1970 AMC 12 Solutions

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1. The fourth power of $\sqrt{1+\sqrt{1+\sqrt{1}}}$ is:

A $\sqrt{2}+\sqrt{3}$

B $\frac{1}{2}(7+3\sqrt{5})$

C $1+2\sqrt{3}$

D 3

E $3+2\sqrt{2}$

Solution:

The innermost radical is $\sqrt{1}=1$, so the given number is $\sqrt{1+\sqrt{2}}$. Its fourth power is

$$\begin{aligned}(1+\sqrt{2})^2 &= 1+2\sqrt{2}+2 \\ &= 3+2\sqrt{2}.\end{aligned}$$

Therefore, the correct answer is **E**.

2. A square and a circle have equal perimeters. The ratio of the area of the circle to the area of the square is:

- A $\frac{4}{\pi}$
- B $\frac{\pi}{\sqrt{2}}$
- C $\frac{4}{1}$
- D $\frac{\sqrt{2}}{\pi}$
- E $\frac{\pi}{4}$

Solution:

If the common perimeter is p, then the circle has radius $\frac{p}{2\pi}$ and the square has side $\frac{p}{4}$. Therefore

$$\frac{\pi\left(\frac{p}{2\pi}\right)^2}{\left(\frac{p}{4}\right)^2} = \frac{\frac{p^2}{4\pi}}{\frac{p^2}{16}} = \frac{4}{\pi}.$$

Therefore, the correct answer is **A**.

3. If $x = 1 + 2^p$ and $y = 1 + 2^{-p}$, then y in terms of x is:

A $\dfrac{x+1}{x-1}$

B $\dfrac{x+2}{x-1}$

C $\dfrac{x}{x-1}$

D $2 - x$

E $\dfrac{x-1}{x}$

Solution:

From $x = 1 + 2^p$ we get $2^p = x - 1$, so

$$\begin{aligned} y &= 1 + 2^{-p} \\ &= 1 + \frac{1}{x-1} = \frac{x}{x-1}. \end{aligned}$$

Therefore, the correct answer is **C**.

4. Let S be the set of all numbers which are the sum of the squares of three consecutive integers. Then we can say that:

- A No member of S is divisible by 2
- B No member of S is divisible by 3 but some member is divisible by 11
- C No member of S is divisible by 3 or by 5
- D No member of S is divisible by 3 or by 7
- E None of these

Solution:

The sum is

$$\begin{aligned}(n-1)^2 + n^2 + (n+1)^2 \\ = 3n^2 + 2.\end{aligned}$$

It is always congruent to $2 \pmod 3$, so no member of S is divisible by 3. On the other hand, taking $n = 5$ gives $3(25) + 2 = 77$, which is divisible by 11.

Therefore, the correct answer is **B**.

5. If $f(x) = \dfrac{x^4 + x^2}{x + 1}$, then $f(i)$, where $i = \sqrt{-1}$, is equal to:

A $1 + i$

B 1

C -1

D 0

E $-1 - i$

Solution:

Because $i^2 = -1$ and $i^4 = 1$,

$$f(i) = \frac{i^4 + i^2}{i + 1} = \frac{1 - 1}{i + 1} = 0.$$

Therefore, the correct answer is **D**.

6. The smallest value of $x^2 + 8x$ for real values of x is:

A -16.25

B -16

C -15

D -8

E None of these

Solution:

Completing the square gives

$$x^2 + 8x = (x+4)^2 - 16.$$

Since $(x+4)^2 \geq 0,$ the minimum occurs at $x = -4$ and equals $-16.$

Therefore, the correct answer is **B**.

7. Inside square $ABCD$ with side s, quarter-circle arcs with radii s and centers at A and B are drawn. These arcs intersect at a point X inside the square. How far is X from side CD?

A $\frac{1}{2}s(\sqrt{3}+4)$

B $\frac{1}{2}s\sqrt{3}$

C $\frac{1}{2}s(1+\sqrt{3})$

D $\frac{1}{2}s(\sqrt{3}-1)$

E $\frac{1}{2}s(2-\sqrt{3})$

Solution:

Since $AX = BX = AB = s$, triangle ABX is equilateral. Thus the perpendicular distance from X to AB is $\frac{s\sqrt{3}}{2}$. The distance between the parallel sides AB and CD is s, so the requested distance is

$$s - \frac{s\sqrt{3}}{2} = \frac{s}{2}(2-\sqrt{3}).$$

Therefore, the correct answer is **E**.

