1972 AMC 12 Solutions

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1. The lengths in inches of the three sides of each of four triangles I, II, III, and IV are as follows:

$$\begin{array}{ll} \mathrm{I} & 3, 4, 5 \\ \mathrm{II} & 4, 7\frac{1}{2}, 8\frac{1}{2} \\ \mathrm{III} & 7, 24, 25 \\ \mathrm{IV} & 3\frac{1}{2}, 4\frac{1}{2}, 5\frac{1}{2} \end{array}$$

Of these four given triangles, the only right triangles are:

A I and II

B I and III

C I and IV

D I, II, and III

E I, II, and IV

Solution:

For triangles $\mathrm{I}, \mathrm{II}, \mathrm{III}$, the relevant equalities are

$$\begin{aligned} 3^2 + 4^2 &= 5^2, \\ 4^2 + \left(\frac{15}{2}\right)^2 &= \left(\frac{17}{2}\right)^2, \\ 7^2 + 24^2 &= 25^2. \end{aligned}$$

For IV,

$$\begin{aligned} \left(\frac{7}{2}\right)^2 + \left(\frac{9}{2}\right)^2 &= \frac{130}{4}, \\ \left(\frac{11}{2}\right)^2 &= \frac{121}{4}. \end{aligned}$$

These values are unequal. Thus precisely I, II, and III are right triangles.

Therefore, the correct answer is **D**.

2. If a dealer could get his goods for 8% less while keeping his selling price fixed, his profit, based on cost, would be increased to $(x+10)\%$ from his present profit of $x\%$, which is:

A 12%

B 15%

C 30%

D 50%

E 75%

Solution:

The fixed selling price gives

$$1+0.01x=0.92 \\ +0.0092(x+10).$$

Simplifying yields

$$1+0.01x=1.012+0.0092x,$$

so $0.0008x=0.012$ and $x=15.$

Therefore, the correct answer is **B**.

3. If $x = \dfrac{1 - i\sqrt{3}}{2}$, where $i = \sqrt{-1}$, then $\dfrac{1}{x^2 - x}$ is equal to:

A -2

B -1

C $1 + i\sqrt{3}$

D 1

E 2

Solution:

Directly,

$$x^2 - x = \left(\frac{1 - i\sqrt{3}}{2}\right)^2 - \frac{1 - i\sqrt{3}}{2} = -1.$$

Its reciprocal is therefore also -1.

Therefore, the correct answer is **B**.

4. The number of solutions to $\{1, 2\} \subseteq X \subseteq \{1, 2, 3, 4, 5\}$, where X is a subset of $\{1, 2, 3, 4, 5\}$, is:

A 2

B 4

C 6

D 8

E None of these

Solution:

Every valid X contains $1, 2$, while each of $3, 4, 5$ may be chosen independently. Hence there are

$$2^3 = 8$$

possible sets.

Therefore, the correct answer is **D**.

5. From among $2^{\frac{1}{2}}, 3^{\frac{1}{3}}, 8^{\frac{1}{8}}, 9^{\frac{1}{9}}$, those which have the greatest and the next to the greatest values, in that order, are:

A $3^{\frac{1}{3}}, 2^{\frac{1}{2}}$

B $3^{\frac{1}{3}}, 8^{\frac{1}{8}}$

C $3^{\frac{1}{3}}, 9^{\frac{1}{9}}$

D $8^{\frac{1}{8}}, 9^{\frac{1}{9}}$

E None of these

Solution:

Raising to convenient common powers gives

$$(3^{\frac{1}{3}})^6 = 9 > 8 = (2^{\frac{1}{2}})^6,$$

$$(2^{\frac{1}{2}})^8 = 16 > 8 = (8^{\frac{1}{8}})^8,$$

and

$$(2^{\frac{1}{2}})^{18} = 512 > 81 = (9^{\frac{1}{9}})^{18}.$$

Thus $3^{\frac{1}{3}}$ is greatest and $2^{\frac{1}{2}}$ is next.

Therefore, the correct answer is **A**.

