1967 AMC 12 Solutions

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1. The three-digit number $2a3$ is added to the number 326 to give the three-digit number $5b9$. If $5b9$ is divisible by 9, then $a+b$ equals:

A 2

B 4

C 6

D 8

E 9

Solution:

Divisibility by 9 requires $5+b+9$ to be a multiple of 9. Since b is a digit, this gives $b=4$. The addition is then $2a3+326=549$, so $a=2$. Hence $a+b=6$.

Therefore, the correct answer is **C**.

2. An equivalent of the expression

$$\left(\frac{x^2+1}{x}\right)\left(\frac{y^2+1}{y}\right) + \left(\frac{x^2-1}{y}\right)\left(\frac{y^2-1}{x}\right),$$
$$xy \neq 0,$$

is:

A 1

B $2xy$

C $2x^2y^2+2$

D $2xy+\frac{2}{xy}$

E $\frac{2x}{y}+\frac{2y}{x}$

Solution:

The first product is

$$xy+\frac{x}{y}+\frac{y}{x}+\frac{1}{xy},$$

while the second is

$$xy-\frac{x}{y}-\frac{y}{x}+\frac{1}{xy}.$$

Adding gives $2xy+\frac{2}{xy}$.

Therefore, the correct answer is **D**.

3. The side of an equilateral triangle is s. A circle is inscribed in the triangle and a square is inscribed in the circle. The area of the square is:

A $\dfrac{s^2}{24}$

B $\dfrac{s^2}{6}$

C $\dfrac{s^2\sqrt{2}}{6}$

D $\dfrac{s^2\sqrt{3}}{6}$

E $\dfrac{s^2}{3}$

Solution:

The inradius is $r = \frac{s\sqrt{3}}{6}$. The inscribed square has diagonal $2r$, so its area is half the square of that diagonal:

$$\begin{aligned}\frac{(2r)^2}{2} &= 2r^2 \\ &= 2\left(\frac{s\sqrt{3}}{6}\right)^2 \\ &= \frac{s^2}{6}.\end{aligned}$$

Therefore, the correct answer is **B**.

4. Given $\dfrac{\log a}{p} = \dfrac{\log b}{q} = \dfrac{\log c}{r} = \log x$, all logarithms to the same base and $x \neq 1$. If $\dfrac{b^2}{ac} = x^y$, then y is:

A $\dfrac{q^2}{p+r}$

B $\dfrac{p+r}{2q}$

C $2q - p - r$

D $2q - pr$

E $q^2 - pr$

Solution:

The given equalities imply $a = x^p,\ b = x^q$, and $c = x^r$. Therefore

$$\frac{b^2}{ac} = x^{2q-p-r}.$$

Since this is x^y and $x \neq 1$, we have $y = 2q - p - r$.

Therefore, the correct answer is **C**.

5. A triangle is circumscribed about a circle of radius r inches. If the perimeter of the triangle is P inches and the area is K square inches, then $\frac{P}{K}$ is:

A independent of the value of r

B $\dfrac{\sqrt{2}}{r}$

C $\dfrac{2}{\sqrt{r}}$

D $\dfrac{2}{r}$

E $\dfrac{r}{2}$

Solution:

The three radii to the points of tangency split the triangle into smaller triangles whose total area is

$$K = \frac{1}{2}r(a+b+c) = \frac{Pr}{2}.$$

Hence $\frac{P}{K} = \frac{2}{r}$.

Therefore, the correct answer is **D**.

6. If $f(x) = 4^x$, then $f(x+1) - f(x)$ equals:

A 4

B $f(x)$

C $2f(x)$

D $3f(x)$

E $4f(x)$

Solution:

We have $f(x+1) = 4^{x+1} = 4f(x)$. Therefore $f(x+1) - f(x) = 3f(x)$.

Thus, the correct answer is **D**.

7. If $\frac{a}{b} < -\frac{c}{d}$, where a, b, c, d are real numbers and $bd \neq 0$, then:

A a must be negative

B a must be positive

C a must not be zero

D a can be negative or zero, but not positive

E a can be positive, negative, or zero

Solution:

Take $b = d = 1$ and $c = -2$. The inequality becomes $a < 2$, which is satisfied by positive, zero, and negative values of a. Thus none of the three possible signs is forced.

Therefore, the correct answer is **E**.