8. If $a = \log_8 225$ and $b = \log_2 15$, then:

A $a = \frac{b}{2}$

B $a = \frac{2b}{3}$

C $a = b$

D $b = \frac{a}{2}$

E $a = \frac{3b}{2}$

Solution:

Using $225 = 15^2$ and $8 = 2^3$,

$$a = \frac{\log_2(15^2)}{\log_2(2^3)} \\ = \frac{2\log_2 15}{3} = \frac{2b}{3}.$$

Therefore, the correct answer is **B**.

9. Points P and Q are on line segment $AB,$ and both points are on the same side of the midpoint of $AB.$ Point P divides AB in the ratio $2 : 3,$ and Q divides AB in the ratio $3 : 4.$ If $PQ = 2,$ then the length of segment AB is:

- A 12
- B 28
- C 70
- D 75
- E 105

Solution:

The division ratios give

$$AP = \frac{2}{5}AB, \qquad AQ = \frac{3}{7}AB.$$

Because the two points are on the same side of the midpoint,

$$\begin{aligned} PQ &= AQ - AP \\ &= \left(\frac{3}{7} - \frac{2}{5}\right) AB \\ &= \frac{1}{35}AB. \end{aligned}$$

Since $PQ = 2,$ we obtain $AB = 70.$

Therefore, the correct answer is **C**.

10. Let $F = 0.4818181\ldots$ be an infinite repeating decimal with the digits 8 and 1 repeating. When F is written as a fraction in lowest terms, the denominator exceeds the numerator by:

A 13

B 14

C 29

D 57

E 126

Solution:

We have

$$\begin{aligned} F &= 0.4 + 0.081(1 + 0.01 + \cdots) \\ &= \frac{2}{5} + \frac{0.081}{0.99} \\ &= \frac{2}{5} + \frac{9}{110} \\ &= \frac{53}{110}. \end{aligned}$$

The denominator exceeds the numerator by $110 - 53 = 57$.

Therefore, the correct answer is **D**.

11. If two factors of $2x^3 - hx + k$ are $x + 2$ and $x - 1$, the value of $|2h - 3k|$ is:

A 4

B 3

C 2

D 1

E 0

Solution:

The factor theorem gives

$$\begin{aligned} -16 + 2h + k &= 0, \\ 2 - h + k &= 0. \end{aligned}$$

Subtracting the second equation from the first gives $3h = 18$, so $h = 6$ and then $k = 4$. Hence

$$|2h - 3k| = |12 - 12| = 0.$$

Therefore, the correct answer is **E**.

12. A circle with radius r is tangent to sides $AB, AD,$ and CD of rectangle $ABCD$ and passes through the midpoint of diagonal AC. The area of the rectangle, in terms of r, is:

- A $4r^2$
- B $6r^2$
- C $8r^2$
- D $12r^2$
- E $20r^2$

Solution:

Let the circle be tangent to the parallel sides AB and CD. Their separation is its diameter, so $AD = BC = 2r$. Let R be the tangency point on AD, let Q be the center, and let M be the midpoint of AC.

The line RQ is parallel to AB and halfway between AB and CD, so R is the midpoint of AD and the line contains M. Since M lies on the circle and R, Q, M are collinear, $RM = 2r$. Thus RM is a midsegment of triangle ADC, so $DC = 2RM = 4r$. Therefore the area is

$$(2r)(4r) = 8r^2.$$

Therefore, the correct answer is **C**.

13. Given the binary operation $*$ defined by $a * b = a^b$ for all positive numbers a and b. Then for all positive $a, b, c, n,$ we have:

A $a * b = b * a$

B $a * (b * c) = (a * b) * c$

C $a * (b^n) = (a * n) * b$

D $(a * b)^n = a * (bn)$

E None of these

Solution:

For choice D,

$$
\begin{aligned}
(a * b)^n &= (a^b)^n = a^{bn}, \\
a * (bn) &= a^{bn}.
\end{aligned}
$$

Thus that identity always holds. The other choices would assert false general identities such as $a^b = b^a$ or $a^{b^c} = a^{bc}$.

Therefore, the correct answer is **D**.

