1971 AMC 12 Solutions

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1. The number of digits in the number $N = 2^{12} \times 5^8$ is:

   - A 9
   - B 10
   - C 11
   - D 12
   - E 20

Solution:

We have $2^{12}5^8 = 2^4(2^8 5^8)$ and $16 \cdot 10^8 = 1{,}600{,}000{,}000,$ which has 10 digits.

Therefore, the correct answer is **B**.

2. If b men take c days to lay f bricks, then the number of days it will take c men working at the same rate to lay b bricks is:

A fb^2

B $\frac{b}{f^2}$

C $\frac{f^2}{b}$

D $\frac{b^2}{f}$

E $\frac{f}{b^2}$

Solution:

The rate is $\frac{f}{bc}$ bricks per man-day. Thus c men lay $\frac{f}{b}$ bricks per day, so laying b bricks takes

$$\frac{b}{\frac{f}{b}} = \frac{b^2}{f}$$

days.

Therefore, the correct answer is **D**.

3. If the point $(x, -4)$ lies on the straight line joining the points $(0, 8)$ and $(-4, 0)$ in the xy-plane, then x is equal to:

- A -2
- B 2
- C -8
- D 6
- E -6

Solution:

The slope through $(0, 8)$ and $(-4, 0)$ is 2, so the line is $y = 2x + 8$. Substituting $y = -4$ gives $-4 = 2x + 8$, hence $x = -6$.

Therefore, the correct answer is **E**.

4. After simple interest for two months at 5% per annum was credited, a Boy Scout Troop had a total of $\$255.31$ in the Council Treasury. The interest credited was a number of dollars plus the following number of cents:

A 11

B 12

C 13

D 21

E 31

Solution:

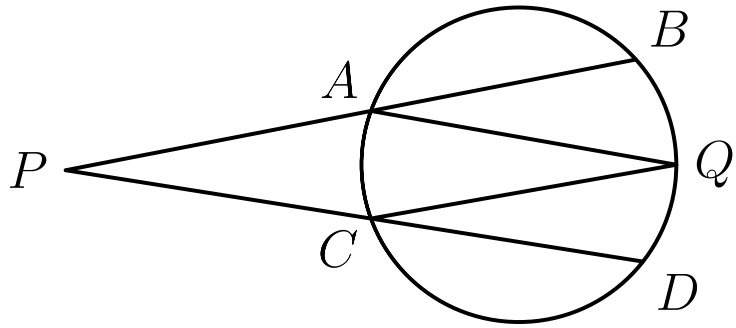
If P is the principal, then

$$\begin{aligned} 255.31 &= P\left(1+\frac{0.05}{6}\right) \\ &= P\frac{121}{120}. \end{aligned}$$

Hence $P = 253.20$, so the interest is $\$2.11$ and its cents part is 11.

Therefore, the correct answer is **A**.

5. Points $A, B, Q, D,$ and C lie on the circle shown, and the measures of arcs BQ and QD are 42° and 38°, respectively. The sum of the measures of angles P and Q is:



A 80°

B 62°

C 40°

D 46°

E None of these

Solution:

Let $m\overset{\frown}{AC} = u$. The external-secant formula gives

$$m\angle P = \frac{1}{2}\left(m\overset{\frown}{BD} - u\right),$$

while the inscribed-angle theorem gives $m\angle Q = \frac{u}{2}$. Therefore

$$\begin{aligned} m\angle P + m\angle Q &= \frac{1}{2}m\overset{\frown}{BD} \\ &= \frac{1}{2}(42^\circ + 38^\circ) \\ &= 40^\circ. \end{aligned}$$

Therefore, the correct answer is **C**.

6. Let $*$ be a symbol denoting the binary operation on the set S of all nonzero real numbers as follows: for any $a, b \in S, a * b = 2ab$. Which statement is not true?

A $*$ is commutative over S

B $*$ is associative over S

C $\frac{1}{2}$ is an identity element for $*$ in S

D Every element of S has an inverse for $*$

E $\frac{1}{2a}$ is an inverse for $*$ of the element a of S

Solution:

The operation is commutative, and

$$(a * b) * c = 4abc = a * (b * c),$$

so it is associative. Its identity is $\frac{1}{2}$. The inverse of a must satisfy $2ax = \frac{1}{2}$, so it is $x = \frac{1}{4a}$, not $\frac{1}{2a}$.

