1969 AMC 12 Solutions

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1. When x is added to both the numerator and the denominator of the fraction $\frac{a}{b}$, $a \neq b$, $b \neq 0$, the value of the fraction is changed to $\frac{c}{d}$. Then x equals:

A $\dfrac{1}{c-d}$

B $\dfrac{ad-bc}{c-d}$

C $\dfrac{ad-bc}{c+d}$

D $\dfrac{bc-ad}{c-d}$

E $\dfrac{bc-ad}{c+d}$

Solution:

From

$$\frac{a+x}{b+x} = \frac{c}{d}$$

we obtain $ad + dx = bc + cx$. Hence $x(d-c) = bc - ad$, so

$$x = \frac{bc-ad}{d-c} = \frac{ad-bc}{c-d}.$$

Therefore, the correct answer is **B**.

2. If an item is sold for x dollars, there is a loss of 15% based on the cost. If, however, the same item is sold for y dollars, there is a profit of 15% based on the cost. The ratio $y : x$ is:

Here a $t\%$ loss or profit based on cost means $\frac{t}{100}$ times the cost.

A $23 : 17$

B $17y : 23$

C $23x : 17$

D dependent upon the cost

E none of these

Solution:

If the cost is C, then $x = 0.85C$ and $y = 1.15C$. Therefore

$$\begin{aligned} y : x &= 1.15C : 0.85C \\ &= 115 : 85 = 23 : 17. \end{aligned}$$

Therefore, the correct answer is **A**.

3. If N, written in base 2, is 11000, the integer immediately preceding N, written in base 2, is:

A 10001

B 10010

C 10011

D 10110

E 10111

Solution:

Subtracting 1 from 11000_2 requires borrowing through the three trailing zeros. Thus

$$11000_2 - 1_2 = 10111_2.$$

Therefore, the correct answer is **E**.

4. Let a binary operation $*$ on ordered pairs of integers be defined by $(a, b) * (c, d) = (a - c, b + d)$. Then, if $(3, 2) * (0, 0)$ and $(x, y) * (3, 2)$ represent identical pairs, x equals:

A -3

B 0

C 2

D 3

E 6

Solution:

The first pair is $(3 - 0, 2 + 0) = (3, 2)$. The second is $(x - 3, y + 2)$. Equality of the first coordinates gives $x - 3 = 3$, so $x = 6$.

Therefore, the correct answer is **E**.

5. If a number $N, N \neq 0,$ diminished by four times its reciprocal, equals a given real constant $R,$ then, for this given $R,$ the sum of all such possible values of N is:

A $\frac{1}{R}$

B R

C 4

D $\frac{1}{4}$

E $-R$

Solution:

Multiplying $N - \frac{4}{N} = R$ by N gives

$$N^2 - RN - 4 = 0.$$

The sum of its two roots is R by Vieta's formulas. Both roots are nonzero because their product is $-4.$

Therefore, the correct answer is **B**.

6. The area of the ring between two concentric circles is $12\frac{1}{2}\pi$ square inches. The length of a chord of the larger circle tangent to the smaller circle, in inches, is:

A $\frac{5}{\sqrt{2}}$

B 5

C $5\sqrt{2}$

D 10

E $10\sqrt{2}$

Solution:

Let the radii be R and r. The ring area gives

$$\pi(R^2 - r^2) = \frac{25}{2}\pi.$$

The radius to the tangent point bisects the chord. If its length is L, then

$$\left(\frac{L}{2}\right)^2 + r^2 = R^2.$$

Thus $L = 2\sqrt{R^2 - r^2}$, so $L = 2\sqrt{\frac{25}{2}} = 5\sqrt{2}$.

Therefore, the correct answer is **C**.

7. If the points $(1, y_1)$ and $(-1, y_2)$ lie on the graph of $y = ax^2 + bx + c$, and $y_1 - y_2 = -6$, then b equals:

A -3

B 0

C 3

D $\sqrt{ac}$

E $\frac{a+c}{2}$

Solution:

We have $y_1 = a + b + c$ and $y_2 = a - b + c$. Therefore $y_1 - y_2 = 2b = -6$, so $b = -3$.

Therefore, the correct answer is **A**.