14. Consider $x^2 + px + q = 0$, where p and q are positive numbers. If the roots of this equation differ by 1, then p equals:

A $\sqrt{4q+1}$

B $q-1$

C $-\sqrt{4q+1}$

D $q+1$

E $\sqrt{4q-1}$

Solution:

The two roots differ in absolute value by

$$\sqrt{p^2-4q}.$$

Hence $p^2 - 4q = 1$, so $p^2 = 4q + 1$. Since p is positive,

$$p = \sqrt{4q+1}.$$

Therefore, the correct answer is **A**.

15. Lines in the xy-plane are drawn through the point $(3, 4)$ and the trisection points of the line segment joining the points $(-4, 5)$ and $(5, -1)$. One of these lines has the equation:

A $3x - 2y - 1 = 0$

B $4x - 5y + 8 = 0$

C $5x + 2y - 23 = 0$

D $x + 7y - 31 = 0$

E $x - 4y + 13 = 0$

Solution:

The displacement from $(-4, 5)$ to $(5, -1)$ is $(9, -6)$. The trisection points are therefore

$$(-4, 5) + \tfrac{1}{3}(9, -6) = (-1, 3)$$

and $(2, 1)$. The line through $(-1, 3)$ and $(3, 4)$ has slope $\frac{1}{4}$, so

$$y - 4 = \frac{1}{4}(x - 3),$$

or $x - 4y + 13 = 0.$

Therefore, the correct answer is **E**.

16. If $F(n)$ is a function such that $F(1) = F(2) = F(3) = 1$, and such that

$$F(n+1) = \frac{F(n)F(n-1)+1}{F(n-2)}$$

for $n \geq 3$, then $F(6)$ is equal to:

A 2

B 3

C 7

D 11

E 26

Solution:

Applying the recurrence successively,

$$F(4) = \frac{1 \cdot 1 + 1}{1} = 2,$$
$$F(5) = \frac{2 \cdot 1 + 1}{1} = 3,$$

and

$$F(6) = \frac{3 \cdot 2 + 1}{1} = 7.$$

Therefore, the correct answer is **C**.

17. If $r > 0$, then for all p and q such that $pq \neq 0$ and $pr > qr$, we have:

- A $-p > -q$
- B $-p > q$
- C $1 > -\frac{q}{p}$
- D $1 < \frac{q}{p}$
- E None of these

Solution:

Since $r > 0$, the hypothesis is equivalent to $p > q$. Then $-p < -q$, so A is false. Taking $(p, q) = (2, 1)$ makes B and D false. Taking $(p, q) = (1, -2)$ makes C false because $-\frac{q}{p} = 2$. Thus none of A-D follows in every case.

Therefore, the correct answer is **E**.

18. $\sqrt{3+2\sqrt{2}}-\sqrt{3-2\sqrt{2}}$ is equal to:

A 2

B $2\sqrt{3}$

C $4\sqrt{2}$

D $\sqrt{6}$

E $2\sqrt{2}$

Solution:

Since

$$3+2\sqrt{2}=(\sqrt{2}+1)^2,$$
$$3-2\sqrt{2}=(\sqrt{2}-1)^2,$$

and both $\sqrt{2}+1$ and $\sqrt{2}-1$ are positive, the difference is

$$(\sqrt{2}+1)-(\sqrt{2}-1)=2.$$

Therefore, the correct answer is **A**.

19. The sum of an infinite geometric series with common ratio r such that $|r| < 1$ is $15,$ and the sum of the squares of the terms of this series is $45.$ The first term of the series is:

A 12

B 10

C 5

D 3

E 2

Solution:

Let the first term be $a.$ Then

$$\frac{a}{1-r} = 15, \qquad \frac{a^2}{1-r^2} = 45.$$

Since $1 - r^2 = (1-r)(1+r)$, dividing the second equation by the first gives $\frac{a}{1+r} = 3.$ Thus

$$a = 15(1-r) = 3(1+r).$$

Solving gives $r = \frac{2}{3}$ and $a = 5.$

Therefore, the correct answer is **C**.