Therefore, the correct answer is **E**.

7. $2^{-(2k+1)} - 2^{-(2k-1)} + 2^{-2k}$ is equal to:

A 2^{-2k}

B $2^{-(2k-1)}$

C $-2^{-(2k+1)}$

D 0

E 2

Solution:

Factoring gives

$$2^{-(2k+1)}(1 - 4 + 2) = -2^{-(2k+1)}.$$

Therefore, the correct answer is **C**.

8. The solution set of $6x^2 + 5x < 4$ is the set of all values of x such that:

A $-2 < x < 1$

B $-\dfrac{4}{3} < x < \dfrac{1}{2}$

C $-\dfrac{1}{2} < x < \dfrac{4}{3}$

D $x < \dfrac{1}{2}$ or $x > -\dfrac{4}{3}$

E $x < -\dfrac{4}{3}$ or $x > \dfrac{1}{2}$

Solution:

The inequality is

$$6x^2 + 5x - 4 < 0,$$
$$(3x + 4)(2x - 1) < 0.$$

The roots are $-\frac{4}{3}$ and $\frac{1}{2}$, and the upward-opening quadratic is negative between them.

Therefore, the correct answer is **B**.

9. An uncrossed belt is fitted without slack around two circular pulleys with radii of 14 inches and 4 inches. If the distance between the points of contact of the belt with the pulleys is 24 inches, then the distance between the centers of the pulleys in inches is:

A 24

B $2\sqrt{119}$

C 25

D 26

E $4\sqrt{35}$

Solution:

The common external tangent and the radii to its contact points form a right triangle whose legs are 24 and $14 - 4 = 10$. Thus the center distance is

$$\sqrt{24^2 + 10^2} = \sqrt{676} = 26.$$

Therefore, the correct answer is **D**.

10. Each of a group of 50 girls is blonde or brunette and is blue-eyed or brown-eyed. If 14 are blue-eyed blondes, 31 are brunettes, and 18 are brown-eyed, then the number of brown-eyed brunettes is:

A 5

B 7

C 9

D 11

E 13

Solution:

There are $50 - 31 = 19$ blondes, of whom $19 - 14 = 5$ are brown-eyed. Hence the number of brown-eyed brunettes is $18 - 5 = 13.$

Therefore, the correct answer is **E**.

11. The numeral 47 in base a represents the same number as 74 in base b. Assuming both bases are positive integers, the least possible value of $a + b$, written as a Roman numeral, is:

- A XIII
- B XV
- C XXI
- D XXIV
- E XVI

Solution:

The equality is $4a + 7 = 7b + 4$, or $4a - 7b = -3$. Since both digits must be valid, $a, b > 7$. Reducing modulo 7 gives $a \equiv 1 \pmod 7$. The first valid value is $a = 15$, which gives $b = 9$. Thus $a + b = 24$, written XXIV.

Therefore, the correct answer is **D**.

12. For each integer $N > 1$, define positive integers to be congruent if they leave the same nonnegative remainder when divided by N. If $69, 90,$ and 125 are congruent in one such system, then in that same system, 81 is congruent to:

- A 3
- B 4
- C 5
- D 7
- E 8

Solution:

The modulus N divides both $90 - 69 = 21$ and $125 - 90 = 35$. Since $N > 1$, this forces $N = 7$. Then $81 \equiv 4 \pmod{7}$.

Therefore, the correct answer is **B**.

13. If $(1.0025)^{10}$ is evaluated correct to 5 decimal places, then the digit in the fifth decimal place is:

A 0

B 1

C 2

D 5

E 8

Solution:

The binomial expansion gives

$$\begin{aligned} &(1+0.0025)^{10} \\ &= 1 + 10(0.0025) \\ &\quad +45(0.0025)^2 \\ &+120(0.0025)^3 + \cdots \\ &= 1.025283125\ldots \end{aligned}$$

which rounds to 1.02528. The fifth decimal digit is 8.

Therefore, the correct answer is **E**.

14. The number $2^{48}-1$ is exactly divisible by two numbers between 60 and 70. These numbers are:

- A $61, 63$
- B $61, 65$
- C $63, 65$
- D $63, 67$
- E $67, 69$

Solution:

Because $6 \mid 48$,

$$63 = 2^6 - 1 \mid 2^{48} - 1.$$

Also $2^{12} - 1 = 4095 = 63 \cdot 65$, and $12 \mid 48$, so $65 \mid 2^{48} - 1$. Thus the two numbers are 63 and 65.

