1973 AMC 12 Solutions

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1. A chord which is the perpendicular bisector of a radius of length 12 in a circle has length

   A $3\sqrt{3}$

   B 27

   C $6\sqrt{3}$

   D $12\sqrt{3}$

   E none of these

Solution:

Let M be the midpoint of the chord and also the midpoint of the radius. The distance from the center O to M is 6, while the circle's radius is 12. If A is one endpoint of the chord, then

$$AM = \sqrt{12^2 - 6^2} = 6\sqrt{3}.$$

The full chord has length $2AM = 12\sqrt{3}$.

Therefore, the correct answer is **D**.

2. One thousand unit cubes are fastened together to form a large cube with edge length 10 units; this is painted and then separated into the original cubes. The number of these unit cubes which have at least one face painted is

A 600

B 520

C 488

D 480

E 400

Solution:

The unpainted cubes are precisely the interior $8 \times 8 \times 8$ cube. Therefore the number having at least one painted face is

$$10^3 - 8^3 = 1000 - 512 = 488.$$

Therefore, the correct answer is **C**.

3. The stronger Goldbach conjecture states that any even integer greater than 7 can be written as the sum of two different prime numbers. For such representations of the even number $126,$ the largest possible difference between the two primes is

Note: The regular Goldbach conjecture states that any even integer greater than 3 is expressible as a sum of two primes. Neither this conjecture nor the stronger version has been settled.

A 112

B 100

C 92

D 88

E 80

Solution:

For primes $p < q$ with $p + q = 126,$ the difference $q - p = 126 - 2p$ is largest when p is as small as possible. The complements of $3, 5, 7,$ and 11 are $123, 121, 119,$ and $115,$ none prime. For $p = 13,$ the complement is $113,$ which is prime. Thus the largest difference is

$$113 - 13 = 100.$$

Therefore, the correct answer is **B**.

4. Two congruent 30°-60°-90° triangles are placed so that they overlap partly and their hypotenuses coincide. If the hypotenuse of each triangle is 12, the area common to both triangles is

A $6\sqrt{3}$

B $8\sqrt{3}$

C $9\sqrt{3}$

D $12\sqrt{3}$

E 24

Solution:

Put the common hypotenuse from $A = (0, 0)$ to $B = (12, 0)$. The two triangles have their third vertices at $(9, 3\sqrt{3})$ and $(3, 3\sqrt{3})$. Their common region is the triangle with base AB and apex where

$$y = \frac{x}{\sqrt{3}} \quad \text{and} \quad y = \frac{12 - x}{\sqrt{3}}$$

meet. This occurs at $x = 6$, with height $2\sqrt{3}$. Hence the common area is

$$\frac{1}{2}(12)(2\sqrt{3}) = 12\sqrt{3}.$$

Therefore, the correct answer is **D**.

5. Of the following five statements, I to V, about the binary operation of averaging (arithmetic mean),

I. Averaging is associative

II. Averaging is commutative

III. Averaging distributes over addition

IV. Addition distributes over averaging

V. Averaging has an identity element

those which are always true are

A All

B I and II only

C II and III only

D II and IV only

E II and V only

Solution:

Let $a * b = \frac{a+b}{2}$. Commutativity is immediate. Also

$$\begin{aligned} a + (b * c) &= \frac{2a + b + c}{2} \\ &= (a + b) * (a + c). \end{aligned}$$

so addition distributes over averaging.

Associativity fails because

$$\begin{aligned} (a * b) * c &= \frac{a + b + 2c}{4}, \\ a * (b * c) &= \frac{2a + b + c}{4}. \end{aligned}$$

Averaging does not distribute over addition since generally

$$a * (b + c) \neq (a * b) + (a * c).$$

Finally, an identity e would require $\frac{e+a}{2} = a$ for every a, or $e = a$, which is impossible for a fixed e. Thus only II and IV always hold.

Therefore, the correct answer is **D**.

6. If 554 is the base b representation of the square of the number whose base b representation is 24, then b, when written in base 10, equals

A 6

B 8

C 12

D 14

E 16

Solution:

In base ten,

$$\begin{aligned} 24_b &= 2b + 4, \\ 554_b &= 5b^2 + 5b + 4. \end{aligned}$$

Therefore

$$\begin{aligned} (2b + 4)^2 &= 5b^2 + 5b + 4, \\ b^2 - 11b - 12 &= 0, \\ (b - 12)(b + 1) &= 0. \end{aligned}$$

A base is positive and must exceed 5, so $b = 12$.