8. Triangle ABC is inscribed in a circle. The measures of the non-overlapping minor arcs AB, BC, and CA are, respectively, $x+75°, 2x+25°, 3x-22°$. Then one interior angle of the triangle, in degrees, is:

A $57\frac{1}{2}$

B 59

C 60

D 61

E 122

Solution:

The arc sum gives

$$\begin{aligned} 6x + 78 &= 360, \\ x &= 47. \end{aligned}$$

The arcs then measure $122°, 119°, 119°$. The angle intercepting the $122°$ arc measures $61°$.

Therefore, the correct answer is **D**.

9. The arithmetic mean (ordinary average) of the fifty-two successive positive integers beginning with 2 is:

A 27

B $27\frac{1}{4}$

C $27\frac{1}{2}$

D 28

E $28\frac{1}{2}$

Solution:

The integers run from 2 through $2+51=53$. Their mean is the average of the first and last:

$$\frac{2+53}{2}=\frac{55}{2}=27\frac{1}{2}.$$

Therefore, the correct answer is **C**.

10. The number of points equidistant from a circle and two parallel tangents to the circle is:

A 0

B 2

C 3

D 4

E infinite

Solution:

Let the circle have center O and radius r. A point equidistant from the two parallel tangents must lie on their midline, which passes through O. If its distance from O is t, its distance from either tangent is r, while its distance from the circle is $|t-r|$. Thus $|t-r|=r$, giving $t=0$ or $t=2r$. There is one point with $t=0$ and two with $t=2r$, for a total of 3.

Therefore, the correct answer is **C**.

11. Given points $P(-1, -2)$ and $Q(4, 2)$ in the xy-plane, point $R(1, m)$ is taken so that $PR + RQ$ is a minimum. Then m equals:

A $-\frac{3}{5}$

B $-\frac{2}{5}$

C $-\frac{1}{5}$

D $\frac{1}{5}$

E either $-\frac{1}{5}$ or $\frac{1}{5}$

Solution:

By the triangle inequality, $PR + RQ \geq PQ$, with equality when R lies on segment PQ. The slope of PQ is $\frac{2-(-2)}{4-(-1)} = \frac{4}{5}$. Moving from $x = -1$ to $x = 1$ raises the y-coordinate by $(\frac{4}{5})(2) = \frac{8}{5}$, so

$$m = -2 + \frac{8}{5} = -\frac{2}{5}.$$

Therefore, the correct answer is **B**.

12. Let $F = \dfrac{6x^2 + 16x + 3m}{6}$ be the square of an expression which is linear in x. Then m has a particular value between:

- A 3 and 4
- B 4 and 5
- C 5 and 6
- D -4 and -3
- E -6 and -5

Solution:

We have

$$F = x^2 + \frac{8}{3}x + \frac{m}{2}.$$

For this to be a square, it must equal $(x + \frac{4}{3})^2$, whose constant term is $\frac{16}{9}$. Hence $\frac{m}{2} = \frac{16}{9}$, so $m = \frac{32}{9}$, which lies between 3 and 4.

Therefore, the correct answer is **A**.

13. A circle with radius r is contained within the region bounded by a circle with radius R. The area bounded by the larger circle is $\frac{a}{b}$ times the area of the region outside the smaller circle and inside the larger circle. Then $R : r$ equals:

- A $\sqrt{a} : \sqrt{b}$
- B $\sqrt{a} : \sqrt{a-b}$
- C $\sqrt{b} : \sqrt{a-b}$
- D $a : \sqrt{a-b}$
- E $b : \sqrt{a-b}$

Solution:

The area condition is

$$\pi R^2 = \frac{a}{b}\pi(R^2 - r^2).$$

Thus $bR^2 = aR^2 - ar^2$, so $ar^2 = (a-b)R^2$. Therefore

$$\frac{R}{r} = \sqrt{\frac{a}{a-b}},$$

and $R : r = \sqrt{a} : \sqrt{a-b}$.

Therefore, the correct answer is **B**.

14. The complete set of x-values satisfying the inequality $\dfrac{x^2-4}{x^2-1} > 0$ is the set of all x such that:

- A $x > 2$ or $x < -2$ or $-1 < x < 1$
- B $x > 2$ or $x < -2$
- C $x > 1$ or $x < -2$
- D $x > 1$ or $x < -1$
- E x is any real number except 1 or -1

Solution:

Factor the expression as

$$\frac{(x-2)(x+2)}{(x-1)(x+1)}.$$

A sign chart at $-2, -1, 1, 2$ shows it is positive on

$$(-\infty, -2) \cup (-1, 1) \cup (2, \infty).$$

The endpoints ± 2 give zero and ± 1 are undefined.

