1968 AMC 12 Solutions

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1. Let P units be the increase in the circumference of a circle resulting from an increase of π units in the diameter. Then P equals:

A $\frac{1}{\pi}$

B π

C $\frac{\pi^2}{2}$

D π^2

E 2π

Solution:

The original circumference is $\pi d.$ After the diameter increases by $\pi,$ it is $\pi(d+\pi).$ The increase is therefore $\pi(d+\pi) - \pi d = \pi^2.$

Therefore, the correct answer is **D**.

2. The real value of x such that 64^{x-1} divided by 4^{x-1} equals 256^{2x} is:

A $-\frac{2}{3}$

B $-\frac{1}{3}$

C 0

D $\frac{1}{4}$

E $\frac{3}{8}$

Solution:

The left side is $(\frac{64}{4})^{x-1} = 16^{x-1}$, while $256^{2x} = (16^2)^{2x} = 16^{4x}$. Hence $x - 1 = 4x$, so $x = -\frac{1}{3}$.

Therefore, the correct answer is **B**.

3. A straight line passing through the point $(0, 4)$ is perpendicular to the line $x - 3y - 7 = 0$. Its equation is:

- A $y + 3x - 4 = 0$
- B $y + 3x + 4 = 0$
- C $y - 3x - 4 = 0$
- D $3y + x - 12 = 0$
- E $3y - x - 12 = 0$

Solution:

The given line has slope $\frac{1}{3}$, so a perpendicular line has slope -3. Through $(0, 4)$ its equation is $y = -3x + 4$, or $y + 3x - 4 = 0$.

Therefore, the correct answer is **A**.

4. Define an operation $*$ for positive real numbers by $a * b = \dfrac{ab}{a+b}$. Then $4 * (4 * 4)$ equals:

A $\dfrac{2}{3}$

B 1

C $\dfrac{4}{3}$

D 2

E $\dfrac{16}{3}$

Solution:

First, $4 * 4 = \frac{16}{8} = 2$. Therefore $4 * (4 * 4) = 4 * 2$, which equals $\frac{8}{4+2} = \frac{4}{3}$.

Therefore, the correct answer is **C**.

5. If $f(n)=\dfrac{1}{3}n(n+1)(n+2)$, then $f(r)-f(r-1)$ equals:

A $r(r+1)$

B $(r+1)(r+2)$

C $\dfrac{1}{3}r(r+1)$

D $\dfrac{1}{3}(r+1)(r+2)$

E $\dfrac{1}{3}r(r+1)(2r+1)$

Solution:

We have $f(r)=\frac{1}{3}r(r+1)(r+2)$ and $f(r-1)=\frac{1}{3}(r-1)r(r+1)$. Factor their difference as $\frac{1}{3}r(r+1)$ times $(r+2)-(r-1)=3$. The result is $r(r+1)$.

Therefore, the correct answer is **A**.

6. Let side AD of convex quadrilateral $ABCD$ be extended through D, and let side BC be extended through C, to meet in point E. Let S represent the degree-sum of angles CDE and DCE, and let S' represent the degree-sum of angles BAD and ABC. If $r = \frac{S}{S'}$, then:

A $r = 1$ sometimes, $r > 1$ sometimes

B $r = 1$ sometimes, $r < 1$ sometimes

C $0 < r < 1$

D $r > 1$

E $r = 1$

Solution:

In triangle ECD, $S = 180^\circ - \angle E$. Since D lies on AE and C lies on BE, triangle EAB gives $S' = 180^\circ - \angle E$ as well. Thus $S = S'$ and $r = 1$.

Therefore, the correct answer is **E**.

7. Let O be the intersection point of medians AP and CQ of triangle ABC. If OQ is 3 inches, then OP, in inches, is:

- A 3
- B $\frac{9}{2}$
- C 6
- D 9
- E undetermined

Solution:

The centroid divides each median in a $2:1$ ratio, so $OQ = 3$ determines $CQ = 9$. It gives no relation between the lengths of the two different medians CQ and AP. Triangles with $CQ = 9$ can have different AP, so OP is undetermined.

