^4,$$

a divisor independently chooses each of three exponents from $0, 1, 2, 3, 4$. There are $5^3 = 125$ divisors. Excluding 1 and 30^4 leaves $125 - 2 = 123.$

Therefore, the correct answer is **C**.

27. If $f(x) = 3x + 2$ for all real x, then the statement:

"$|f(x) + 4| < a$ whenever $|x + 2| < b$ and $a > 0$ and $b > 0$"

is true when

A $b \leq \frac{a}{3}$

B $b > \frac{a}{3}$

C $a \leq \frac{b}{3}$

D $a > \frac{b}{3}$

E The statement is never true.

Solution:

We have

$$|f(x) + 4| = |3x + 6| = 3|x + 2|.$$

If $|x + 2| < b$ and $b \leq \frac{a}{3}$, then $3|x + 2| < 3b \leq a$, as required. If $b > \frac{a}{3}$, one can choose $|x + 2|$ strictly between $\frac{a}{3}$ and b, so the statement fails. Thus the exact condition is $b \leq \frac{a}{3}$.

Therefore, the correct answer is **A**.

28. Which of the following is satisfied by all numbers x of the form

$$x = \frac{a_1}{3} + \frac{a_2}{3^2} + \cdots + \frac{a_{25}}{3^{25}},$$

where a_1 is 0 or 2, a_2 is 0 or $2, \ldots, a_{25}$ is 0 or 2?

A $0 \leq x < \frac{1}{3}$

B $\frac{1}{3} \leq x < \frac{2}{3}$

C $\frac{2}{3} \leq x < 1$

D $0 \leq x < \frac{1}{3}$ or $\frac{2}{3} \leq x < 1$

E $\frac{1}{2} \leq x \leq \frac{3}{4}$

Solution:

If $a_1 = 0$, then

$$0 \leq x \leq \sum_{n=2}^{25} \frac{2}{3^n} < \sum_{n=2}^{\infty} \frac{2}{3^n} = \frac{1}{3}.$$

If $a_1 = 2$, then $x \geq \frac{2}{3}$, while

$$x \leq \sum_{n=1}^{25} \frac{2}{3^n} < \sum_{n=1}^{\infty} \frac{2}{3^n} = 1.$$

Hence every such x lies in one of the two stated outer thirds.

Therefore, the correct answer is **D**.

29. For $p = 1, 2, \ldots, 10$ let S_p be the sum of the first 40 terms of the arithmetic progression whose first term is p and whose common difference is $2p - 1$; then $S_1 + S_2 + \cdots + S_{10}$ is

A $80{,}000$

B $80{,}200$

C $80{,}400$

D $80{,}600$

E $80{,}800$

Solution:

For each p,

$$\begin{aligned} S_p &= \frac{40}{2}\big(2p + 39(2p-1)\big) \\ &= 1600p - 780. \end{aligned}$$

Since $1 + 2 + \cdots + 10 = 55$, therefore

$$\begin{aligned} \sum_{p=1}^{10} S_p &= 1600(55) - 7800 \\ &= 80{,}200. \end{aligned}$$

Therefore, the correct answer is **B**.

30. A line segment is divided so that the lesser part is to the greater part as the greater part is to the whole. If R is the ratio of the lesser part to the greater part, then the value of

$$R^{\left[R^{\left(R^2+R^{-1}\right)}+R^{-1}\right]}+R^{-1}$$

is

A 2

B $2R$

C R^{-1}

D $2+R^{-1}$

E $2+R$

Solution:

Scale the greater part to 1, making the lesser part R. The condition gives $R=\frac{1}{1+R}$, so

$$R^2+R=1, \qquad R^{-1}=R+1.$$

Consequently

$$R^2+R^{-1}=R^2+R+1=2.$$

Working outward, the exponent of the outer R becomes

$$\begin{aligned} R^{R^2+R^{-1}}+R^{-1} &= R^2+R^{-1} \\ &= 2. \end{aligned}$$

The whole expression is therefore $R^2+R^{-1}=2.$

Therefore, the correct answer is **A**.

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