Therefore, the correct answer is **C**.

15. An aquarium on a level table has rectangular faces and is 10 inches wide and 8 inches high. When it was tilted, the water in it just covered an 8-inch by 10-inch end but only three-fourths of the rectangular bottom. The depth of the water when the bottom was again made level was:

A $2\frac{1}{2}$ inches

B 3 inches

C $3\frac{1}{4}$ inches

D $3\frac{1}{2}$ inches

E 4 inches

Solution:

Let the aquarium length be L and the level-water depth be h. In the tilted position the longitudinal cross-section of the water is a triangle with base $\frac{3L}{4}$ and height 8. Therefore

$$10\left(\frac{1}{2}\cdot\frac{3L}{4}\cdot 8\right) = 10Lh,$$

so $h = 3$ inches.

Therefore, the correct answer is **B**.

16. After finding the average of 35 scores, a student carelessly included the average with the 35 scores and found the average of these 36 numbers. The ratio of the second average to the true average was:

A $1 : 1$

B $35 : 36$

C $36 : 35$

D $2 : 1$

E None of these

Solution:

If the true average is x, then the original sum is $35x$. Including x itself gives a sum of $36x$ over 36 numbers, so the new average is still x. The ratio is $1 : 1$.

Therefore, the correct answer is **A**.

17. A circular disk is divided by $2n$ equally spaced radii $(n > 0)$ and one secant line. The maximum number of nonoverlapping areas into which the disk can be divided is:

A $2n + 1$

B $2n + 2$

C $3n - 1$

D $3n$

E $3n + 1$

Solution:

The radii first make $2n$ sectors. A secant not through the center can meet at most one radius in each of n opposite pairs, hence at most n radii. Those intersections divide the secant chord into $n + 1$ pieces, each of which adds one region. The maximum is therefore $2n + (n + 1) = 3n + 1.$

Therefore, the correct answer is **E**.

18. The current in a river flows steadily at 3 miles per hour. A motorboat traveling at a constant rate in still water goes downstream 4 miles and then returns to its starting point. The trip takes one hour, excluding turning time. The ratio of the downstream rate to the upstream rate is:

- A $4:3$
- B $3:2$
- C $5:3$
- D $2:1$
- E $5:2$

Solution:

If v is the still-water speed, then

$$\frac{4}{v+3}+\frac{4}{v-3}=1.$$

This simplifies to $v^2-8v-9=0$, so the positive admissible solution is $v=9.$ The downstream and upstream rates are 12 and $6,$ whose ratio is $2:1.$

Therefore, the correct answer is **D**.

19. If the line $y = mx + 1$ intersects the ellipse $x^2 + 4y^2 = 1$ exactly once, then the value of m^2 is:

A $\frac{1}{2}$

B $\frac{2}{3}$

C $\frac{3}{4}$

D $\frac{4}{5}$

E $\frac{5}{6}$

Solution:

Substitution gives

$$(1 + 4m^2)x^2 + 8mx + 3 = 0.$$

Tangency requires its discriminant to vanish:

$$64m^2 - 12(1 + 4m^2) = 0.$$

Thus $16m^2 = 12$, so $m^2 = \frac{3}{4}$.

Therefore, the correct answer is **C**.

20. The sum of the squares of the roots of the equation $x^2 + 2hx = 3$ is 10. The absolute value of h is equal to:

A -1

B $\frac{1}{2}$

C $\frac{2}{3}$

D 2

E None of these

Solution:

For roots $r, s,$ Vieta's formulas give $r + s = -2h$ and $rs = -3$. Hence

$$\begin{aligned} 4h^2 = (r+s)^2 &= r^2 + s^2 + 2rs \\ &= 10 - 6 = 4. \end{aligned}$$

Therefore $|h| = 1,$ which is not among choices A-D because choice A is $-1.$

Therefore, the correct answer is **E**.

21. If

$$\log_2(\log_3(\log_4 x)) = 0,$$
$$\log_3(\log_4(\log_2 y)) = 0,$$
$$\log_4(\log_2(\log_3 z)) = 0,$$

then $x + y + z$ is equal to:

A 50

B 58

C 89

D 111

E 1296

Solution:

Undoing the logarithms from the outside inward gives

$$x = 4^3 = 64,$$
$$y = 2^4 = 16,$$
$$z = 3^2 = 9.$$

Therefore $x + y + z = 64 + 16 + 9 = 89.$

Therefore, the correct answer is **C**.