6. If $3^{2x} + 9 = 10(3^x)$, then the value of $x^2 + 1$ is:

- A 1 only
- B 5 only
- C 1 or 5
- D 2
- E 10

Solution:

Let $u = 3^x$. Then

$$\begin{aligned} u^2 - 10u + 9 &= (u-1)(u-9) \\ &= 0. \end{aligned}$$

Thus $x = 0$ or $x = 2$, so $x^2 + 1$ is respectively 1 or 5.

Therefore, the correct answer is **C**.

7. If $yz : zx : xy = 1 : 2 : 3$, then $\frac{x}{yz} : \frac{y}{zx}$ is equal to:

A $3{:}2$

B $1{:}2$

C $1{:}4$

D $2{:}1$

E $4{:}1$

Solution:

The requested ratio has quotient

$$\frac{\frac{x}{yz}}{\frac{y}{zx}} = \frac{x^2}{y^2}.$$

Also $yz : zx = y : x = 1 : 2$, so $x : y = 2 : 1$. Therefore $x^2 : y^2 = 4 : 1$.

Therefore, the correct answer is **E**.

8. If $|x - \log y| = x + \log y$, where x and $\log y$ are real, then:

A $x = 0$

B $y = 1$

C $x = 0$ and $y = 1$

D $x(y - 1) = 0$

E None of these

Solution:

If $x - \log y \geq 0$, the equation gives $\log y = 0$, hence $y = 1$. If $x - \log y < 0$, it gives $x = 0$. Therefore every solution satisfies either $x = 0$ or $y = 1$, which is expressed by

$$x(y - 1) = 0.$$

Therefore, the correct answer is **D**.

9. Ann and Sue bought identical boxes of stationery. Ann used hers to write 1-sheet letters and Sue used hers to write 3-sheet letters. Ann used all the envelopes and had 50 sheets of paper left, while Sue used all of the sheets of paper and had 50 envelopes left. The number of sheets of paper in each box was:

A 150

B 125

C 120

D 100

E 80

Solution:

Let S and E denote sheets and envelopes per box. The two accounts give

$$S - E = 50, \qquad E - \frac{S}{3} = 50.$$

Adding yields $\frac{2S}{3} = 100$, so $S = 150$.

Therefore, the correct answer is **A**.

10. For x real, the inequality $1 \leq |x-2| \leq 7$ is equivalent to:

- A $x \leq 1$ or $x \geq 3$
- B $1 \leq x \leq 3$
- C $-5 \leq x \leq 9$
- D $-5 \leq x \leq 1$ or $3 \leq x \leq 9$
- E $-6 \leq x \leq 1$ or $3 \leq x \leq 10$

Solution:

The condition $|x-2| \leq 7$ gives $-5 \leq x \leq 9$. The condition $|x-2| \geq 1$ gives $x \leq 1$ or $x \geq 3$. Their intersection is

$$[-5, 1] \cup [3, 9].$$

Therefore, the correct answer is **D**.

11. The value(s) of y for which the following pair of equations

$$\begin{aligned} x^2 + y^2 - 16 &= 0, \\ x^2 - 3y + 12 &= 0 \end{aligned}$$

may have a real common solution, are:

- A 4 only
- B $-7, 4$
- C $0, 4$
- D no y
- E all y

Solution:

From the second equation, $x^2 = 3y - 12$. Substitution into the first gives

$$\begin{aligned} y^2 + 3y - 28 &= (y-4)(y+7) \\ &= 0. \end{aligned}$$

If $y = 4$, then $x = 0$. If $y = -7$, then $x^2 = -33$, so no real x exists. Thus only $y = 4$ is possible.

Therefore, the correct answer is **A**.

12. The number of cubic feet in the volume of a cube is the same as the number of square inches in its surface area. The length of the edge expressed as a number of feet is:

A 6

B 864

C 1728

D 6×1728

E 2304

Solution:

Let the edge length be f feet, or $12f$ inches. The stated numerical equality is

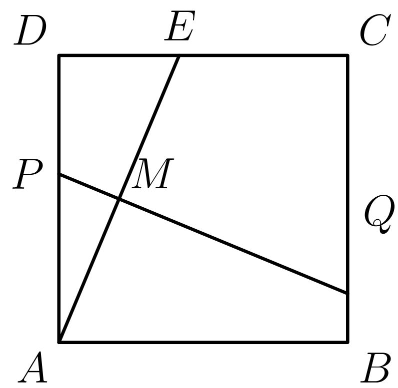
$$f^3 = 6(12f)^2.$$

Since $f > 0$, division by f^2 gives

$$f = 6 \cdot 144 = 864.$$

Therefore, the correct answer is **B**.