Therefore, the correct answer is **E**.

8. A positive number is mistakenly divided by 6 instead of being multiplied by 6. Based on the correct answer, the error thus committed, to the nearest percent, is:

- A 100
- B 97
- C 83
- D 17
- E 3

Solution:

The correct result is $6N$, while the mistaken result is $\frac{N}{6}$. Relative to the correct result, the error is $\frac{6N-\frac{N}{6}}{6N} = \frac{35}{36} \approx 97.2\%$.

Therefore, the correct answer is **B**.

9. The sum of the real values of x satisfying $|x+2| = 2|x-2|$ is:

A $\frac{1}{3}$

B $\frac{2}{3}$

C 6

D $6\frac{1}{3}$

E $6\frac{2}{3}$

Solution:

Squaring gives $(x+2)^2 = 4(x-2)^2$, or $3x^2 - 20x + 12 = 0$. Both roots satisfy the original absolute-value equation, and their sum is $\frac{20}{3} = 6\frac{2}{3}$.

Therefore, the correct answer is **E**.

10. Assume that, for a certain school, it is true that

I: Some students are not honest.

II: All fraternity members are honest.

A necessary conclusion is:

A Some students are fraternity members

B Some fraternity members are not students

C Some students are not fraternity members

D No fraternity member is a student

E No student is a fraternity member

Solution:

Statement I guarantees a dishonest student. Statement II says every fraternity member is honest, so that dishonest student cannot be a fraternity member. Hence some student is not a fraternity member.

Therefore, the correct answer is **C**.

11. If an arc of 60° on circle I has the same length as an arc of 45° on circle II, the ratio of the area of circle I to that of circle II is:

- A $16 : 9$
- B $9 : 16$
- C $4 : 3$
- D $3 : 4$
- E none of these

Solution:

Equal arc lengths give $60r_1 = 45r_2$, so $\frac{r_1}{r_2} = \frac{3}{4}$. Circle areas scale as the squares of the radii, giving $A_1 : A_2 = 9 : 16.$

Therefore, the correct answer is **B**.

12. A circle passes through the vertices of a triangle with side-lengths $7\frac{1}{2}, 10, 12\frac{1}{2}$. The radius of the circle is:

A $\frac{15}{4}$

B 5

C $\frac{25}{4}$

D $\frac{35}{4}$

E $\frac{15\sqrt{2}}{2}$

Solution:

The sides are $\frac{5}{2}(3,4,5)$, so the triangle is right with hypotenuse $\frac{25}{2}$. The hypotenuse is the circumdiameter, making the radius $\frac{25}{4}$.

Therefore, the correct answer is **C**.

13. If m and n are the roots of $x^2 + mx + n = 0, m \neq 0, n \neq 0,$ then the sum of the roots is:

A $-\frac{1}{2}$

B -1

C $\frac{1}{2}$

D 1

E undetermined

Solution:

Because the roots are $m, n,$ Vieta gives $m + n = -m$ and $mn = n.$ Since $n \neq 0,$ the second equation gives $m = 1,$ and then $n = -2.$ Their sum is $-1.$

Therefore, the correct answer is **B**.

14. If x and y are nonzero numbers such that $x = 1 + \frac{1}{y}$ and $y = 1 + \frac{1}{x}$, then y equals:

A $x - 1$

B $1 - x$

C $1 + x$

D $-x$

E x

Solution:

The equations give $xy = y + 1$ and $xy = x + 1$. Therefore $x + 1 = y + 1$, so $y = x$.

Therefore, the correct answer is **E**.