22. If w is one of the imaginary roots of $x^3 = 1$, then $(1 - w + w^2)(1 + w - w^2)$ is equal to:

A 4

B w

C 2

D w^2

E 1

Solution:

Because $w \neq 1$ and $w^3 = 1$, we have $1 + w + w^2 = 0$. Thus

$$1 - w + w^2 = -2w,$$
$$1 + w - w^2 = -2w^2.$$

Their product is $4w^3 = 4$.

Therefore, the correct answer is **A**.

23. Teams A and B are playing a series of games. If either team has an equal chance to win any game, and Team A must win two games while Team B must win three games to win the series, then the odds favoring Team A to win the series are:

- A 11 to 5
- B 5 to 2
- C 8 to 3
- D 3 to 2
- E 13 to 6

Solution:

Team A loses in the sequences BBB, ABBB, BABB, and BBAB. Their total probability is

$$\frac{1}{8} + 3\left(\frac{1}{16}\right) = \frac{5}{16}.$$

Thus Team A wins with probability $\frac{11}{16}$, so the odds in its favor are $11 : 5$.

Therefore, the correct answer is **A**.

24. Pascal's triangle is an array of positive integers, shown below, in which the first row is 1, the second row is two 1's, each row begins and ends with 1, and each other entry is the sum of the two entries above it.

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \\ & & & & \text{etc.} & & & & \end{array}$$

The quotient of the number of entries in the first n rows which are not 1's and the number of 1's is:

A $\dfrac{n^2-n}{2n-1}$

B $\dfrac{n^2-n}{4n-2}$

C $\dfrac{n^2-2n}{2n-1}$

D $\dfrac{n^2-3n+2}{4n-2}$

E None of these

Solution:

The first n rows contain $1+2+\cdots+n=\frac{n(n+1)}{2}$ entries. There are $2n-1$ boundary 1's, so the number of other entries is

$$
\begin{aligned}
&\frac{n(n+1)}{2}-(2n-1) \\
&=\frac{n^2+n-4n+2}{2} \\
&=\frac{n^2-3n+2}{2}.
\end{aligned}
$$

Dividing by $2n-1$ gives $\frac{n^2-3n+2}{4n-2}$.

Therefore, the correct answer is **D**.

25. A teenage boy wrote his own age after his father's. From this new four-place number, he subtracted the absolute value of the difference of their ages to get $4{,}289$. The sum of their ages was:

A 48

B 52

C 56

D 59

E 64

Solution:

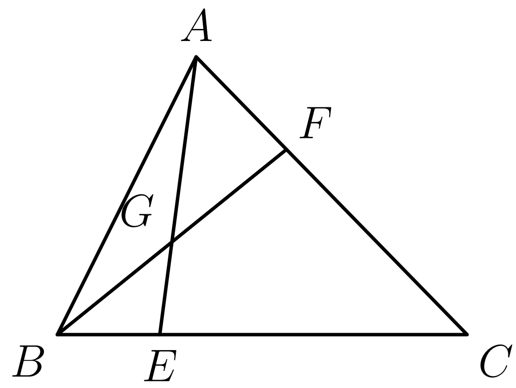
Let the father and boy be f and b years old. Since $f > b$,

$$
\begin{aligned}
100f+b-(f-b)&=4289, \\
99f+2b&=4289.
\end{aligned}
$$

The boy is a teenager, so $13 \le b \le 19$. Reducing modulo 9 gives $2b \equiv 5 \pmod 9$, hence $b=16$. Then $f=\frac{4289-32}{99}=43$, and $f+b=59$.

Therefore, the correct answer is **D**.

26. In triangle ABC, point F divides side AC in the ratio $1 : 2$. Let E be the point where side BC meets AG, where G is the midpoint of BF. Then E divides BC in the ratio:



A $1 : 4$

B $1 : 3$

C $2 : 5$

D $4 : 11$

E $3 : 8$

Solution:

Use mass points. Since $AF : FC = 1 : 2$, assign masses 2 and 1 to A and C, so the mass at F is 3. Because G is the midpoint of BF, the mass at B is also 3. Therefore

$$BE : EC = m_C : m_B = 1 : 3.$$

Therefore, the correct answer is **B**.