13. Inside square $ABCD$ with sides of length 12 inches, segment AE is drawn, where E is the point on DC which is 5 inches from D. The perpendicular bisector of AE is drawn and intersects AE, AD, and BC at points M, P, and Q, respectively. The ratio of segment PM to MQ is:



A $5{:}12$

B $5{:}13$

C $5{:}19$

D $1{:}4$

E $5{:}21$

Solution:

Because M is the midpoint of AE, its horizontal distance from AD is $\frac{5}{2}$. Its distance from BC is

$$12 - \frac{5}{2} = \frac{19}{2}.$$

The two right triangles cut from line PQ by the horizontal through M are similar, so their hypotenuses are in the ratio of these horizontal legs:

$$PM : MQ = \frac{5}{2} : \frac{19}{2} = 5 : 19.$$

Therefore, the correct answer is **C**.

14. A triangle has angles of $30°$ and $45°$. If the side opposite the $45°$ angle has length 8, then the side opposite the $30°$ angle has length:

A 4

B $4\sqrt{2}$

C $4\sqrt{3}$

D $4\sqrt{6}$

E 6

Solution:

If s is the side opposite $30°$, the Law of Sines gives

$$\frac{s}{\sin 30°} = \frac{8}{\sin 45°}.$$

Therefore

$$s = 8\frac{\frac{1}{2}}{\frac{\sqrt{2}}{2}} = 4\sqrt{2}.$$

Therefore, the correct answer is **B**.

15. A contractor estimated that one of his two bricklayers would take 9 hours to build a certain wall and the other 10 hours. However, he knew from experience that when they worked together, their combined output fell by 10 bricks per hour. Being in a hurry, he put both men on the job and found that it took exactly 5 hours to build the wall. The number of bricks in the wall was:

A 500

B 550

C 900

D 950

E 960

Solution:

If the wall contains N bricks, the individual rates are $\frac{N}{9}$ and $\frac{N}{10}$. Thus

$$5\left(\frac{N}{9}+\frac{N}{10}-10\right)=N.$$

Multiplying by 18 and simplifying gives $19N-900=18N$, hence $N=900.$

Therefore, the correct answer is **C**.

16. There are two positive numbers that may be inserted between 3 and 9 such that the first three are in geometric progression while the last three are in arithmetic progression. The sum of those two positive numbers is:

A $13\frac{1}{2}$

B $11\frac{1}{4}$

C $10\frac{1}{2}$

D 10

E $9\frac{1}{2}$

Solution:

The progression conditions give

$$x^2 = 3y, \qquad 2y = x + 9.$$

Eliminating y yields

$$\begin{aligned} 2x^2 - 3x - 27 &= (2x-9)(x+3) \\ &= 0. \end{aligned}$$

Positivity gives $x = \frac{9}{2}$, and then $y = \frac{27}{4}$. Their sum is

$$\frac{9}{2} + \frac{27}{4} = \frac{45}{4} = 11\frac{1}{4}.$$

Therefore, the correct answer is **B**.

17. A piece of string is cut in two at a point selected at random. The probability that the longer piece is at least x times as large as the shorter piece is:

A $\frac{1}{2}$

B $\frac{2}{x}$

C $\frac{1}{x+1}$

D $\frac{1}{x}$

E $\frac{2}{x+1}$

Solution:

Take the string to have length 1. Near one endpoint, the condition is

$$1 - t \geq xt,$$

so $t \leq \frac{1}{x+1}$. The same interval occurs at the other endpoint. Their total length, hence the probability, is

$$\frac{2}{x+1}.$$

Therefore, the correct answer is **E**.

18. Let $ABCD$ be a trapezoid with the measure of base AB twice that of base DC, and let E be the point of intersection of the diagonals. If the measure of diagonal AC is 11, then that of segment EC is equal to:

A $3\frac{2}{3}$

B $3\frac{3}{4}$

C 4

D $3\frac{1}{2}$

E 3

Solution:

Since $AB \parallel DC$, triangles ABE and CDE are similar. Hence

$$AE : EC = AB : DC = 2 : 1.$$

Thus $AC = AE + EC = 3EC = 11$, so

$$EC = \frac{11}{3} = 3\frac{2}{3}.$$

Therefore, the correct answer is **A**.