8. To m ounces of an $m\%$ solution of acid, x ounces of water are added to yield an $(m-10)\%$ solution. If $m > 25$, then x is:

A $\frac{10m}{m-10}$

B $\frac{5m}{m-10}$

C $\frac{m}{m-10}$

D $\frac{5m}{m-20}$

E not determined by the given information

Solution:

Conservation of acid gives

$$\frac{m^2}{100} = \frac{m-10}{100}(m+x).$$

Hence $m^2 = (m-10)(m+x)$, so $10m = (m-10)x$ and

$$x = \frac{10m}{m-10}.$$

Therefore, the correct answer is **A**.

9. Let K, in square units, be the area of a trapezoid such that the shorter base, the altitude, and the longer base, in that order, are in arithmetic progression. Then:

A K must be an integer

B K must be a rational fraction

C K must be an irrational number

D K must be an integer or a rational fraction

E taken alone neither (A) nor (B) nor (C) nor (D) is true

Solution:

Let the shorter base, altitude, and longer base be $a-d,\ a,$ and $a+d.$ Then

$$K = \frac{1}{2}a\big((a-d)+(a+d)\big)$$
$$= a^2.$$

Depending on $a,$ this can be an integer, a nonintegral rational number, or an irrational number. No one of choices (A) through (D) must hold.

Therefore, the correct answer is **E**.

10. If

$$\frac{a}{10^x - 1} + \frac{b}{10^x + 2} = \frac{2 \cdot 10^x + 3}{(10^x - 1)(10^x + 2)}$$

is an identity for positive rational values of x, then the value of $a - b$ is:

A $\frac{4}{3}$

B $\frac{5}{3}$

C 2

D $\frac{11}{4}$

E 3

Solution:

Clearing denominators gives

$$a(10^x + 2) + b(10^x - 1) = 2 \cdot 10^x + 3.$$

Therefore $a + b = 2$ and $2a - b = 3$. Solving gives $a = \frac{5}{3},\ b = \frac{1}{3}$, so $a - b = \frac{4}{3}$.

Therefore, the correct answer is **A**.

11. If the perimeter of rectangle $ABCD$ is 20 inches, the least value of diagonal AC, in inches, is:

- A 0
- B $\sqrt{50}$
- C 10
- D $\sqrt{200}$
- E none of these

Solution:

Let adjacent sides be x and $10 - x$. Then

$$\begin{aligned} AC^2 &= x^2 + (10 - x)^2 \\ &= 2(x-5)^2 + 50. \end{aligned}$$

This is minimized at $x = 5$, giving $AC = \sqrt{50}$.

Therefore, the correct answer is **B**.

12. If the (convex) area bounded by the x-axis and the lines $y = mx + 4, x = 1,$ and $x = 4$ is $7,$ then m equals:

A $-\frac{1}{2}$

B $-\frac{2}{3}$

C $-\frac{3}{2}$

D -2

E none of these

Solution:

The bounded region is a trapezoid with parallel sides $m + 4$ and $4m + 4,$ separated by $3.$ Thus

$$\frac{3}{2}\big((m+4) + (4m+4)\big) = 7.$$

Hence $15m + 24 = 14,$ so $m = -\frac{2}{3}.$ The endpoint heights are positive, as required for the stated convex region.

Therefore, the correct answer is **B**.

13. A triangle ABC is to be constructed given side a (opposite angle A), angle B, and h_c, the altitude from C. If N is the number of noncongruent solutions, then N

A. is 1

B. is 2

C. must be zero

D. must be infinite

E. must be zero or infinite

Solution:

Fix $BC = a$ and draw the ray from B that makes the prescribed angle B. Its distance from C is fixed. If that distance is not h_c, there is no solution. If it is h_c, then A may be chosen at infinitely many positions on the ray, producing infinitely many noncongruent triangles. Thus N must be zero or infinite.

Therefore, the correct answer is **E**.

14. Let $f(t) = \dfrac{t}{1-t}, t \neq 1.$ If $y = f(x),$ then x can be expressed as:

A $f\left(\dfrac{1}{y}\right)$

B $-f(y)$

C $-f(-y)$

D $f(-y)$

E $f(y)$

Solution:

From $y = \frac{x}{1-x}$ we obtain $y - yx = x,$ so

$$x = \frac{y}{1+y}.$$

Since $f(-y) = -\frac{y}{1+y},$ this is $-f(-y).$

Therefore, the correct answer is **C**.