Therefore, the correct answer is **A**.

15. In a circle with center at O and radius r, chord AB is drawn with length equal to r units. From O a perpendicular to AB meets AB at M. From M a perpendicular to OA meets OA at D. In terms of r, the area of triangle MDA, in appropriate square units, is:

A $\frac{3r^2}{16}$

B $\frac{\pi r^2}{16}$

C $\frac{\pi r^2\sqrt{2}}{8}$

D $\frac{r^2\sqrt{3}}{32}$

E $\frac{r^2\sqrt{6}}{48}$

Solution:

Triangle OAB is equilateral. Since $OM \perp AB$, M is the midpoint of AB, so $AM = \frac{r}{2}$ and $\angle OAM = 60°$. In right triangle AMD, the hypotenuse is $AM = \frac{r}{2}$, giving

$$AD = \frac{r}{4}, \qquad MD = \frac{r\sqrt{3}}{4}.$$

Hence

$$\begin{aligned}[MDA] &= \frac{1}{2}(AD)(MD) \\ &= \frac{r^2\sqrt{3}}{32}.\end{aligned}$$

Therefore, the correct answer is **D**.

16. When $(a-b)^n, n \geq 2, ab \neq 0,$ is expanded by the binomial theorem, it is found that, when $a = kb,$ where k is a positive integer, the sum of the second and third terms is zero. Then n equals:

A $\frac{1}{2}k(k-1)$

B $\frac{1}{2}k(k+1)$

C $2k-1$

D $2k$

E $2k+1$

Solution:

The sum of the second and third terms is

$$-na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2.$$

Substituting $a = kb$ and dividing by the nonzero quantity $na^{n-2}b$ gives

$$-k + \frac{n-1}{2} = 0.$$

Therefore $n = 2k+1.$

Therefore, the correct answer is **E**.

17. The equation $2^{2x} - 8 \cdot 2^x + 12 = 0$ is satisfied by:

A $\log 3$

B $\frac{1}{2}\log 6$

C $1 + \log \frac{3}{4}$

D $1 + \frac{\log 3}{\log 2}$

E none of these

Solution:

Set $t = 2^x$. Then

$$\begin{aligned} t^2 - 8t + 12 &= 0, \\ (t-2)(t-6) &= 0. \end{aligned}$$

The root $t = 6$ gives

$$\begin{aligned} x &= \log_2 6 \\ &= 1 + \log_2 3 \\ &= 1 + \frac{\log 3}{\log 2}. \end{aligned}$$

(The equation is also satisfied by $x = 1$, which is not a listed choice.)

Therefore, the correct answer is **D**.

18. The number of points common to the graphs of

$$(x-y+2)(3x+y-4)=0$$

and

$$(x+y-2)(2x-5y+7)=0$$

is:

A 2

B 4

C 6

D 16

E infinite

Solution:

The first graph is the pair $x-y+2=0,\ 3x+y-4=0$, and the second is $x+y-2=0,\ 2x-5y+7=0$. Each line in the first pair meets each line in the second pair. Solving the four pairings gives four distinct points:

$$\begin{gathered}(0,2), \quad (-1,1), \\ (1,1), \quad \left(\frac{13}{17}, \frac{29}{17}\right).\end{gathered}$$

Thus there are 4 common points.

Therefore, the correct answer is **B**.

19. The number of distinct ordered pairs (x, y), where x and y have positive integral values satisfying the equation $x^4y^4 - 10x^2y^2 + 9 = 0$, is:

- A 0
- B 3
- C 4
- D 12
- E infinite

Solution:

Let $u = x^2y^2$. Then

$$u^2 - 10u + 9 = 0,$$
$$(u-1)(u-9) = 0.$$

Since x, y are positive integers, this means $xy = 1$ or $xy = 3$. The pairs are $(1,1), (1,3), (3,1)$, for a total of 3.

Therefore, the correct answer is **B**.

20. Let P equal the product of $3{,}659{,}893{,}456{,}789{,}325{,}678$ and $342{,}973{,}489{,}379{,}256$. The number of digits in P is:

A 36

B 35

C 34

D 33

E 32

Solution:

The factors satisfy

$$\begin{aligned} 3.6 \cdot 10^{18} < A < 3.7 \cdot 10^{18}, \\ 3.4 \cdot 10^{14} < B < 3.5 \cdot 10^{14}. \end{aligned}$$

Hence

$$12.24 \cdot 10^{32} < AB < 12.95 \cdot 10^{32},$$

so $10^{33} < P < 10^{34}$. Therefore P has 34 digits.