19. The sum of the first n terms of the sequence

$$1,\ (1+2),\ (1+2+2^2),\ \dots,\ (1+2+2^2+\dots+2^{n-1})$$

in terms of n is:

A 2^n

B $2^n - n$

C $2^{n+1} - n$

D $2^{n+1} - n - 2$

E $n \cdot 2^n$

Solution:

The k-th term is the geometric sum $2^k - 1$. Therefore the desired total is

$$\sum_{k=1}^{n}(2^k - 1) = (2^{n+1} - 2) - n$$
$$= 2^{n+1} - n - 2.$$

Therefore, the correct answer is **D**.

20. If $\tan x = \dfrac{2ab}{a^2 - b^2}$, where $a > b > 0$ and $0^\circ < x < 90^\circ$, then $\sin x$ is equal to:

A $\dfrac{a}{b}$

B $\dfrac{b}{a}$

C $\dfrac{\sqrt{a^2 - b^2}}{2a}$

D $\dfrac{\sqrt{a^2 - b^2}}{2ab}$

E $\dfrac{2ab}{a^2 + b^2}$

Solution:

Use a right triangle with opposite leg $2ab$ and adjacent leg $a^2 - b^2$. Its hypotenuse is

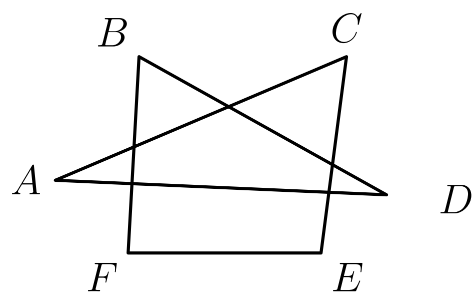
$$\begin{aligned} &\sqrt{(2ab)^2 + (a^2 - b^2)^2} \\ &= \sqrt{(a^2 + b^2)^2} \\ &= a^2 + b^2. \end{aligned}$$

Hence

$$\sin x = \frac{2ab}{a^2 + b^2}.$$

Therefore, the correct answer is **E**.

21. If the sum of the measures in degrees of angles $A, B, C, D, E,$ and F in the figure is $90n,$ then n is equal to:



A 2

B 3

C 4

D 5

E 6

Solution:

Let $P = AD \cap BF$ and $Q = AD \cap CE.$ In quadrilateral $EFPQ,$

$$E + F + \angle P + \angle Q = 360^\circ.$$

The triangles BPD and AQC give

$$B + D = \angle P, \qquad A + C = \angle Q,$$

using the supplementary angles at P and $Q.$ Adding yields

$$\begin{gathered} A + B + C + D + E + F \\ = 360^\circ = 90^\circ \cdot 4. \end{gathered}$$

Thus $n = 4.$

Therefore, the correct answer is **C**.

22. If $a \pm bi\,(b \neq 0)$ are imaginary roots of the equation $x^3 + qx + r = 0$, where $a, b, q,$ and r are real numbers, then q in terms of a and b is:

A $a^2 + b^2$

B $2a^2 - b^2$

C $b^2 - a^2$

D $b^2 - 2a^2$

E $b^2 - 3a^2$

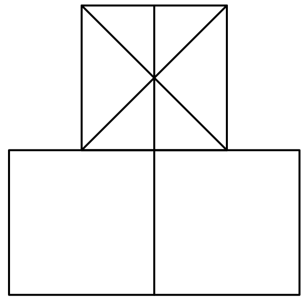
Solution:

The conjugate root is $a - bi$. Since the sum of the three roots is 0, the third root is $-2a$. By Vieta's formulas,

$$\begin{aligned} q &= (a+bi)(a-bi) \\ &\quad - 2a(a+bi) \\ &\quad - 2a(a-bi) \\ &= a^2 + b^2 - 4a^2 \\ &= b^2 - 3a^2. \end{aligned}$$

Therefore, the correct answer is **E**.