15. The difference in the areas of two similar triangles is 18 square feet, and the ratio of the larger area to the smaller is the square of an integer. The area of the smaller triangle, in square feet, is an integer, and one of its sides is 3 feet. The corresponding side of the larger triangle, in feet, is:

A 12

B 9

C $6\sqrt{2}$

D 6

E $3\sqrt{2}$

Solution:

Let the side-scale factor be the integer $k > 1$ and the smaller area be the integer T. Then

$$T(k^2 - 1) = 18.$$

Thus $k^2 - 1$ is a positive divisor of 18. The only resulting square k^2 is 4, so $k = 2$. The corresponding side is $3k = 6$.

Therefore, the correct answer is **D**.

16. Let the product $(12)(15)(16)$, each factor written in base b, equal 3146 in base b. Let $s = 12 + 15 + 16$, each term expressed in base b. Then s, in base b, is:

A 43

B 44

C 45

D 46

E 47

Solution:

The product equation is

$$(b+2)(b+5)(b+6)$$
$$= 3b^3 + b^2 + 4b + 6.$$

It simplifies to

$$b^3 - 6b^2 - 24b - 27 = 0,$$
$$(b-9)(b^2+3b+3) = 0.$$

Thus the valid base is $b = 9$. The required sum is $3b + 13 = 4b + 4$, which is 44_b.

Therefore, the correct answer is **B**.

17. If r_1 and r_2 are the distinct real roots of $x^2 + px + 8 = 0$, then it must follow that:

A $|r_1 + r_2| > 4\sqrt{2}$

B $|r_1| > 3$ or $|r_2| > 3$

C $|r_1| > 2$ and $|r_2| > 2$

D $r_1 < 0$ and $r_2 < 0$

E $|r_1 + r_2| < 4\sqrt{2}$

Solution:

The discriminant condition is $p^2 - 32 > 0$, so $|p| > 4\sqrt{2}$. By Vieta's formulas, $r_1 + r_2 = -p$, and therefore

$$|r_1 + r_2| = |p| > 4\sqrt{2}.$$

Therefore, the correct answer is **A**.

18. If $x^2 - 5x + 6 < 0$ and $P = x^2 + 5x + 6,$ then

A P can take any real value

B $20 < P < 30$

C $0 < P < 20$

D $P < 0$

E $P > 30$

Solution:

The inequality $(x-2)(x-3) < 0$ gives $2 < x < 3.$ On this interval $P = x^2 + 5x + 6$ is increasing, with endpoint values $P(2) = 20$ and $P(3) = 30.$ The endpoints are excluded, so $20 < P < 30.$

Therefore, the correct answer is **B**.

19. The area of a rectangle remains unchanged when it is made $2\frac{1}{2}$ inches longer and $\frac{2}{3}$ inch narrower, or when it is made $2\frac{1}{2}$ inches shorter and $\frac{4}{3}$ inch wider. Its area, in square inches, is:

A 30

B $\frac{80}{3}$

C 24

D $\frac{45}{2}$

E 20

Solution:

The two conditions are

$$lw = \left(l + \frac{5}{2}\right)\left(w - \frac{2}{3}\right),$$
$$lw = \left(l - \frac{5}{2}\right)\left(w + \frac{4}{3}\right).$$

After expanding, these give $-4l + 15w = 10$ and $8l - 15w = 20$. Hence $l = \frac{15}{2},\ w = \frac{8}{3}$, and $lw = 20$.

Therefore, the correct answer is **E**.

20. A circle is inscribed in a square of side m, then a square is inscribed in that circle, then a circle is inscribed in the latter square, and so on. If S_n is the sum of the areas of the first n circles so inscribed, then, as n grows beyond all bounds, S_n approaches:

A $\frac{\pi m^2}{2}$

B $\frac{3\pi m^2}{8}$

C $\frac{\pi m^2}{3}$

D $\frac{\pi m^2}{4}$

E $\frac{\pi m^2}{8}$

Solution:

The first circle has area $\pi(\frac{m}{2})^2 = \frac{\pi m^2}{4}$. Each inscribed square reduces the next circle's squared radius, and hence its area, by a factor of $\frac{1}{2}$. Thus the limiting sum is

$$\frac{\pi m^2}{4}\left(1+\frac{1}{2}+\frac{1}{4}+\cdots\right)$$
$$=\frac{\pi m^2}{2}.$$

Therefore, the correct answer is **A**.