Therefore, the correct answer is **C**.

21. If the graph of $x^2 + y^2 = m$ is tangent to that of $x + y = \sqrt{2m}$, then:

A m must equal $\frac{1}{2}$

B m must equal $\frac{1}{\sqrt{2}}$

C m must equal $\sqrt{2}$

D m must equal 2

E m may be any nonnegative real number

Solution:

The circle has radius $\sqrt{m}$. The distance from the origin to the line is

$$\frac{|-\sqrt{2m}|}{\sqrt{1^2+1^2}} = \sqrt{m}.$$

Thus the line is tangent for every $m \geq 0$ (with the $m = 0$ case degenerate at the origin).

Therefore, the correct answer is **E**.

22. Let K be the measure of the area bounded by the x-axis, the line $x = 8$, and the curve defined by

$$f = \{(x, y) \mid \\ y = x \\ \text{when } 0 \leq x \leq 5, \\ y = 2x - 5 \\ \text{when } 5 \leq x \leq 8\}.$$

Then K is:

- A 21.5
- B 36.4
- C 36.5
- D 44
- E less than 44 but arbitrarily close to it

Solution:

From $x = 0$ to 5, the region is a triangle of area $\frac{1}{2}(5)(5) = 12.5$. From $x = 5$ to 8, the endpoint heights are 5 and 11, so the trapezoid has area

$$\frac{1}{2}(5 + 11)(3) = 24.$$

Thus $K = 12.5 + 24 = 36.5$.

Therefore, the correct answer is **C**.

23. For any integer n greater than 1, the number of prime numbers greater than $n! + 1$ and less than $n! + n$ is:

Here $n! = 1 \cdot 2 \cdots (n-1)n.$

A 0

B 1

C $\frac{n}{2}$ for n even, $\frac{n+1}{2}$ for n odd

D $n-1$

E n

Solution:

Every integer in the interval has the form $n! + k$ for some $2 \leq k \leq n-1.$ Because $k \mid n!$, we also have $k \mid (n! + k)$, and $n! + k > k.$ Thus every such integer is composite. (For $n = 2$, the interval is empty.) Hence there are no primes in the interval.

Therefore, the correct answer is **A**.

24. When the natural numbers P and P', with $P > P'$, are divided by the natural number D, the remainders are R and R', respectively. When PP' and RR' are divided by D, the remainders are r and r', respectively. Then:

A $r > r'$ always

B $r < r'$ always

C $r > r'$ sometimes, and $r < r'$ sometimes

D $r > r'$ sometimes, and $r = r'$ sometimes

E $r = r'$ always

Solution:

By definition of the remainders,

$$P \equiv R \pmod{D},$$
$$P' \equiv R' \pmod{D}.$$

Multiplying gives $PP' \equiv RR' \pmod{D}$. Since each has a unique remainder between 0 and $D-1$, their remainders must be equal: $r = r'$.

Therefore, the correct answer is **E**.

25. If it is known that $\log_2 a + \log_2 b \geq 6$, then the least value that can be taken on by $a + b$ is:

A $2\sqrt{6}$

B 6

C $8\sqrt{2}$

D 16

E none of these

Solution:

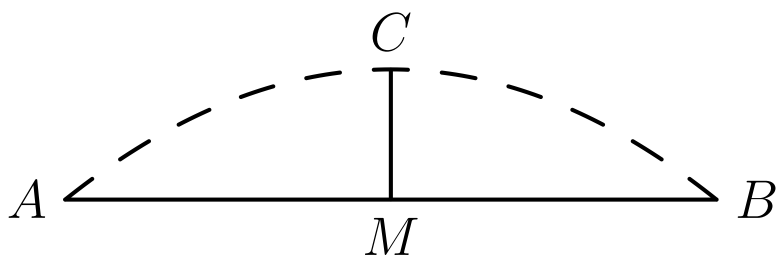
The logarithm condition gives $\log_2(ab) \geq 6$, so $ab \geq 64$. By AM-GM,

$$a + b \geq 2\sqrt{ab} \geq 2\sqrt{64} = 16.$$

Equality is attained at $a = b = 8$.

Therefore, the correct answer is **D**.