Therefore, the correct answer is **C**.

7. The sum of all the integers between 50 and 350 which end in 1 is

A 5880

B 5539

C 5208

D 4877

E 4566

Solution:

The sequence is

$$51, 61, \ldots, 341.$$

It has

$$\frac{341-51}{10}+1=30$$

terms. Its sum is therefore

$$\begin{aligned} \frac{30(51+341)}{2} &= 15 \cdot 392 \\ &= 5880. \end{aligned}$$

Therefore, the correct answer is **A**.

8. If 1 pint of paint is needed to paint a statue 6 ft. high, then the number of pints it will take to paint (to the same thickness) 540 statues similar to the original but only 1 ft. high is

A 90

B 72

C 45

D 30

E 15

Solution:

A 1-ft. statue has linear scale $\frac{1}{6}$ relative to the original, so its surface area and paint requirement are $\frac{1}{36}$ as large. The 540 small statues therefore require

$$540 \cdot \frac{1}{36} = 15$$

pints.

Therefore, the correct answer is **E**.

9. In $\triangle ABC$ with right angle at C, altitude CH and median CM trisect the right angle. If the area of $\triangle CHM$ is K, then the area of $\triangle ABC$ is

A $6K$

B $4\sqrt{3}K$

C $3\sqrt{3}K$

D $3K$

E $4K$

Solution:

Since $CH \perp AB$, both $\triangle CHM$ and $\triangle CHB$ are right at H. The trisection gives equal acute angles at C, and the triangles share side CH, so they are congruent. Hence $HM = HB$.

Because M is the midpoint of AB, $AM = MB$. Along the hypotenuse the order is A, M, H, B, and $MB = MH + HB = 2MH$. Thus

$$AB = 2MB = 4MH.$$

Triangles ABC and CHM have bases on the same line and the same altitude from C, so their areas are in the ratio $AB : HM = 4 : 1$. Therefore the area of $\triangle ABC$ is $4K$.

Therefore, the correct answer is **E**.

10. If n is a real number, then the simultaneous system

$$\begin{cases} nx + y = 1, \\ ny + z = 1, \\ x + nz = 1 \end{cases}$$

has no solution if and only if n is equal to

A -1

B 0

C 1

D 0 or 1

E $\frac{1}{2}$

Solution:

Adding the equations gives

$$(n+1)(x+y+z) = 3.$$

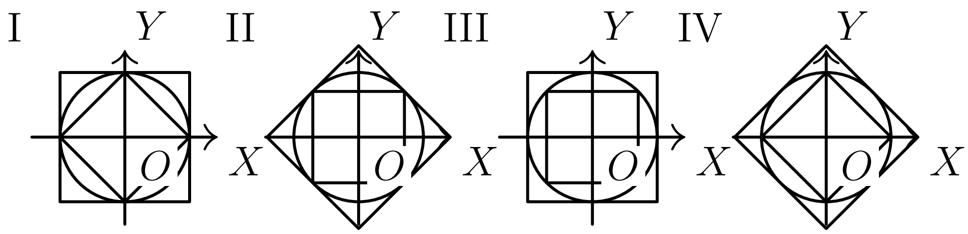
When $n = -1$, this becomes $0 = 3$, so no solution exists. If $n \neq -1$, the symmetric assignment

$$x = y = z = \frac{1}{n+1}$$

satisfies all three equations. Hence the system has no solution exactly when $n = -1$.

Therefore, the correct answer is **A**.

11. A circle with a circumscribed and an inscribed square centered at the origin O of a rectangular coordinate system with positive x- and y-axes OX and OY is shown in each figure I to IV below.



The inequalities

$$|x| + |y| \le \sqrt{2(x^2 + y^2)} \\ \le 2\,\mathrm{Max}(|x|, |y|).$$

are represented geometrically by the figure numbered

Geometric representation: An inequality $f(x, y) \le g(x, y)$ for all x, y is represented by a figure showing, for a typical real number a, the containment

$$\{(x, y) : g(x, y) \le a\} \\ \subset \{(x, y) : f(x, y) \le a\}.$$

A. I

B. II

C. III

D. IV

E. none of these

Solution:

For a fixed positive a, the set

$$2\,\mathrm{Max}(|x|, |y|) \le a$$

is an axis-aligned square. The set

$$\sqrt{2(x^2+y^2)} \leq a$$

is its circumscribed circle, and

$$|x|+|y| \leq a$$

is the diamond circumscribed about that circle. Thus the required nesting is an inner axis-aligned square, then a circle, then an outer diamond, which is figure II.