23. The radius of the smallest circle containing the symmetric figure composed of 3 unit squares shown is:



A $\sqrt{2}$

B $\sqrt{1.25}$

C 1.25

D $\dfrac{5\sqrt{17}}{16}$

E None of these

Solution:

Put the midpoint of the bottom edge at $O=(0,0)$. A lower outer corner is $A=(1,0)$, and an upper outer corner is $B=(\frac{1}{2},2)$. By symmetry the center of the smallest enclosing circle is $P=(0,k)$. At the optimum both types of outer corner lie on the circle, so

$$1+k^2=\left(\frac{1}{2}\right)^2+(2-k)^2.$$

This gives $k=\frac{13}{16}$. Hence

$$r^2=1+\left(\frac{13}{16}\right)^2=\frac{425}{256},$$
$$r=\frac{5\sqrt{17}}{16}.$$

Therefore, the correct answer is **D**.

24. A man walked a certain distance at a constant rate. If he had gone $\frac{1}{2}$ mile per hour faster, he would have walked the distance in four-fifths of the time; if he had gone $\frac{1}{2}$ mile per hour slower, he would have been $2\frac{1}{2}$ hours longer on the road. The distance in miles he walked was:

- A $13\frac{1}{2}$
- B 15
- C $17\frac{1}{2}$
- D 20
- E 25

Solution:

Let the actual speed and time be v, t. The faster case gives

$$vt = \left(v + \frac{1}{2}\right)\frac{4t}{5},$$

so $v = 2$. The slower case then gives

$$2t = \frac{3}{2}\left(t + \frac{5}{2}\right),$$

whence $t = \frac{15}{2}$. The distance was

$$vt = 2 \cdot \frac{15}{2} = 15.$$

Therefore, the correct answer is **B**.

25. Inscribed in a circle is a quadrilateral having sides of lengths $25, 39, 52,$ and 60 taken consecutively. The diameter of this circle has length:

A 62

B 63

C 65

D 66

E 69

Solution:

The consecutive sides may be labeled

$$AB = 25, \quad BC = 39,$$
$$CD = 52, \quad DA = 60.$$

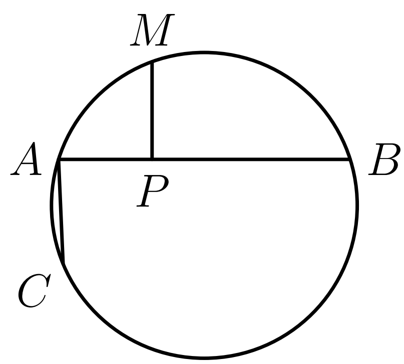
Since

$$25^2 + 60^2 = 65^2 = 39^2 + 52^2,$$

triangles ABD and BCD are right triangles with common hypotenuse $BD = 65.$ Their union is the cyclic quadrilateral, and a right triangle's hypotenuse is a diameter of its circumcircle. Thus the diameter is $65.$

Therefore, the correct answer is **C**.

26. In the circle shown, M is the midpoint of arc CAB, and segment MP is perpendicular to chord AB at P. If the measure of chord AC is x and that of segment AP is $x+1$, then segment PB has measure equal to:



A $3x+2$

B $3x+1$

C $2x+3$

D $2x+2$

E $2x+1$

Solution:

Choose N on arc MB so that arcs MN and CA are equal, and drop $NQ \perp AB$. Then $MN = AC = x$. The remaining equal arcs AM and NB give equal corresponding chord projections, so $QB = AP = x+1$. Also $MNPQ$ is a rectangle, hence $PQ = MN = x$. Therefore

$$\begin{aligned} PB &= PQ + QB \\ &= x + (x+1) \\ &= 2x+1. \end{aligned}$$

Therefore, the correct answer is **E**.

27. If the area of $\triangle ABC$ is 64 square inches and the geometric mean (mean proportional) between sides AB and AC is 12 inches, then $\sin A$ is equal to:

A $\frac{\sqrt{3}}{2}$

B $\frac{3}{5}$

C $\frac{4}{5}$

D $\frac{8}{9}$

E $\frac{15}{17}$

Solution:

The geometric-mean condition says

$$AB \cdot AC = 12^2 = 144.$$

Therefore

$$\begin{aligned} 64 &= \frac{1}{2}(AB)(AC)\sin A \\ &= 72 \sin A, \end{aligned}$$

so $\sin A = \frac{8}{9}$.