20. Lines HK and BC lie in a plane. M is the midpoint of line segment BC, and BH and CK are perpendicular to HK. Then we:

A always have $MH = MK$

B always have $MH > BK$

C sometimes have $MH = MK$ but not always

D always have $MH > MB$

E always have $BH < BC$

Solution:

Choose coordinates

$$\begin{aligned} H &= (h, 0), \quad & B &= (h, u), \\ K &= (k, 0), \quad & C &= (k, v). \end{aligned}$$

Then

$$M = \left(\frac{h+k}{2}, \frac{u+v}{2}\right).$$

The horizontal displacements from M to H and K are opposites, while the vertical displacements are equal. Therefore $MH^2 = MK^2$, so $MH = MK$ always.

Therefore, the correct answer is **A**.

21. On an auto trip, the distance read from the instrument panel was 450 miles. With snow tires on for the return trip over the same route, the reading was 440 miles. Find, to the nearest hundredth of an inch, the increase in radius of the wheels if the original radius was 15 inches.

- A 0.33
- B 0.34
- C 0.35
- D 0.38
- E 0.66

Solution:

For a fixed true distance, the odometer reading is proportional to the number of wheel revolutions. Thus

$$450r_1 = 440r_2, \qquad \frac{r_2}{r_1} = \frac{45}{44}.$$

With $r_1 = 15$, the increase is

$$\begin{aligned} r_2 - r_1 &= 15\left(\frac{45}{44} - 1\right) \\ &= \frac{15}{44} \\ &\approx 0.3409. \end{aligned}$$

To the nearest hundredth this is 0.34 inch.

Therefore, the correct answer is **B**.

22. If the sum of the first $3n$ positive integers is 150 more than the sum of the first n positive integers, then the sum of the first $4n$ positive integers is:

- A 300
- B 350
- C 400
- D 450
- E 600

Solution:

Let $S_m = \frac{m(m+1)}{2}$. Then

$$\begin{aligned} 2(S_{3n} - S_n) &= 3n(3n+1) \\ &\quad - n(n+1) \\ &= 8n^2 + 2n. \end{aligned}$$

Since $S_{3n} - S_n = 150$, this gives $4n^2 + n = 150$. Hence $(n-6)(4n+25) = 0$, and the positive solution is $n = 6$. Therefore

$$S_{4n} = S_{24} = \frac{24 \cdot 25}{2} = 300.$$

Therefore, the correct answer is **A**.

23. The number $10!$ (10 is written in base 10), when written in the base 12 system, ends in exactly k zeros. The value of k is:

A 1

B 2

C 3

D 4

E 5

Solution:

The prime exponents in $10!$ are

$$v_2(10!) = 5 + 2 + 1 = 8,$$
$$v_3(10!) = 3 + 1 = 4.$$

Each factor of $12 = 2^2 \cdot 3$ uses two factors of 2 and one of 3. Hence the number of trailing base-12 zeros is

$$\min\left(\left\lfloor \frac{8}{2} \right\rfloor, 4\right) = 4.$$

Therefore, the correct answer is **D**.

24. An equilateral triangle and a regular hexagon have equal perimeters. If the area of the triangle is 2, then the area of the hexagon is:

A 2

B 3

C 4

D 6

E 12

Solution:

Let the hexagon side be s. Its perimeter is $6s$, so the equilateral triangle has side $2s$. The large triangle consists of 4 equilateral triangles of side s, while the hexagon consists of 6. Their area ratio is therefore $\frac{6}{4} = \frac{3}{2}$. Since the triangle's area is 2, the hexagon's area is 3.

Therefore, the correct answer is **B**.

25. For every real number x, let $\lfloor x \rfloor$ be the greatest integer which is less than or equal to x. If the postal rate for first class mail is six cents for every ounce or portion thereof, then the cost in cents of first-class postage on a letter weighing W ounces is always:

A $6W$

B $6\lfloor W \rfloor$

C $6(\lfloor W \rfloor - 1)$

D $6(\lfloor W \rfloor + 1)$

E $-6\lfloor -W \rfloor$

Solution:

Charging for every ounce or portion thereof means the number of charged ounces is $\lceil W \rceil$. The floor and ceiling functions satisfy

$$\lceil W \rceil = -\lfloor -W \rfloor.$$

Thus the cost is

$$6\lceil W \rceil = -6\lfloor -W \rfloor.$$

Therefore, the correct answer is **E**.