Therefore, the correct answer is **B**.

12. The average (arithmetic mean) age of a group consisting of doctors and lawyers is 40. If the doctors average 35 and the lawyers 50 years old, then the ratio of the number of doctors to the number of lawyers is

- A $3{:}2$
- B $3{:}1$
- C $2{:}3$
- D $2{:}1$
- E $1{:}2$

Solution:

If there are d doctors and ℓ lawyers, then

$$35d + 50\ell = 40(d+\ell).$$

Thus $5d = 10\ell$, so

$$d : \ell = 2 : 1.$$

Therefore, the correct answer is **D**.

13. The fraction

$$\frac{2(\sqrt{2}+\sqrt{6})}{3\sqrt{2+\sqrt{3}}}$$

is equal to

A $\frac{2\sqrt{2}}{3}$

B 1

C $\frac{2\sqrt{3}}{3}$

D $\frac{4}{3}$

E $\frac{16}{9}$

Solution:

The fraction is positive, and its square is

$$\begin{aligned}\frac{4(\sqrt{2}+\sqrt{6})^2}{9(2+\sqrt{3})} \\ &= \frac{4(8+4\sqrt{3})}{9(2+\sqrt{3})} \\ &= \frac{16(2+\sqrt{3})}{9(2+\sqrt{3})} \\ &= \frac{16}{9}.\end{aligned}$$

Therefore the original fraction is $\frac{4}{3}$.

Therefore, the correct answer is **D**.

14. Each valve $A, B,$ and $C,$ when open, releases water into a tank at its own constant rate. With all three valves open, the tank fills in 1 hour, with only valves A and C open it takes 1.5 hour, and with only valves B and C open it takes 2 hours. The number of hours required with only valves A and B open is

A 1.1

B 1.15

C 1.2

D 1.25

E 1.75

Solution:

Let a, b, c be the hourly rates in tankfuls. Then

$$\begin{aligned} a+b+c &= 1, \\ a+c &= \frac{2}{3}, \\ b+c &= \frac{1}{2}. \end{aligned}$$

Twice the first equation minus the other two gives

$$a+b = 2-\frac{2}{3}-\frac{1}{2}=\frac{5}{6}.$$

Thus valves A and B fill the tank in

$$\frac{1}{a+b}=\frac{6}{5}=1.2$$

hours.

Therefore, the correct answer is **C**.

15. A sector with acute central angle θ is cut from a circle of radius 6. The radius of the circle circumscribed about the sector is

- A $3\cos\theta$
- B $3\sec\theta$
- C $3\cos\frac{1}{2}\theta$
- D $3\sec\frac{1}{2}\theta$
- E 3

Solution:

The two radii and the sector's chord form an isosceles triangle with equal sides 6 and vertex angle θ. Its base has length $12\sin(\frac{\theta}{2})$. If R is the triangle's circumradius, the extended sine rule gives

$$2R = \frac{12\sin(\frac{\theta}{2})}{\sin\theta}.$$

Since $\sin\theta = 2\sin(\frac{\theta}{2})\cos(\frac{\theta}{2})$,

$$R = 3\sec\frac{\theta}{2}.$$

Therefore, the correct answer is **D**.

16. If the sum of all the angles except one of a convex polygon is 2190°, then the number of sides of the polygon must be

A 13

B 15

C 17

D 19

E 21

Solution:

If the omitted angle is x°, then convexity gives $0 < x < 180$. Hence

$$180(n-2) = 2190 + x,$$

so

$$\frac{2190}{180} < n - 2 < \frac{2370}{180}.$$

This is

$$12\frac{1}{6} < n - 2 < 13\frac{1}{6}.$$

The only possible integer is $n - 2 = 13$, so $n = 15$.

Therefore, the correct answer is **B**.