27. A box contains chips, each of which is red, white, or blue. The number of blue chips is at least half the number of white chips and at most one-third the number of red chips. The number which are white or blue is at least 55. The minimum number of red chips is:

A 24

B 33

C 45

D 54

E 57

Solution:

Let the counts be r, w, b. The conditions give

$$\begin{aligned} w &\leq 2b, \\ r &\geq 3b, \\ w + b &\geq 55. \end{aligned}$$

Hence $3b \geq w + b \geq 55$, so $b \geq 19$ and $r \geq 57$. Equality is possible with $(w, b, r) = (36, 19, 57)$, so the minimum is 57.

Therefore, the correct answer is **E**.

28. Nine lines parallel to the base of a triangle divide the other sides each into 10 equal segments and the area into 10 distinct parts. If the area of the largest of these parts is 38, then the area of the original triangle is:

A 180

B 190

C 200

D 210

E 240

Solution:

If the whole area is K, the triangle above the bottom strip is similar to the original with scale $\frac{9}{10}$, so its area is $\frac{81K}{100}$. Thus the largest strip has area

$$K - \frac{81K}{100} = \frac{19K}{100} = 38,$$

giving $K = 200.$

Therefore, the correct answer is **C**.

29. Given the progression

$$10^{\frac{1}{11}}, 10^{\frac{2}{11}}, 10^{\frac{3}{11}},$$
$$10^{\frac{4}{11}}, \ldots, 10^{\frac{n}{11}},$$

the least positive integer n such that the product of the first n terms exceeds $100{,}000$ is:

A 7

B 8

C 9

D 10

E 11

Solution:

The product is

$$10^{\frac{1+2+\cdots+n}{11}} = 10^{\frac{n(n+1)}{22}}.$$

It exceeds $100{,}000 = 10^5$ exactly when $n(n+1) > 110$. For $n = 10$ there is equality, while $n = 11$ works.

Therefore, the correct answer is **E**.

30. Given the linear fractional transformation

$$f_1(x) = \frac{2x-1}{x+1},$$

define $f_{n+1}(x) = f_1(f_n(x))$ for $n = 1, 2, 3, \ldots$. Assuming $f_{35}(x) = f_5(x)$, it follows that $f_{28}(x)$ is equal to:

A x

B $\frac{1}{x}$

C $\frac{x-1}{x}$

D $\frac{1}{1-x}$

E None of these

Solution:

The transformation is invertible. From $f_{35} = f_5$, composing with f_5^{-1} shows that f_{30} is the identity. Solving $y = \frac{2x-1}{x+1}$ for x gives

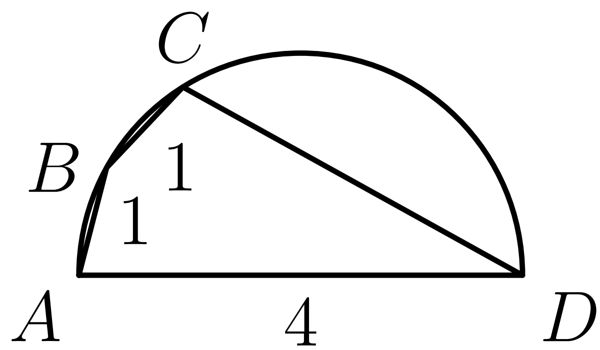
$$g(y) = f_1^{-1}(y) = \frac{y+1}{2-y}.$$

Since iterates have period 30, $f_{28} = f_{-2} = g^2$. Direct composition gives

$$g(g(x)) = \frac{1}{1-x}.$$

Therefore, the correct answer is **D**.

31. Quadrilateral $ABCD$ is inscribed in a circle with side AD, a diameter of length 4. If sides AB and BC each have length 1, then side CD has length:



A $\frac{7}{2}$

B $\frac{5\sqrt{2}}{2}$

C $\sqrt{11}$

D $\sqrt{13}$

E $2\sqrt{3}$

Solution:

The circle has radius 2. Let the central angles subtending the equal chords AB and BC each be $2t$. Then

$$1 = 4\sin t,$$

so $\sin t = \frac{1}{4}$. The remaining central angle from C to D along the semicircle is $\pi - 4t$, hence

$$\begin{aligned} CD &= 4\sin\left(\frac{\pi - 4t}{2}\right) \\ &= 4\cos(2t) \\ &= 4\left(1 - 2\cdot\frac{1}{16}\right) \\ &= \frac{7}{2}. \end{aligned}$$

Therefore, the correct answer is **A**.