15. Let P be the product of any three consecutive positive odd integers. The largest integer dividing all such P is:

A 15

B 6

C 5

D 3

E 1

Solution:

Three consecutive odd integers occupy three consecutive residues modulo $3,$ so one is divisible by $3.$ Thus every product is divisible by $3.$ The products $1 \cdot 3 \cdot 5 = 15$ and $7 \cdot 9 \cdot 11 = 693$ have greatest common divisor $3,$ so no larger integer always divides the product.

Therefore, the correct answer is **D**.

16. If x is such that $\dfrac{1}{x} < 2$ and $\dfrac{1}{x} > -3$, then:

A $-\dfrac{1}{3} < x < \dfrac{1}{2}$

B $-\dfrac{1}{3} < x < 3$

C $x > \dfrac{1}{2}$

D $x > \dfrac{1}{2}$ or $-\dfrac{1}{3} < x < 0$

E $x > \dfrac{1}{2}$ or $x < -\dfrac{1}{3}$

Solution:

For $x > 0, \frac{1}{x} < 2$ gives $x > \frac{1}{2}$, while the other inequality is automatic. For $x < 0, \frac{1}{x} > -3$ gives $x < -\frac{1}{3}$, while the first is automatic. Thus $x > \frac{1}{2}$ or $x < -\frac{1}{3}$.

Therefore, the correct answer is **E**.

17. Let $f(n) = \dfrac{x_1 + x_2 + \cdots + x_n}{n}$, where n is a positive integer. If $x_k = (-1)^k, k = 1, 2, \ldots, n$, the set of possible values of $f(n)$ is:

A $\{0\}$

B $\left\{\dfrac{1}{n}\right\}$

C $\left\{0, -\dfrac{1}{n}\right\}$

D $\left\{0, \dfrac{1}{n}\right\}$

E $\left\{1, \dfrac{1}{n}\right\}$

Solution:

If n is even, the terms pair to give sum 0, so $f(n) = 0$. If n is odd, one final -1 remains, so $f(n) = -\frac{1}{n}$. The possible values are therefore $\{0, -\frac{1}{n\}}$.

Therefore, the correct answer is **C**.

18. Side AB of triangle ABC has length 8 inches. Line DEF is drawn parallel to AB so that D is on segment AC, and E is on segment BC. Line AE extended bisects angle FEC. If DE has length 5 inches, then the length of CE, in inches, is:

- A $\frac{51}{4}$
- B 13
- C $\frac{53}{4}$
- D $\frac{40}{3}$
- E $\frac{27}{2}$

Solution:

Because $FE \parallel AB, \angle FEA = \angle BAE$. The bisector condition gives $\angle FEA = \angle AEC = \angle AEB$, so triangle ABE is isosceles and $BE = AB = 8$. Similar triangles CDE and CAB give $\frac{CE}{CE+8} = \frac{5}{8}$. Hence $8CE = 5CE + 40$, so $CE = \frac{40}{3}$.

Therefore, the correct answer is **D**.

19. Let n be the number of ways that 10 dollars can be changed into dimes and quarters, with at least one of each coin being used. Then n equals:

- A 40
- B 38
- C 21
- D 20
- E 19

Solution:

The equation $10d + 25q = 1000$ reduces to $2d + 5q = 200$. Thus q must be even; write $q = 2k$, giving $d = 100 - 5k$. Positivity requires $k = 1, 2, \ldots, 19$, so there are 19 ways.

Therefore, the correct answer is **E**.

20. The measures of the interior angles of a convex polygon of n sides are in arithmetic progression. If the common difference is 5° and the largest angle is 160°, then n equals:

A 9

B 10

C 12

D 16

E 32

Solution:

The smallest angle is $160 - 5(n-1)$. The angle sum is $\frac{n}{2}$ times $320 - 5(n-1)$, and it equals $180(n-2)$. This simplifies to

$$n^2 + 7n - 144 = 0,$$
$$(n-9)(n+16) = 0.$$

Thus $n = 9$.

Therefore, the correct answer is **A**.