17. If θ is an acute angle and

$$\sin\frac{1}{2}\theta = \sqrt{\frac{x-1}{2x}},$$

then $\tan\theta$ equals

A x

B $\dfrac{1}{x}$

C $\dfrac{\sqrt{x-1}}{x+1}$

D $\dfrac{\sqrt{x^2-1}}{x}$

E $\sqrt{x^2-1}$

Solution:

The half-angle identity gives

$$\begin{aligned}\cos\theta &= 1-2\sin^2\frac{\theta}{2}\\ &= 1-\frac{x-1}{x}\\ &= \frac{1}{x}.\end{aligned}$$

Therefore

$$\tan^2\theta = \sec^2\theta - 1 = x^2-1.$$

Because θ is acute, its tangent is positive, so

$$\tan\theta = \sqrt{x^2-1}.$$

Therefore, the correct answer is **E**.

18. If $p \geq 5$ is a prime number, then 24 divides $p^2 - 1$ without remainder

- A never
- B sometimes only
- C always
- D only if $p = 5$
- E none of these

Solution:

A prime $p \geq 5$ is odd, so $p - 1$ and $p + 1$ are consecutive even integers. One is divisible by 4, making their product divisible by 8. Among the three consecutive integers $p - 1, p, p + 1$, one is divisible by 3. It cannot be p, since $p \geq 5$ is prime, so 3 also divides $(p - 1)(p + 1)$. Because 3 and 8 are relatively prime,

$$24 \mid (p-1)(p+1) = p^2 - 1$$

always.

Therefore, the correct answer is **C**.

19. Define $n_a!$ for positive n and a to be

$$n_a! = n(n-a)(n-2a) \\ \cdot (n-3a)\cdots(n-ka),$$

where k is the greatest integer for which $n > ka$. Then the quotient

$$\frac{72_8!}{18_2!}$$

is equal to

- A 4^5
- B 4^6
- C 4^8
- D 4^9
- E 4^{12}

Solution:

The two products are

$$72_8! = 72 \cdot 64 \cdots 8 = 8^9(9!), \\ 18_2! = 18 \cdot 16 \cdots 2 = 2^9(9!).$$

Their quotient is therefore

$$\frac{8^9(9!)}{2^9(9!)} = 4^9.$$

Therefore, the correct answer is **D**.

20. A cowboy is 4 miles south of a stream which flows due east. He is also 8 miles west and 7 miles north of his cabin. He wishes to water his horse at the stream and return home. The shortest distance (in miles) he can travel and accomplish this is

A $4+\sqrt{185}$

B 16

C 17

D 18

E $\sqrt{32}+\sqrt{137}$

Solution:

Take the stream as the x-axis and put the cowboy at $C=(0,-4)$. His cabin is then $H=(8,-11)$. Reflect C across the stream to $D=(0,4)$. For any point S on the stream, $CS=DS$, so minimizing $CS+SH$ is the same as minimizing $DS+SH$. This occurs when D, S, H are collinear. The minimum distance is

$$DH=\sqrt{8^2+15^2}=\sqrt{289}=17.$$

Therefore, the correct answer is **C**.

21. The number of sets of two or more consecutive positive integers whose sum is 100 is

- A 1
- B 2
- C 3
- D 4
- E 5

Solution:

For $k \geq 2$ consecutive positive integers starting at a,

$$200 = k(2a + k - 1).$$

Positivity gives $k < \sqrt{200}$, so the possible divisors k of 200 are

$$2, 4, 5, 8, 10.$$

From

$$2a = \frac{200}{k} - k + 1,$$

only $k = 5$ and $k = 8$ give positive even right-hand sides. They yield the sets $18, 19, 20, 21, 22$ and $9, 10, \ldots, 16$. Hence there are 2 sets.

Therefore, the correct answer is **B**.

22. The set of all real solutions of the inequality

$$|x-1|+|x+2|<5$$

is

A $\{x : -3 < x < 2\}$

B $\{x : -1 < x < 2\}$

C $\{x : -2 < x < 1\}$

D $\left\{x : -\frac{3}{2} < x < \frac{7}{2}\right\}$

E $\varnothing$ (empty)

Solution:

For $-2 \leq x \leq 1,$ the sum of the distances from x to -2 and 1 is $3.$ If x lies a distance u outside this interval, the sum is $3+2u.$ Thus

$$3+2u<5 \quad \Longleftrightarrow \quad u<1.$$

Extending the interval $[-2,1]$ by 1 at each end gives

$$-3<x<2.$$

Therefore, the correct answer is **A**.