Therefore, the correct answer is **D**.

28. A circular disc with diameter D is placed on an 8×8 checkerboard with width D so that the centers coincide. The number of checkerboard squares which are completely covered by the disc is:

A 48

B 44

C 40

D 36

E 32

Solution:

The 28 border squares are not fully covered. Consider the remaining 6×6 interior grid and measure distances in square side lengths, so the disc has radius 4. The four outer corners of this interior grid are at distance

$$\sqrt{3^2 + 3^2} = 3\sqrt{2} > 4$$

from the center, so those four corner squares are not fully covered. Every other interior square lies within the disc: its farthest possible corner is at distance at most

$$\sqrt{3^2 + 2^2} = \sqrt{13} < 4.$$

Hence $36 - 4 = 32$ squares are completely covered.

Therefore, the correct answer is **E**.

29. If $f(x) = \log\left(\frac{1+x}{1-x}\right)$ for $-1 < x < 1$, then $f\left(\frac{3x+x^3}{1+3x^2}\right)$ in terms of $f(x)$ is:

A $-f(x)$

B $2f(x)$

C $3f(x)$

D $[f(x)]^2$

E $[f(x)]^3 - f(x)$

Solution:

Let $u = \frac{3x+x^3}{1+3x^2}$. Then

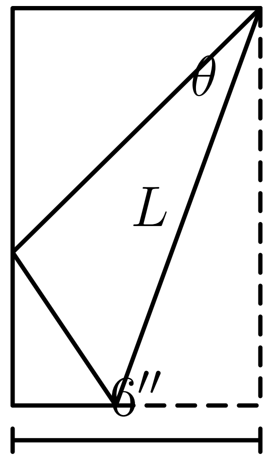
$$\begin{aligned}\frac{1+u}{1-u} &= \frac{1+3x^2+3x+x^3}{1+3x^2-3x-x^3} \\ &= \left(\frac{1+x}{1-x}\right)^3.\end{aligned}$$

Consequently

$$f(u) = 3\log\left(\frac{1+x}{1-x}\right) = 3f(x).$$

Therefore, the correct answer is **C**.

30. A rectangular piece of paper 6 inches wide is folded as in the diagram so that one corner touches the opposite side. The length in inches of the crease L in terms of angle θ is:



A $3\sec^2\theta\csc\theta$

B $6\sin\theta\sec\theta$

C $3\sec\theta\csc\theta$

D $6\sec\theta\csc^2\theta$

E None of these

Solution:

Let h be the sheet's height. The congruent right triangles created by reflecting the folded corner across the crease give

$$\frac{6}{h} = \sin(2\theta) = 2\sin\theta\cos\theta,$$

so $h = \frac{3}{\sin\theta\cos\theta}$. They also give $\frac{L}{h} = \sec\theta$. Hence

$$\begin{aligned} L &= h\sec\theta \\ &= \frac{3\sec\theta}{\sin\theta\cos\theta} \\ &= 3\sec^2\theta\csc\theta. \end{aligned}$$

Therefore, the correct answer is **A**.

31. When the number 2^{1000} is divided by $13,$ the remainder in the division is:

A 1

B 2

C 3

D 7

E 11

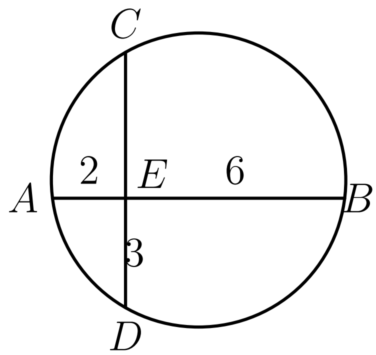
Solution:

Since $2^6 = 64 \equiv -1 \pmod{13},$

$$\begin{aligned} 2^{1000} &= 2^{6\cdot 166+4} \\ &\equiv (-1)^{166} 2^4 \\ &\equiv 16 \\ &\equiv 3 \pmod{13}. \end{aligned}$$

Therefore, the correct answer is **C**.