21. If all the operations in $S = 1! + 2! + 3! + \cdots + 99!$ are correctly performed, the units digit in the value of S is:

A 9

B 8

C 5

D 3

E 0

Solution:

For $k \geq 5, k!$ is divisible by $10.$ Only the first four terms matter. Their sum is $1 + 2 + 6 + 24 = 33,$ whose units digit is $3.$

Therefore, the correct answer is **D**.

22. A segment of length 1 is divided into four segments. Then there exists a simple quadrilateral with the four segments as sides if and only if each segment is:

A equal to $\frac{1}{4}$

B equal to or greater than $\frac{1}{8}$ and less than $\frac{1}{2}$

C greater than $\frac{1}{8}$ and less than $\frac{1}{2}$

D greater than $\frac{1}{8}$ and less than $\frac{1}{4}$

E less than $\frac{1}{2}$

Solution:

A simple nondegenerate quadrilateral exists exactly when each side is less than the sum of the other three. Since the total is 1, this condition is $s < 1 - s$, or $s < \frac{1}{2}$, for every segment.

Therefore, the correct answer is **E**.

23. If all the logarithms are real numbers, the equality

$$\begin{aligned} \log(x+3) + \log(x-1) \\ = \log(x^2 - 2x - 3) \end{aligned}$$

is satisfied for:

- A all real values of x
- B no real values of x
- C all real values of x except $x = 0$
- D no real values of x except $x = 0$
- E all real values of x except $x = 1$

Solution:

Combining logarithms would require $(x+3)(x-1) = x^2 - 2x - 3$. This gives $x = 0$. But then $x - 1 = -1$, so the logarithm on the left is not real. Hence there are no real solutions.

Therefore, the correct answer is **B**.

24. A painting $18'' \times 24''$ is to be placed into a wooden frame with the longer dimension vertical. The wood at the top and bottom is twice as wide as the wood on the sides. If the frame area equals that of the painting itself, the ratio of the smaller to the larger dimension of the framed painting is:

A $1:3$

B $1:2$

C $2:3$

D $3:4$

E $1:1$

Solution:

The outside dimensions are $18 + 2x$ and $24 + 4x$. Since the frame area equals the painting area, $(18 + 2x)(24 + 4x) = 2(18)(24)$. This gives $x^2 + 15x - 54 = 0$, so $x = 3$. The dimensions are 24 and 36, with ratio $2 : 3$.

Therefore, the correct answer is **C**.

25. Ace runs with constant speed and Flash runs x times as fast, $x > 1$. Flash gives Ace a head start of y yards, and, at a given signal, they start off in the same direction. Then the number of yards Flash must run to catch Ace is:

A xy

B $\dfrac{y}{x+y}$

C $\dfrac{xy}{x-1}$

D $\dfrac{x+y}{x+1}$

E $\dfrac{x+y}{x-1}$

Solution:

If Ace runs at speed v, Flash runs at xv, so their closing speed is $(x-1)v$. The catch-up time is $\frac{y}{(x-1)v}$, during which Flash runs $\frac{xv \cdot y}{(x-1)v} = \frac{xy}{x-1}$.

Therefore, the correct answer is **C**.

26. Let $S = 2 + 4 + 6 + \cdots + 2N$, where N is the smallest positive integer such that $S > 1{,}000{,}000$. Then the sum of the digits of N is:

A 27

B 12

C 6

D 2

E 1

Solution:

We have $S = N(N+1)$. For $N = 999$, this is $999{,}000 < 1{,}000{,}000$, while for $N = 1000$ it is $1{,}001{,}000$. Thus $N = 1000$, whose digit sum is 1.

Therefore, the correct answer is **E**.