26. A parabolic arch has a height of 16 inches and a span of 40 inches. The height, in inches, of the arch at a point 5 inches from the center M, is:



A 1

B 15

C $15\frac{1}{3}$

D $15\frac{1}{2}$

E $15\frac{3}{4}$

Solution:

Place M at the origin with the span on the x-axis. The arch has equation $y = 16 - ax^2$. Since $(20, 0)$ lies on it, $0 = 16 - 400a$, so $a = \frac{1}{25}$. At $x = 5$,

$$y = 16 - \frac{25}{25} = 15.$$

Therefore, the correct answer is **B**.

27. A particle moves so that its speed for the second and subsequent miles varies inversely as the integral number of miles already traveled. For each subsequent mile the speed is constant. If the second mile is traversed in 2 hours, then the time, in hours, needed to traverse the nth mile is:

A $\frac{2}{n-1}$

B $\frac{n-1}{2}$

C $\frac{2}{n}$

D $2n$

E $2(n-1)$

Solution:

For the nth mile, $v_n = \frac{k}{n-1}$. Since the distance is one mile, the time is

$$T_n = \frac{1}{v_n} = \frac{n-1}{k}.$$

The condition $T_2 = 2$ gives $\frac{1}{k} = 2$. Therefore $T_n = 2(n-1)$.

Therefore, the correct answer is **E**.

28. Let n be the number of points P interior to the region bounded by a circle with radius 1, such that the sum of the squares of the distances from P to the endpoints of a given diameter is 3. Then n is:

- A 0
- B 1
- C 2
- D 4
- E infinite

Solution:

Let $P = (x, y)$ and take the diameter endpoints as $(-1, 0)$ and $(1, 0)$. The condition becomes

$$PA^2 + PB^2 = 3,$$
$$2x^2 + 2y^2 + 2 = 3.$$

Thus $x^2 + y^2 = \frac{1}{2}$. This is an entire circle of radius $\frac{1}{\sqrt{2}}$, lying inside the given unit circle. It contains infinitely many points.

Therefore, the correct answer is **E**.

29. If $x = t^{\frac{1}{t-1}}$ and $y = t^{\frac{t}{t-1}}, t > 0, t \neq 1,$ a relation between x and y is:

A $y^x = x^{\frac{1}{y}}$

B $y^{\frac{1}{x}} = x^y$

C $y^x = x^y$

D $x^x = y^y$

E none of these

Solution:

Dividing the definitions gives

$$\frac{y}{x} = t^{\frac{t-1}{t-1}} = t.$$

Also,

$$y = t^{\frac{t}{t-1}} = \left(t^{\frac{1}{t-1}}\right)^t = x^t.$$

Substituting $t = \frac{y}{x}$ gives $y = x^{\frac{y}{x}}$. Raising both sides to the power x yields $y^x = x^y$.

Therefore, the correct answer is **C**.

30. Let P be a point of hypotenuse AB (or its extension) of isosceles right triangle ABC. Let $s = AP^2 + PB^2$. Then:

A $s < 2CP^2$ for a finite number of positions of P

B $s < 2CP^2$ for an infinite number of positions of P

C $s = 2CP^2$ only if P is the midpoint of AB or an endpoint of AB

D $s = 2CP^2$ always

E $s > 2CP^2$ if P is a trisection point of AB

Solution:

Take $A = (-a, 0)$, $B = (a, 0)$ and $C = (0, a)$, $P = (p, 0)$. This describes every point on the hypotenuse line. Then

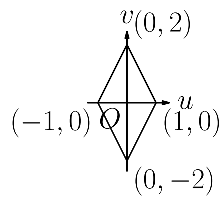
$$\begin{aligned} s &= (p+a)^2 + (p-a)^2 \\ &= 2p^2 + 2a^2, \\ 2CP^2 &= 2(p^2 + a^2) \\ &= 2p^2 + 2a^2. \end{aligned}$$

Thus $s = 2CP^2$ for every position of P.

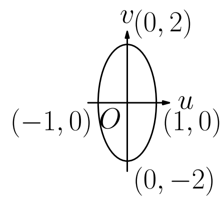
Therefore, the correct answer is **D**.