32. Chords AB and CD in the circle shown intersect at E and are perpendicular to each other. If segments $AE, EB,$ and ED have measures $2, 6,$ and 3 respectively, then the length of the diameter of the circle is:



A $4\sqrt{5}$

B $\sqrt{65}$

C $2\sqrt{17}$

D $3\sqrt{7}$

E $6\sqrt{2}$

Solution:

The intersecting-chords theorem gives

$$2 \cdot 6 = CE \cdot 3,$$

so $CE = 4$. Put $E = (0,0), A = (-2,0), B = (6,0), C = (0,4),$ and $D = (0,-3)$. The perpendicular bisectors of AB and CD meet at

$$O = \left(2, \frac{1}{2}\right).$$

Thus

$$r^2 = OA^2 = 4^2 + \left(\frac{1}{2}\right)^2 = \frac{65}{4},$$

and the diameter is $2r = \sqrt{65}$.

Therefore, the correct answer is **B**.

33. The minimum value of the quotient of a (base ten) number of three different nonzero digits divided by the sum of its digits is:

- A 9.7
- B 10.1
- C 10.5
- D 10.9
- E 20.5

Solution:

Let Q denote the quotient. Then

$$Q = 1 + \frac{99H + 9T}{H + T + U}.$$

Interchanging U with a larger digit in either earlier position decreases the quotient, so the units digit must be the largest. Increasing that units digit lowers a quotient greater than 1, so $U = 9$. Then

$$\frac{T + 11H}{T + H + 9} = 1 + \frac{10H - 9}{T + H + 9}.$$

This is minimized by taking the largest available $T = 8$ and then the smallest $H = 1$. The number is 189, and

$$\frac{189}{1 + 8 + 9} = \frac{189}{18} = 10.5.$$

Therefore, the correct answer is **C**.

34. Three times Dick's age plus Tom's age equals twice Harry's age. Double the cube of Harry's age is equal to three times the cube of Dick's age added to the cube of Tom's age. Their respective ages are relatively prime to each other. The sum of the squares of their ages is:

A 42

B 46

C 122

D 290

E 326

Solution:

The equations are

$$3D + T = 2H,$$
$$2H^3 = 3D^3 + T^3.$$

Rewrite them as

$$2(H - D) = D + T$$

and

$$2(H - D)(H^2 + HD + D^2) = (D + T) \cdot (D^2 - DT + T^2).$$

Canceling the equal positive factors gives

$$H^2 + HD + DT - T^2 = 0,$$
$$(H + T)(H + D - T) = 0,$$

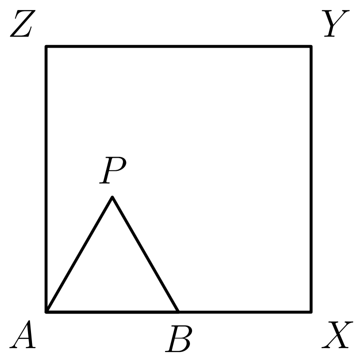
so $T = H + D$. The linear equation then gives $H = 4D$, and pairwise relative primality forces

$$(D, H, T) = (1, 4, 5).$$

The requested sum is $1^2 + 4^2 + 5^2 = 42.$

Therefore, the correct answer is **A**.

35. Equilateral triangle ABP with side AB of length 2 inches is placed inside square $AXYZ$ with side of length 4 inches so that B is on side AX. The triangle is rotated clockwise about B, then P, and so on along the sides of the square until the triangle returns to its original position. The length of the path in inches traversed by vertex P is equal to:



A $\frac{20\pi}{3}$

B $\frac{32\pi}{3}$

C 12π

D $\frac{40\pi}{3}$

E 15π

Solution:

One circuit of the square uses 8 pivots and changes the triangle's orientation by $\frac{2}{3}$ of a full turn. Therefore 3 circuits, or 24 pivots, are required to restore the entire triangle. In 8 of those pivots the rotation is about P, so P does not move. In the other 16, eight arcs subtend 120° and eight subtend 30°, all with radius 2. Hence the total path length is

$$\begin{aligned}8\left(\frac{1}{3}\cdot 4\pi\right)+8\left(\frac{1}{12}\cdot 4\pi\right)\\ =\frac{32\pi}{3}+\frac{8\pi}{3}\\ =\frac{40\pi}{3}.\end{aligned}$$

Therefore, the correct answer is **D**.

Problems: https://live.poshenloh.com/past-contests/amc12/1972