26. The number of distinct points in the xy-plane common to the graphs of

$$(x+y-5)(2x-3y+5)=0$$

and

$$(x-y+1)(3x+2y-12)=0$$

is:

- A 0
- B 1
- C 2
- D 3
- E 4
- F infinite

Solution:

The first graph is the union of

$$x+y=5, \qquad 2x-3y=-5,$$

and those two lines meet at $(2,3)$. The second graph is the union of

$$x-y=-1, \qquad 3x+2y=12,$$

and these also meet at $(2,3)$. In fact, substituting $(2,3)$ satisfies all four line equations. Since the four lines have distinct slopes, no other point can belong to a line from each graph. There is exactly one common point.

Therefore, the correct answer is **B**.

27. In a triangle, the area is numerically equal to the perimeter. What is the radius of the inscribed circle?

A 2

B 3

C 4

D 5

E 6

Solution:

If r is the inradius and s the semiperimeter, the triangle's area is $K = rs$. Its perimeter is $2s$. The given equality yields

$$rs = 2s.$$

Since $s > 0$, we obtain $r = 2$.

Therefore, the correct answer is **A**.

28. In triangle ABC, the median from vertex A is perpendicular to the median from vertex B. If the lengths of sides AC and BC are 6 and 7, respectively, then the length of side AB is:

A $\sqrt{17}$

B 4

C $4\frac{1}{2}$

D $2\sqrt{5}$

E $4\frac{1}{4}$

Solution:

Place the centroid at the origin. Let the position vectors of A and B be u and v. The two medians lie along u and v, so $u \cdot v = 0$. Also the centroid condition gives $C = -u - v$.

Let $U = |u|^2$ and $V = |v|^2$. Then

$$AC^2 = |2u + v|^2 = 4U + V = 36$$

and

$$\begin{aligned} BC^2 &= |u + 2v|^2 \\ &= U + 4V = 49. \end{aligned}$$

Adding appropriate multiples gives $U + V = 17$. Finally,

$$AB^2 = |u - v|^2 = U + V = 17,$$

so $AB = \sqrt{17}$.

Therefore, the correct answer is **A**.

29. It is now between $10{:}00$ and $11{:}00$ o'clock, and six minutes from now, the minute hand of a watch will be exactly opposite the place where the hour hand was three minutes ago. What is the exact time now?

A $10{:}05\frac{5}{11}$

B $10{:}07\frac{1}{2}$

C $10{:}10$

D $10{:}15$

E $10{:}17\frac{1}{2}$

Solution:

Let x be the number of minutes after $10{:}00$ now. Six minutes from now, the minute hand is $x+6$ minute spaces clockwise from 12. Three minutes ago, the hour hand was $\frac{x-3}{12}$ minute spaces past the 10, so the point opposite it was

$$20+\frac{x-3}{12}$$

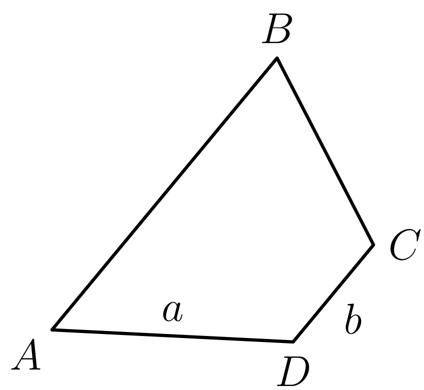
minute spaces past 12. Hence

$$x+6=20+\frac{x-3}{12}.$$

Solving gives $x=15$, so the time is $10{:}15$.

Therefore, the correct answer is **D**.

30. In the accompanying figure, segments AB and CD are parallel, the measure of angle D is twice that of angle B, and the measures of segments AD and CD are a and b, respectively. Then the measure of AB is equal to:



A $\frac{1}{2}a + 2b$

B $\frac{3}{2}b + \frac{3}{4}a$

C $2a - b$

D $4b - \frac{1}{2}a$

E $a + b$

Solution:

Let the bisector of $\angle D$ meet AB at P. Because $\angle D = 2\angle B$, each half has measure $\angle B$. Since $AB \parallel CD$, the alternate interior angle $\angle APD$ also equals $\angle PDC$. Thus triangle APD is isosceles and

$$AP = AD = a.$$

Also $PB \parallel CD$ and $BC \parallel PD$, so $PBCD$ is a parallelogram. Hence $PB = CD = b$. Therefore

$$AB = AP + PB = a + b.$$

Therefore, the correct answer is **E**.