31. Let $OABC$ be a unit square in the xy-plane with $O(0,0), A(1,0), B(1,1),$ and $C(0,1)$. Let $u = x^2 - y^2$ and $v = 2xy$ define a transformation of the xy-plane into the uv-plane. The transform (or image) of the square is:

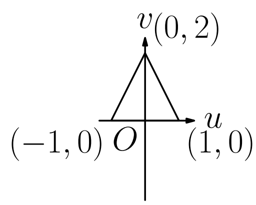
A



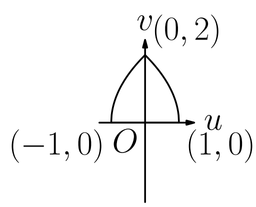
B



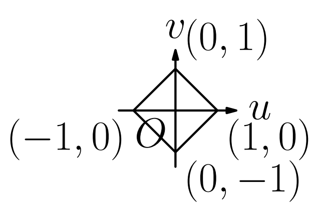
C



D



E



Solution:

The vertices map as

$$
\begin{aligned}
O &\mapsto (0,0),\\
A &\mapsto (1,0),\\
B &\mapsto (0,2),\\
C &\mapsto (-1,0).
\end{aligned}
$$

Side OA maps to the segment from $(0,0)$ to $(1,0)$, and CO maps to the segment from $(-1,0)$ to $(0,0)$. On $AB,\ x=1$, so $u=1-\frac{v^2}{4}$; on $BC,\ y=1$, so $u=\frac{v^2}{4}-1$. These are the two upper parabolic arcs joining $(\pm1,0)$ to $(0,2)$.

Therefore, the correct answer is **D**.

32. Let a sequence $\{u_n\}$ be defined by $u_1=5$ and the relation $u_{n+1}-u_n=3+4(n-1), n=1,2,3,\ldots$. If u_n is expressed as a polynomial in n, the algebraic sum of its coefficients is:

- A 3
- B 4
- C 5
- D 6
- E 11

Solution:

For any polynomial $p(n)$, the sum of its coefficients is $p(1)$. Here the polynomial represents u_n, and the initial condition gives $u_1=5$. Therefore the coefficient sum is 5.

Therefore, the correct answer is **C**.

33. Let S_n and T_n be the respective sums of the first n terms of two arithmetic series. If $S_n : T_n = (7n+1) : (4n+27)$ for all n, the ratio of the eleventh term of the first series to the eleventh term of the second series is:

- A $4:3$
- B $3:2$
- C $7:4$
- D $78:71$
- E undetermined

Solution:

For an arithmetic sequence, the eleventh term is the average of the first 21 terms. Thus the respective eleventh terms are $\frac{S_{21}}{21}$ and $\frac{T_{21}}{21}$. Their ratio is

$$\frac{S_{21}}{T_{21}} = \frac{7(21)+1}{4(21)+27} = \frac{148}{111} = \frac{4}{3}.$$

Therefore, the correct answer is **A**.

34. The remainder R obtained by dividing x^{100} by $x^2 - 3x + 2$ is a polynomial of degree less than $2.$ Then R may be written as:

A $2^{100} - 1$

B $2^{100}(x-1) - (x-2)$

C $2^{100}(x-3)$

D $x(2^{100}-1) + 2(2^{99}-1)$

E $2^{100}(x+1) - (x+2)$

Solution:

Write $R(x) = ax + b$. Since the divisor is $(x-1)(x-2)$, evaluating the division identity at its roots gives

$$R(1) = 1, \qquad R(2) = 2^{100}.$$

Therefore

$$\begin{aligned} R(x) &= 1 + (2^{100}-1)(x-1) \\ &= 2^{100}(x-1) - (x-2). \end{aligned}$$

Therefore, the correct answer is **B**.

35. Let $L(m)$ be the x-coordinate of the left endpoint of the intersection of the graphs of $y = x^2 - 6$ and $y = m$, where $-6 < m < 6$. Let $r = \frac{[L(-m) - L(m)]}{m}$. Then, as m is made arbitrarily close to zero, the value of r is:

A arbitrarily close to zero

B arbitrarily close to $\frac{1}{\sqrt{6}}$

C arbitrarily close to $\frac{2}{\sqrt{6}}$

D arbitrarily large

E undetermined

Solution:

The left intersection satisfies

$$L(m) = -\sqrt{6+m}.$$

Hence

$$\begin{aligned} r &= \frac{-\sqrt{6-m} + \sqrt{6+m}}{m} \\ &= \frac{2}{\sqrt{6+m} + \sqrt{6-m}}. \end{aligned}$$

As m approaches 0, this approaches $\frac{2}{2\sqrt{6}} = \frac{1}{\sqrt{6}}$.

Therefore, the correct answer is **B**.

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