23. There are two cards; one is red on both sides and the other is red on one side and blue on the other. The cards have the same probability $\left(\frac{1}{2}\right)$ of being chosen, and one is chosen and placed on the table. If the upper side of the card on the table is red, then the probability that the under-side is also red is

A $\frac{1}{4}$

B $\frac{1}{3}$

C $\frac{1}{2}$

D $\frac{2}{3}$

E $\frac{3}{4}$

Solution:

Among outcomes having a red upper side, either of the two red faces of the red-red card or the red face of the red-blue card can be uppermost. These three visible red faces are equally likely. The underside is red in the first two cases and blue in the third, so the conditional probability is

$$\frac{2}{3}.$$

Therefore, the correct answer is **D**.

24. The check for a luncheon of 3 sandwiches, 7 cups of coffee and one piece of pie came to $\$3.15$. The check for a luncheon consisting of 4 sandwiches, 10 cups of coffee and one piece of pie came to $\$4.20$ at the same place. The cost of a luncheon consisting of one sandwich, one cup of coffee and one piece of pie at the same place will come to

- A $\$1.70$
- B $\$1.65$
- C $\$1.20$
- D $\$1.05$
- E $\$0.95$

Solution:

Let s, c, p be the prices of a sandwich, coffee, and pie. The checks give

$$\begin{aligned} 3s + 7c + p &= 3.15, \\ 4s + 10c + p &= 4.20. \end{aligned}$$

Three times the first equation minus twice the second gives

$$\begin{aligned} s + c + p &= 3(3.15) - 2(4.20) \\ &= 1.05. \end{aligned}$$

The requested luncheon costs $\$1.05$.

Therefore, the correct answer is **D**.

25. A circular grass plot 12 feet in diameter is cut by a straight gravel path 3 feet wide, one edge of which passes through the center of the plot. The number of square feet in the remaining grass area is

A $36\pi - 34$

B $30\pi - 15$

C $36\pi - 33$

D $35\pi - 9\sqrt{3}$

E $30\pi - 9\sqrt{3}$

Solution:

The plot has radius $6.$ At the path's other edge, the perpendicular distance from the center is $3,$ so the radius to an intersection point makes a 30° angle with the edge through the center. Half of the path consists of a 30° sector of radius 6 and a right triangle with legs 3 and $3\sqrt{3}.$ Its area is

$$\frac{30}{360}\pi(6^2) + \frac{1}{2}(3)(3\sqrt{3})$$
$$= 3\pi + \frac{9\sqrt{3}}{2}.$$

Thus the whole path has area $6\pi + 9\sqrt{3},$ and the remaining grass area is

$$36\pi - (6\pi + 9\sqrt{3}) = 30\pi - 9\sqrt{3}.$$

Therefore, the correct answer is **E**.

26. The number of terms in an A.P. (Arithmetic Progression) is even. The sums of the odd- and even-numbered terms are 24 and 30, respectively. If the last term exceeds the first by 10.5, the number of terms in the A.P. is

A 20

B 18

C 12

D 10

E 8

Solution:

Let the progression have $2n$ terms and common difference d. Pairing each odd-numbered term with its successor shows that

$$nd = 30 - 24 = 6.$$

The difference between the last and first terms is

$$(2n-1)d = 10.5.$$

Since $2nd = 12$, subtraction gives $d = 1.5$. Hence $n = \frac{6}{1.5} = 4$, and the progression has $2n = 8$ terms.

Therefore, the correct answer is **E**.

27. Cars A and B travel the same distance. Car A travels half that distance at u miles per hour and half at v miles per hour. Car B travels half the time at u miles per hour and half at v miles per hour. The average speed of Car A is x miles per hour and that of Car B is y miles per hour. Then we always have

A $x \leq y$

B $x \geq y$

C $x = y$

D $x < y$

E $x > y$

Solution:

Car A's equal-distance average and Car B's equal-time average are

$$x = \frac{2uv}{u+v}, \qquad y = \frac{u+v}{2}.$$

Their difference is

$$\begin{aligned} y - x &= \frac{(u+v)^2 - 4uv}{2(u+v)} \\ &= \frac{(u-v)^2}{2(u+v)} \\ &\geq 0. \end{aligned}$$

Thus $x \leq y$, with equality possible when $u = v$.

Therefore, the correct answer is **A**.