31. If a number is selected at random from the set of all five-digit numbers in which the sum of the digits is equal to 43, what is the probability that this number will be divisible by 11?

A $\frac{2}{5}$

B $\frac{1}{5}$

C $\frac{1}{6}$

D $\frac{1}{11}$

E $\frac{1}{15}$

Solution:

The maximum five-digit digit sum is 45, so a sum of 43 has total deficit 2 from 99999. Either one digit is 7 and the others are 9, giving 5 numbers, or two digits are 8 and the others are 9, giving $\binom{5}{2} = 10$ numbers. Thus there are 15 numbers total.

Using the divisibility-by-11 test, exactly

$$97999, \qquad 99979, \qquad 98989$$

are divisible by 11. The probability is therefore $\frac{3}{15} = \frac{1}{5}$.

Therefore, the correct answer is **B**.

32. A and B travel around a circular track at uniform speeds in opposite directions, starting from diametrically opposite points. If they start at the same time, meet first after B has travelled 100 yards, and meet a second time 60 yards before A completes one lap, then the circumference of the track in yards is:

- A 400
- B 440
- C 480
- D 560
- E 880

Solution:

Let the circumference be $2C$. At the first meeting, B has traveled 100 yards and A has traveled $C - 100$, so their speed ratio is

$$\frac{v_A}{v_B} = \frac{C - 100}{100}.$$

At the second meeting, A is 60 yards short of one lap, so it has traveled $2C - 60$. By then their combined distance is $3C$, so B has traveled $C + 60$. Hence

$$\frac{C - 100}{100} = \frac{2C - 60}{C + 60}.$$

Cross-multiplying gives $C = 240$, so the circumference is $2C = 480$ yards.

Therefore, the correct answer is **C**.

33. Find the sum of the digits of all the numerals in the sequence $1, 2, 3, 4, \ldots, 10000.$

A $180{,}001$

B $154{,}756$

C $45{,}001$

D $154{,}755$

E $270{,}001$

Solution:

Write the integers from 0 through 9999 using four digits with leading zeros. In each of the four positions, every digit $0, 1, \ldots, 9$ appears 1000 times. Thus their total digit sum is

$$\begin{aligned} &4 \cdot 1000(0 + 1 + \cdots + 9) \\ &= 4000 \cdot 45 \\ &= 180000. \end{aligned}$$

Replacing 0 by 10000 adds 1, so the requested sum is 180001.

Therefore, the correct answer is **A**.

34. The greatest integer that will divide $13{,}511$, $13{,}903$, and $14{,}589$ and leave the same remainder is:

A 28

B 49

C 98

D an odd multiple of 7 greater than 49

E an even multiple of 7 greater than 98

Solution:

A divisor leaves the same remainder on all three numbers exactly when it divides their differences. The two successive differences are

$$13903 - 13511 = 392,$$
$$14589 - 13903 = 686.$$

Therefore the greatest possible divisor is

$$\begin{aligned} &\gcd(392, 686) \\ &= \gcd(392, 294) \\ &= \gcd(294, 98) = 98. \end{aligned}$$

Therefore, the correct answer is **C**.

35. A retiring employee receives an annual pension proportional to the square root of the number of years of his service. Had he served a years more, his pension would have been p dollars greater, whereas, had he served b years more ($b \neq a$), his pension would have been q dollars greater than the original annual pension. Find his annual pension in terms of $a, b, p,$ and q.

A $\dfrac{p^2 - q^2}{2(a-b)}$

B $\dfrac{(p-q)^2}{2\sqrt{ab}}$

C $\dfrac{ap^2 - bq^2}{2(ap-bq)}$

D $\dfrac{aq^2 - bp^2}{2(bp-aq)}$

E $\sqrt{(a-b)(p-q)}$

Solution:

Let the current pension be $X = k\sqrt{n}$. Since $X + p = k\sqrt{n+a}$ and $X + q = k\sqrt{n+b}$, squaring and using $X^2 = k^2 n$ gives the first line below. Multiplying its equations by b and a, respectively, and subtracting gives the second:

$$\begin{aligned} 2pX + p^2 &= k^2 a, \\ 2qX + q^2 &= k^2 b, \\ 2X(bp - aq) &= aq^2 - bp^2. \end{aligned}$$

Hence $X = \dfrac{aq^2 - bp^2}{2(bp-aq)}$.

Therefore, the correct answer is **D**.

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