21. In right triangle ABC the hypotenuse $AB = 5$ and leg $AC = 3$. The bisector of angle A meets the opposite side in A_1. A second right triangle PQR is then constructed with hypotenuse $PQ = A_1B$ and leg $PR = A_1C$. If the bisector of angle P meets the opposite side in P_1, the length of PP_1 is:

A $\frac{3\sqrt{6}}{4}$

B $\frac{3\sqrt{5}}{4}$

C $\frac{3\sqrt{3}}{4}$

D $\frac{3\sqrt{2}}{2}$

E $\frac{15\sqrt{2}}{16}$

Solution:

Since $BC = 4$, the angle-bisector theorem gives $A_1B : A_1C = 5 : 3$. Hence $A_1B = \frac{5}{2}$ and $A_1C = \frac{3}{2}$, so triangle PQR is a half-scale copy of ABC.

For the angle bisector from A in the original triangle,

$$AA_1^2 = 5 \cdot 3 \left(1 - \frac{4^2}{8^2}\right)$$
$$= \frac{45}{4},$$

so $AA_1 = \frac{3\sqrt{5}}{2}$. Therefore PP_1 is half of this, or $\frac{3\sqrt{5}}{4}$.

Thus, the correct answer is **B**.

22. For natural numbers, when P is divided by D, the quotient is Q and the remainder is R. When Q is divided by D', the quotient is Q' and the remainder is R'. Then, when P is divided by DD', the remainder is:

A $R + R'D$

B $R' + RD$

C RR'

D R

E R'

Solution:

Substitution gives

$$\begin{aligned} P &= (Q'D' + R')D + R \\ &= Q'(DD') + (R'D + R). \end{aligned}$$

The remainder bounds imply $R'D + R < DD'$, so this is indeed the remainder.

Therefore, the correct answer is **A**.

23. If x is real and positive and grows beyond all bounds, then $\log_3(6x-5)-\log_3(2x+1)$ approaches:

- A 0
- B 1
- C 3
- D 4
- E no finite number

Solution:

The difference is

$$\log_3\left(\frac{6x-5}{2x+1}\right).$$

The fraction approaches 3, so the expression approaches $\log_3 3 = 1.$

Therefore, the correct answer is **B**.

24. The number of solution-pairs in positive integers of the equation $3x + 5y = 501$ is:

A 33

B 34

C 35

D 100

E none of these

Solution:

Modulo 5, the equation gives $3x \equiv 1$, so $x \equiv 2 \pmod 5$. Write $x = 2 + 5k$. Then

$$y = \frac{501 - 3x}{5} = 99 - 3k.$$

Positivity requires $k = 0, 1, \ldots, 32$, giving 33 pairs.

Therefore, the correct answer is **A**.

25. For every odd number $p > 1$ we have:

A $(p-1)^{\frac{p-1}{2}} - 1$ is divisible by $p-2$

B $(p-1)^{\frac{p-1}{2}} + 1$ is divisible by p

C $(p-1)^{\frac{p-1}{2}}$ is divisible by p

D $(p-1)^{\frac{p-1}{2}} + 1$ is divisible by $p+1$

E $(p-1)^{\frac{p-1}{2}} - 1$ is divisible by $p-1$

Solution:

Because $p > 1$ is odd, $n = \frac{p-1}{2}$ is a positive integer. Modulo $p-2$, we have $p - 1 \equiv 1$. Therefore

$$\begin{aligned}(p-1)^n - 1 &\equiv 1^n - 1 \\ &\equiv 0 \pmod{p-2}.\end{aligned}$$

Thus choice (A) always holds.

Therefore, the correct answer is **A**.

26. If one uses only the tabular information $10^3 = 1000, 10^4 = 10{,}000, 2^{10} = 1024,$ $2^{11} = 2048, 2^{12} = 4096, 2^{13} = 8192,$ then the strongest statement one can make for $\log_{10} 2$ is that it lies between:

A $\frac{3}{10}$ and $\frac{4}{11}$

B $\frac{3}{10}$ and $\frac{4}{12}$

C $\frac{3}{10}$ and $\frac{4}{13}$

D $\frac{3}{10}$ and $\frac{40}{132}$

E $\frac{3}{11}$ and $\frac{40}{132}$

Solution:

From $10^3 < 2^{10}$ we get $3 < 10 \log_{10} 2,$ so $\log_{10} 2 > \frac{3}{10}$. From $2^{13} < 10^4$ we get $13 \log_{10} 2 < 4,$ so $\log_{10} 2 < \frac{4}{13}$. This is the narrowest listed interval justified by the table.