27. Let

$$S_n = 1 - 2 + 3 - 4 + \cdots$$
$$+ (-1)^{n-1} n.$$

Here $n = 1, 2, \ldots$. Then $S_{17} + S_{33} + S_{50}$ equals:

A 0

B 1

C 2

D -1

E -2

Solution:

Pairing gives $S_n = -\frac{n}{2}$ for even n and $S_n = \frac{n+1}{2}$ for odd n. Hence $S_{17} = 9,\ S_{33} = 17,\ S_{50} = -25$, and their sum is 1.

Therefore, the correct answer is **B**.

28. If the arithmetic mean of a and b is double their geometric mean, with $a > b > 0$, then a possible value for the ratio $\frac{a}{b}$, to the nearest integer, is:

- A 5
- B 8
- C 11
- D 14
- E none of these

Solution:

Let $t = \frac{a}{b} > 1$. The condition gives $\frac{t+1}{2} = 2\sqrt{t}$, so $t^2 - 14t + 1 = 0$. The root greater than 1 is $t = 7 + 4\sqrt{3} \approx 13.93$, whose nearest integer is 14.

Therefore, the correct answer is **D**.

29. Given the three numbers $x, y = x^x, z = x^{(x^x)}$, with $0.9 < x < 1.0$. Arranged in order of increasing magnitude, they are:

A x, z, y

B x, y, z

C y, x, z

D y, z, x

E z, x, y

Solution:

Since $0 < x < 1$ and $x < 1$, raising x to the exponent x gives $x < y = x^x < 1$. For a base between 0 and 1, a larger exponent gives a smaller value. Because $x < y < 1$, we have $x = x^1 < x^y = z < x^x = y$. Thus $x < z < y$.

Therefore, the correct answer is **A**.

30. Convex polygons P_1 and P_2 are drawn in the same plane with n_1 and n_2 sides, respectively, $n_1 \leq n_2$. If P_1 and P_2 do not have any line segment in common, then the maximum number of intersections of P_1 and P_2 is:

A $2n_1$

B $2n_2$

C $n_1 n_2$

D $n_1 + n_2$

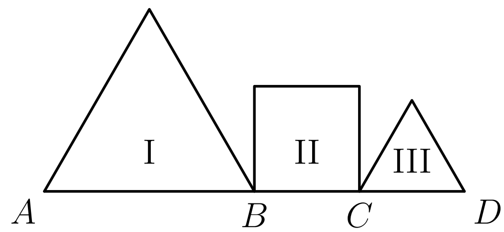
E none of these

Solution:

Each side of P_1 lies on a line, and its intersection with the convex region P_2 is a single segment or empty. Thus that side crosses the boundary of P_2 at most twice. Across n_1 sides there are at most $2n_1$, and a suitable thin convex n_1-gon crossing a convex n_2-gon attains this bound.

Therefore, the correct answer is **A**.

31. In this diagram, not drawn to scale, figures I and III are equilateral triangular regions with respective areas of $32\sqrt{3}$ and $8\sqrt{3}$ square inches. Figure II is a square region with area 32 square inches. Let the length of segment AD be decreased by $12\frac{1}{2}\%$ of itself, while the lengths of AB and CD remain unchanged. The percent decrease in the area of the square is:



A $12\frac{1}{2}$

B 25

C 50

D 75

E $87\frac{1}{2}$

Solution:

From $(\frac{\sqrt{3}}{4})s^2 = 32\sqrt{3}, AB = 8\sqrt{2}.$ The square has $BC = 4\sqrt{2},$ and the smaller equilateral triangle also has $CD = 4\sqrt{2}.$ Thus $AD = 16\sqrt{2}.$ Decreasing it by $\frac{1}{8}$ removes $2\sqrt{2}$ from BC, whose new length is $2\sqrt{2}.$ The square's area falls from 32 to $8,$ a 75% decrease.

Therefore, the correct answer is **D**.