28. If a, b, and c are in geometric progression (G.P.) with $1 < a < b < c$ and $n > 1$ is an integer, then $\log_a n, \log_b n, \log_c n$ form a sequence

A which is a G.P.

B which is an arithmetic progression (A.P.)

C in which the reciprocals of the terms form an A.P.

D in which the second and third terms are the nth powers of the first and second respectively

E none of these

Solution:

By change of base,

$$\frac{1}{\log_a n} = \log_n a,$$
$$\frac{1}{\log_b n} = \log_n b,$$
$$\frac{1}{\log_c n} = \log_n c.$$

Since a, b, c are in geometric progression, $b^2 = ac$. Taking logarithms to base n gives

$$2 \log_n b = \log_n a + \log_n c.$$

Hence the reciprocals of the three given terms form an arithmetic progression.

Therefore, the correct answer is **C**.

29. Two boys start moving from the same point A on a circular track but in opposite directions. Their speeds are 5 ft. per sec. and 9 ft. per sec. If they start at the same time and finish when they first meet at the point A again, then the number of times they meet, excluding the start and finish, is

A 13

B 25

C 44

D infinity

E none of these

Solution:

Let the track length be L. Since $\gcd(5, 9) = 1$, the first positive time when both boys are back at A is $t = L$: they have completed 5 and 9 laps. Their relative speed is $5 + 9 = 14$, so before time L they meet at

$$t = \frac{kL}{14}, \qquad k = 1, 2, \ldots, 13.$$

Thus there are 13 meetings excluding the start and finish.

Therefore, the correct answer is **A**.

30. Let $[t]$ denote the greatest integer $\leq t$, where $t \geq 0$, and

$$S = \{(x, y) : (x - T)^2 + y^2 \leq T^2\},$$
$$T = t - [t].$$

Then we have

A the point $(0, 0)$ does not belong to S for any t

B $0 \leq \text{Area } S \leq \pi$ for all t

C S is contained in the first quadrant for all $t \geq 5$

D the center of S for any t is on the line $y = x$

E none of the other statements is true

Solution:

The fractional part satisfies $0 \leq T < 1$. The set S is the closed disk centered at $(T, 0)$ with radius T. Its area is

$$\pi T^2,$$

so $0 \leq \text{Area } S < \pi$, which in particular gives $0 \leq \text{Area } S \leq \pi$. The origin lies on every such disk, the disk extends below the x-axis when $T > 0$, and its center is generally not on $y = x$.

Therefore, the correct answer is **B**.

31. In the following equation, each of the letters represents uniquely a different digit in base ten:

$$(YE) \cdot (ME) = TTT.$$

The sum $E + M + T + Y$ equals

A 19

B 20

C 21

D 22

E 24

Solution:

Since

$$TTT = 111T = 3 \cdot 37 \cdot T,$$

the prime 37 divides one of YE and ME. A two-digit multiple of 37 is 37 or 74. The value 74 is impossible: the other two-digit factor ending in 4 is at least 14, and $74 \cdot 14 > 999$. Hence one factor is 37, so $E = 7$.

The units digit of the product is the units digit of 7^2, so $T = 9$. Thus the product is 999, and the other factor is

$$\frac{999}{37} = 27.$$

The four digits are $2, 3, 7, 9$, whose sum is 21.

Therefore, the correct answer is **C**.

32. The volume of a pyramid whose base is an equilateral triangle of side length 6 and whose other edges are each of length $\sqrt{15}$ is

A 9

B $\frac{9}{2}$

C $\frac{27}{2}$

D $\frac{9\sqrt{3}}{2}$

E none of these

Solution:

The base area is

$$\frac{\sqrt{3}}{4}(6^2) = 9\sqrt{3}.$$

Because the apex is equally distant from all three base vertices, its perpendicular projection is the base circumcenter. The circumradius of the equilateral base is $\frac{6}{\sqrt{3}} = 2\sqrt{3}$. If h is the pyramid's height, then

$$h^2 + (2\sqrt{3})^2 = (\sqrt{15})^2,$$

so $h = \sqrt{3}$. The volume is

$$\frac{1}{3}(9\sqrt{3})(\sqrt{3}) = 9.$$

Therefore, the correct answer is **A**.