Therefore, the correct answer is **C**.

27. Two candles of the same length are made of different materials so that one burns out completely at a uniform rate in 3 hours and the other in 4 hours. At what time P.M. should the candles be lighted so that, at 4 P.M., one stub is twice the length of the other?

A $1{:}24$

B $1{:}28$

C $1{:}36$

D $1{:}40$

E $1{:}48$

Solution:

After t hours, the faster and slower candles have fractions $1 - \frac{t}{3}$ and $1 - \frac{t}{4}$ remaining. The slower stub must be twice the faster:

$$1 - \frac{t}{4} = 2\left(1 - \frac{t}{3}\right).$$

Thus $t = \frac{12}{5} = 2$ hours 24 minutes. Counting back from 4 P.M. gives $1{:}36$ P.M.

Therefore, the correct answer is **C**.

28. Given the two hypotheses: I Some Mems are not Ens and II No Ens are Vees. If “some” means “at least one”, we can conclude that:

- A Some Mems are not Vees
- B Some Vees are not Mems
- C No Mem is a Vee
- D Some Mems are Vees
- E Neither (A) nor (B) nor (C) nor (D) is deducible from the given statements

Solution:

The hypotheses say only that some element of M lies outside N, and that $N \cap V = \varnothing$. A model with $M = V$ makes (A), (B), and (C) false while satisfying the hypotheses. A model with $M \cap V = \varnothing$ makes (D) false. Hence none of choices (A) through (D) is forced.

Therefore, the correct answer is **E**.

29. AB is a diameter of a circle. Tangents AD and BC are drawn so that AC and BD intersect in a point on the circle. If $AD = a$ and $BC = b, a \neq b,$ the diameter of the circle is:

A $|a - b|$

B $\frac{1}{2}(a + b)$

C $\sqrt{ab}$

D $\frac{ab}{a+b}$

E $\frac{1}{2} \cdot \frac{ab}{a+b}$

Solution:

Let $d = AB$ be the diameter. Since the intersection of AC and BD lies on the circle, those two lines are perpendicular by Thales' theorem. Also the tangents AD and BC are parallel. The resulting right triangles ADB and BCA are similar, so

$$\frac{d}{a} = \frac{b}{d}.$$

Hence $d^2 = ab$ and $d = \sqrt{ab}.$

Therefore, the correct answer is **C**.

30. A dealer bought n radios for d dollars, d, a positive integer. He contributed two radios to a community bazaar at half their cost. The rest he sold at a profit of $\$8$ on each radio sold. If the overall profit was $\$72$, then the least possible value of n for the given information is:

- A 18
- B 16
- C 15
- D 12
- E 11

Solution:

The $n-2$ regular sales bring $(n-2)(\frac{d}{n}+8)$, while the two bazaar radios bring $\frac{d}{n}$ together. Since the total intake is $d+72$,

$$(n-2)\left(\frac{d}{n}+8\right)+\frac{d}{n}$$
$$=d+72.$$

This reduces to $d=8n(n-11)$. Positivity requires $n>11$, and $n=12$ gives the positive integer $d=96$.

Therefore, the correct answer is **D**.

31. Let $D = a^2 + b^2 + c^2$, where a, b are consecutive integers and $c = ab$. Then $\sqrt{D}$ is:

- A always an even integer
- B sometimes an odd integer, sometimes not
- C always an odd integer
- D sometimes rational, sometimes not
- E always irrational

Solution:

With $b = a + 1$ and $c = a(a+1)$,

$$\begin{aligned} D &= a^2 + (a+1)^2 + a^2(a+1)^2 \\ &= (a^2 + a + 1)^2. \end{aligned}$$

Thus $\sqrt{D} = a^2 + a + 1$. Since $a(a+1)$ is even, this integer is always odd.

Therefore, the correct answer is **C**.