32. A and B move uniformly along two straight paths intersecting at right angles in point O. When A is at O, B is 500 yards short of O. In 2 minutes they are equidistant from O, and in 8 minutes more they are again equidistant from O. Then the ratio of A's speed to B's speed is:

A $4 : 5$

B $5 : 6$

C $2 : 3$

D $5 : 8$

E $1 : 2$

Solution:

Let the speeds of A, B be u, v. At $t = 2, 2u = 500 - 2v$, so $u + v = 250$. At $t = 10$, B has passed O, and $10u = 10v - 500$, so $v - u = 50$. Hence $u = 100,\ v = 150$, and $u : v = 2 : 3$.

Therefore, the correct answer is **C**.

33. A number N has three digits when expressed in base 7. When N is expressed in base 9 the digits are reversed. Then the middle digit is:

A 0

B 1

C 3

D 4

E 5

Solution:

Let $N = (abc)_7 = (cba)_9$. Then $49a + 7b + c = 81c + 9b + a$, so $24a - b = 40c$. With $1 \leq a \leq 6,\ 0 \leq b \leq 6$, and $1 \leq c \leq 6$, the only possibility is $a = 5, c = 3, b = 0$. Indeed, $(503)_7 = (305)_9 = 248$.

Therefore, the correct answer is **A**.

34. With 400 members voting, the House of Representatives defeated a bill. A re-vote, with the same members voting, resulted in passage of the bill by twice the margin by which it was originally defeated. The number voting for the bill on the re-vote was $\frac{12}{11}$ of the number voting against it originally. How many more members voted for the bill the second time than voted for it the first time?

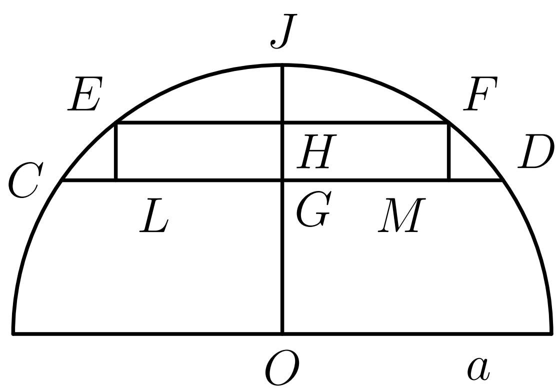
- A 75
- B 60
- C 50
- D 45
- E 20

Solution:

Let F, G be the original and second numbers voting for the bill. The ratio condition gives $G = \frac{12}{11}(400 - F)$. The passage margin is twice the original defeat margin, so $2G - 400 = 2(400 - 2F)$, or $G + 2F = 600$. Solving gives $F = 180,\ G = 240$, so 60 more members voted for it.

Therefore, the correct answer is **B**.

35. In this diagram the center of the circle is O, the radius is a inches, chord EF is parallel to chord CD, O, G, H, J are collinear, and G is the midpoint of CD. Let K (square inches) represent the area of trapezoid $CDFE$ and let R (square inches) represent the area of rectangle $ELMF$. Then, as CD and EF are translated upward so that OG increases toward the value a, while JH always equals HG, the ratio $K : R$ becomes arbitrarily close to:



A 0

B 1

C $\sqrt{2}$

D $\frac{1}{\sqrt{2}} + \frac{1}{2}$

E $\frac{1}{\sqrt{2}} + 1$

Solution:

Let $JH = HG = x$. Then $OG = a - 2x$ and $OH = a - x$. The half-chords are

$$GD = 2\sqrt{x(a-x)},$$
$$HF = \sqrt{x(2a-x)}.$$

Since both figures have height x,

$$\begin{aligned}\frac{K}{R} &= \frac{x(GD+HF)}{2x(HF)} \\ &= \frac{1}{2} + \frac{GD}{2HF}.\end{aligned}$$

As x approaches 0, the latter fraction approaches

$$\frac{2\sqrt{ax}}{2\sqrt{2ax}} = \frac{1}{\sqrt{2}}.$$

Thus $K : R$ approaches $\frac{1}{\sqrt{2}} + \frac{1}{2}$.

Therefore, the correct answer is **D**.

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