33. When one ounce of water is added to a mixture of acid and water, the new mixture is 20% acid. When one ounce of acid is added to the new mixture, the result is $33\frac{1}{3}\%$ acid. The percentage of acid in the original mixture is

A 22%

B 24%

C 25%

D 30%

E $33\frac{1}{3}\%$

Solution:

Let the original mixture contain x ounces of water and y ounces of acid. The two additions give

$$\frac{y}{x+y+1} = \frac{1}{5},$$
$$\frac{y+1}{x+y+2} = \frac{1}{3}.$$

These simplify to

$$x+1 = 4y, \qquad x = 2y+1.$$

Hence $y = 1$ and $x = 3$. The original acid percentage was

$$100 \cdot \frac{y}{x+y} = 100 \cdot \frac{1}{4} = 25\%.$$

Therefore, the correct answer is **C**.

34. A plane flew straight against a wind between two towns in 84 minutes and returned with that wind in 9 minutes less than it would take in still air. The number of minutes (two answers) for the return trip was

A 54 or 18

B 60 or 15

C 63 or 12

D 72 or 36

E 75 or 20

Solution:

Let the distance be d and the return time be x minutes. The against-wind and with-wind speeds are $\frac{d}{84}$ and $\frac{d}{x}$. Their average is the still-air speed $\frac{d}{x+9}$. Therefore

$$\frac{2}{x+9} = \frac{1}{84} + \frac{1}{x}.$$

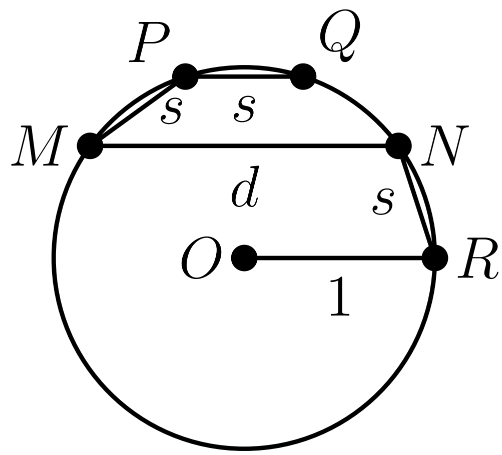
Clearing denominators gives

$$\begin{gathered} 168x = (x+9)(x+84), \\ x^2 - 75x + 756 = 0, \\ (x-63)(x-12) = 0. \end{gathered}$$

Both positive values are consistent with the stated conditions, so the two return times are 63 and 12 minutes.

Therefore, the correct answer is **C**.

35. In the unit circle shown in the figure, chords PQ and MN are parallel to the unit radius OR of the circle with center at O. Chords MP, PQ, and NR are each s units long and chord MN is d units long.



Of the three equations

$$\begin{array}{rl} \text{I.} & d - s = 1, \\ \text{II.} & ds = 1, \\ \text{III.} & d^2 - s^2 = \sqrt{5} \end{array}$$

those which are necessarily true are

- A. I only
- B. II only
- C. III only
- D. I and II only
- E. I, II, and III

Solution:

Let K be the left endpoint of the horizontal diameter. Reflection across the vertical diameter shows that $KM = NR = s$ and $QN = MP = s$. Together with the given equal chords, the upper semicircle is split into five equal arcs. Each subtends 36° at O.

Thus

$$\begin{aligned} s &= 2\sin 18^\circ, \\ d &= 2\sin 54^\circ \\ &= 2\cos 36^\circ. \end{aligned}$$

Using the double-angle identities,

$$\begin{aligned} d &= 2(1 - 2\sin^2 18^\circ) \\ &= 2 - s^2, \\ s &= 2\cos 72^\circ \\ &= 4\cos^2 36^\circ - 2 \\ &= d^2 - 2. \end{aligned}$$

Adding these equations gives

$$d + s = d^2 - s^2 = (d - s)(d + s).$$

Since $d + s > 0,$ it follows that $d - s = 1.$ Substituting $d = s + 1$ into $d = 2 - s^2$ yields

$$s^2 + s = 1, \qquad s = \frac{\sqrt{5} - 1}{2}.$$

Therefore

$$ds = s(s + 1) = 1$$

and

$$\begin{aligned} d^2 - s^2 &= (d - s)(d + s) \\ &= 2s + 1 \\ &= \sqrt{5}. \end{aligned}$$

All three equations are necessarily true.

Therefore, the correct answer is **E**.

Problems: https://live.poshenloh.com/past-contests/amc12/1973