32. In quadrilateral $ABCD$ with diagonals AC and BD intersecting at $O, BO = 4,$ $OD = 6, AO = 8, OC = 3,$ and $AB = 6.$ The length of AD is:

A 9

B 10

C $6\sqrt{3}$

D $8\sqrt{2}$

E $\sqrt{166}$

Solution:

In triangle $AOB,$

$$\begin{aligned}\cos\angle AOB &= \frac{8^2+4^2-6^2}{2\cdot 8\cdot 4}\\ &= \frac{11}{16}.\end{aligned}$$

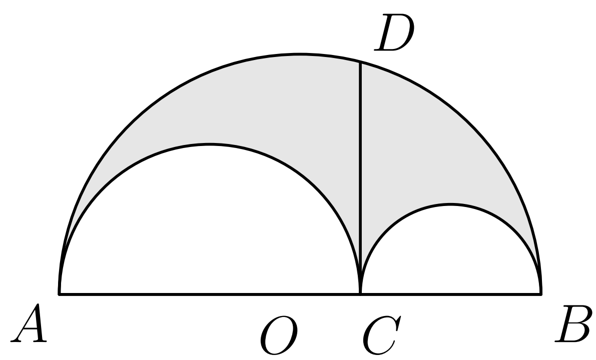
Therefore $\cos\angle AOD = -\frac{11}{16}$. Applying the law of cosines in triangle AOD gives

$$\begin{aligned}AD^2 &= 8^2+6^2\\ &\quad + 2(8)(6)\frac{11}{16}\\ &= 166.\end{aligned}$$

Hence $AD = \sqrt{166}$.

Therefore, the correct answer is **E**.

33. In this diagram semi-circles are constructed on diameters AB, AC, and CB, so that they are mutually tangent. If $CD \perp AB$, then the ratio of the shaded area to the area of a circle with CD as radius is:



A $1 : 2$

B $1 : 3$

C $\sqrt{3} : 7$

D $1 : 4$

E $\sqrt{2} : 6$

Solution:

Let $AC = u$ and $CB = v$. The shaded area is the large semicircle minus the two smaller ones:

$$\frac{\pi}{8}\left((u+v)^2 - u^2 - v^2\right) = \frac{\pi uv}{4}.$$

Since D lies on the semicircle with diameter AB, triangle ADB is right, and its altitude satisfies $CD^2 = uv$. A circle of radius CD therefore has area πuv. The required ratio is $1 : 4$.

Therefore, the correct answer is **D**.

34. Points D, E, F are taken respectively on sides AB, BC, and CA of triangle ABC so that $AD : DB = 1 : n$, $BE : CE = 1 : n$, and $CF : FA = 1 : n$. The ratio of the area of triangle DEF to that of triangle ABC is:

A $\dfrac{n^2 - n + 1}{(n+1)^2}$

B $\dfrac{1}{(n+1)^2}$

C $\dfrac{2n^3}{(n+1)^2}$

D $\dfrac{n^3}{(n+1)^2}$

E $\dfrac{n(n-1)}{n+1}$

Solution:

At each vertex, the two adjacent side fractions used by the corner triangle are $\frac{1}{n+1}$ and $\frac{n}{n+1}$. Thus each corner triangle has area $\frac{n}{(n+1)^2}$ times $[ABC]$. Therefore

$$\begin{aligned}\frac{[DEF]}{[ABC]} &= 1 - \frac{3n}{(n+1)^2} \\ &= \frac{n^2 - n + 1}{(n+1)^2}.\end{aligned}$$

Therefore, the correct answer is **A**.

35. The roots of $64x^3 - 144x^2 + 92x - 15 = 0$ are in arithmetic progression. The difference between the largest and smallest roots is:

A 2

B 1

C $\frac{1}{2}$

D $\frac{3}{8}$

E $\frac{1}{4}$

Solution:

Let the roots be $t-d, t, t+d$. Their sum is $3t = \frac{144}{64} = \frac{9}{4}$, so $t = \frac{3}{4}$. Their product is

$$t(t^2 - d^2) = \frac{15}{64}.$$

Substituting $t = \frac{3}{4}$ gives $d^2 = \frac{1}{4}$, so the difference between the extreme roots is $2|d| = 1$.

Therefore, the correct answer is **B**.

36. Given a geometric progression of five terms, each a positive integer less than 100. The sum of the five terms is 211. If S is the sum of those terms in the progression which are squares of integers, then S is:

A 0

B 91

C 133

D 195

E 211

Solution:

Let the common ratio be $\frac{c}{d}$ in lowest terms and the middle term be a. Since all five terms are integers, $c^2d^2 \mid a$, so write $a = kc^2d^2$. The sum 211 is then divisible by k. Since 211 is prime and the terms are positive, $k = 1$.