32. If

$$s = (1 + 2^{-\frac{1}{32}})(1 + 2^{-\frac{1}{16}}) \cdot (1 + 2^{-\frac{1}{8}})(1 + 2^{-\frac{1}{4}}) \cdot (1 + 2^{-\frac{1}{2}}),$$

then s is equal to:

A $\frac{1}{2}(1 - 2^{-\frac{1}{32}})^{-1}$

B $(1 - 2^{-\frac{1}{32}})^{-1}$

C $1 - 2^{-\frac{1}{32}}$

D $\frac{1}{2}(1 - 2^{-\frac{1}{32}})$

E $\frac{1}{2}$

Solution:

Let $x = 2^{-\frac{1}{32}}$. Then

$$\begin{aligned}(1 - x)s &= 1 - x^{32} \\ &= 1 - \frac{1}{2} = \frac{1}{2}.\end{aligned}$$

Therefore

$$s = \frac{1}{2}(1 - 2^{-\frac{1}{32}})^{-1}.$$

Therefore, the correct answer is **A**.

33. If P is the product of n quantities in geometric progression, S their sum, and S' the sum of their reciprocals, then P in terms of S, S', and n is:

A $(SS')^{\frac{n}{2}}$

B $(\frac{S}{S'})^{\frac{n}{2}}$

C $(SS')^{n-2}$

D $(\frac{S}{S'})^{n}$

E $(\frac{S'}{S})^{\frac{n-1}{2}}$

Solution:

Write the terms as $a, ar, \ldots, ar^{n-1}$. Reversing the reciprocal sum gives

$$S' = \frac{S}{a^2 r^{n-1}},$$

so $\frac{S}{S'} = a^2 r^{n-1}$. Meanwhile

$$\begin{aligned} P &= a^n r^{\frac{n(n-1)}{2}} \\ &= \left(a^2 r^{n-1}\right)^{\frac{n}{2}} \\ &= \left(\frac{S}{S'}\right)^{\frac{n}{2}}. \end{aligned}$$

Therefore, the correct answer is **B**.

34. An ordinary clock in a factory is running slow so that the minute hand passes the hour hand at the usual dial positions $(12 \text{ o'clock},$ etc.) but only every 69 minutes. At time and one-half for overtime, the extra pay to which a $\$4.00$-per-hour worker should be entitled after working a normal 8-hour day by that slow-running clock is:

A $\$2.30$

B $\$2.60$

C $\$2.80$

D $\$3.00$

E $\$3.30$

Solution:

Successive hand-overlaps are $\frac{720}{11}$ real minutes apart, while this slow clock displays 69 minutes between them. Thus 12 displayed hours correspond to

$$11 \cdot 69 = 759 \text{ real minutes,}$$
$$759 \text{ minutes} = 12 \text{ h } 39 \text{ min.}$$

Eight displayed hours therefore take 8 h 26 min, so the overtime is 26 minutes. At $\$6$ per hour, that pays $\$2.60$.

Therefore, the correct answer is **B**.

35. Each circle in an infinite sequence with decreasing radii is tangent externally to the one following it and to both sides of a given right angle. The ratio of the area of the first circle to the sum of the areas of all the other circles in the sequence is:

A $(4+3\sqrt{2}):4$

B $9\sqrt{2}:2$

C $(16+12\sqrt{2}):1$

D $(2+2\sqrt{2}):1$

E $(3+2\sqrt{2}):1$

Solution:

If consecutive radii are $r > r'$, their centers lie on the angle bisector at distances $r\sqrt{2}$ and $r'\sqrt{2}$ from the vertex. External tangency gives

$$\sqrt{2}(r-r') = r + r',$$

so $\frac{r'}{r} = 3 - 2\sqrt{2}$. The ratio of successive areas is

$$q = (3-2\sqrt{2})^2.$$

Therefore the first area divided by the sum of all later areas is

$$\begin{aligned} \frac{1-q}{q} &= \frac{1}{q} - 1 \\ &= (3+2\sqrt{2})^2 - 1 \\ &= 16 + 12\sqrt{2}. \end{aligned}$$

Therefore, the correct answer is **C**.

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