Thus

$$d^4 + d^3c + d^2c^2 \\ +dc^3 + c^4 = 211.$$

The bound gives $c, d < 4$. If either is 1, the possible sums are 31 or 121, not 211. Coprimality therefore leaves $c, d = 2, 3$ in either order. The progression is $16, 24, 36, 54, 81$. Its square terms sum to $16 + 36 + 81 = 133$.

Therefore, the correct answer is **C**.

37. Segments $AD = 10, BE = 6, CF = 24$ are drawn from the vertices of triangle ABC, each perpendicular to a straight line RS, not intersecting the triangle. Points D, E, F are the intersection points of RS with the perpendiculars. If x is the length of the perpendicular segment GH drawn to RS from the intersection point G of the medians of the triangle, then x is:

A $\frac{40}{3}$

B 16

C $\frac{56}{3}$

D $\frac{80}{3}$

E undetermined

Solution:

Because RS does not intersect the triangle, the three perpendicular distances have the same sign. Signed distance to a fixed line is affine, and the centroid is the average of the vertices. Therefore its distance is the average

$$x = \frac{10 + 6 + 24}{3} = \frac{40}{3}.$$

Therefore, the correct answer is **A**.

38. Given a set S consisting of two undefined elements "pib" and "maa", and the four postulates:

P_1 : Every pib is a collection of maas.

P_2 : Any two distinct pibs have one and only one maa in common.

P_3 : Every maa belongs to two and only two pibs.

P_4 : There are exactly four pibs.

Consider the three theorems:

T_1 : There are exactly six maas.

T_2 : There are exactly three maas in each pib.

T_3 : For each maa there is exactly one other maa not in the same pib with it.

The theorems which are deducible from the postulates are:

A T_3 only

B T_2 and T_3 only

C T_1 and T_2 only

D T_1 and T_3 only

E all

Solution:

Label the four pibs $1, 2, 3, 4$. By P_2 and P_3, every maa is exactly the common maa of one unordered pair of pibs. Thus there are $\binom{4}{2} = 6$ maas, proving T_1.

A fixed pib i contains the three maas ij with $j \neq i$, proving T_2. For maa ij, the unique maa sharing neither pib is the one belonging to the complementary pair of pibs, proving T_3. All three follow.

Therefore, the correct answer is **E**.

39. Given the sets of consecutive integers $\{1\}, \{2,3\}, \{4,5,6\}, \{7,8,9,10\}, \ldots,$ where each set contains one more element than the preceding one, and where the first element of each succeeding set is one more than the last element of the preceding set. Let S_n be the sum of the elements in the nth set. Then S_{21} equals:

A 1113

B 4641

C 5082

D 53361

E none of these

Solution:

The nth set ends at $\frac{n(n+1)}{2}$ and contains n consecutive integers. Its sum is

$$S_n = n\left(\frac{n(n+1)}{2}\right) - \frac{n(n-1)}{2} = \frac{n(n^2+1)}{2}.$$

Thus $S_{21} = \frac{21(442)}{2} = 4641.$

Therefore, the correct answer is **B**.

40. Located inside equilateral triangle ABC is a point P such that $PA = 6, PB = 8$, and $PC = 10$. To the nearest integer the area of triangle ABC is:

A 159

B 131

C 95

D 79

E 50

Solution:

Rotate P by 60° about A to P'. Since this rotation sends C to B, we have $P'B = PC = 10, PP' = PA = 6$, and $PB = 8$. Thus triangle $PP'B$ is right. Also triangle APP' is equilateral, so $\angle APB = 60^\circ + 90^\circ = 150^\circ$.

If the equilateral triangle has side s, the law of cosines in triangle APB gives

$$\begin{aligned} s^2 &= 6^2 + 8^2 - 2(6)(8)\cos 150^\circ \\ &= 100 + 48\sqrt{3}. \end{aligned}$$

Its area is

$$\frac{\sqrt{3}}{4}s^2 = 36 + 25\sqrt{3},$$

whose nearest integer is 79.

Therefore, the correct answer is **D**.

Problems: https://live.poshenloh.com/past-contests/amc12/1967