1965 AMC 12 Solutions

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1. The number of real values of x satisfying the equation $2^{2x^2-7x+5} = 1$ is:

A 0

B 1

C 2

D 4

E more than 4

Solution:

Since the base is 2, the exponent must be zero. Factoring gives

$$2x^2 - 7x + 5 = 0,$$
$$(2x-5)(x-1) = 0.$$

Thus $x = \frac{5}{2}$ or $x = 1$, giving two real values.

Therefore, the correct answer is **C**.

2. A regular hexagon is inscribed in a circle. The ratio of the length of a side of the hexagon to the length of the shorter of the arcs intercepted by the side is:

A $1:1$

B $1:6$

C $1:\pi$

D $3:\pi$

E $6:\pi$

Solution:

If the radius is r, the hexagon side is r. Its intercepted minor arc has length $\frac{2\pi r}{6} = \frac{\pi r}{3}$. The ratio is $r : \left(\frac{\pi r}{3}\right) = 3 : \pi$.

Therefore, the correct answer is **D**.

3. The expression $81^{-(2^{-2})}$ has the same value as:

A $\frac{1}{81}$

B $\frac{1}{3}$

C 3

D 81

E 81^4

Solution:

Since $2^{-2} = \frac{1}{4}$,

$$\begin{aligned} 81^{-(2^{-2})} &= 81^{-\frac{1}{4}} \\ &= (3^4)^{-\frac{1}{4}} = \frac{1}{3}. \end{aligned}$$

Therefore, the correct answer is **B**.

4. Line l_2 intersects line l_1 and line l_3 is parallel to l_1. The three lines are distinct and lie in a plane. The number of points equidistant from all three lines is:

- A 0
- B 1
- C 2
- D 4
- E 8

Solution:

Points equidistant from parallel lines l_1 and l_3 lie on the unique line midway between them. Points equidistant from intersecting lines l_1 and l_2 lie on either of their two angle bisectors. Each angle bisector meets the midway parallel once, giving two points.

Therefore, the correct answer is **C**.

5. When the repeating decimal $0.363636\ldots$ is written in simplest fractional form, the sum of the numerator and denominator is:

- A 15
- B 45
- C 114
- D 135
- E 150

Solution:

Let $z = 0.363636\ldots$. Then $100z - z = 36$, so $z = \frac{36}{99} = \frac{4}{11}$. The requested sum is $4 + 11 = 15$.

Therefore, the correct answer is **A**.

6. If $10^{\log_{10} 9} = 8x + 5$, then x equals:

A 0

B $\frac{1}{2}$

C $\frac{5}{8}$

D $\frac{9}{8}$

E $\frac{2\log_{10} 3 - 5}{8}$

Solution:

The exponential and logarithm cancel, so $10^{\log_{10} 9} = 9$. Thus $8x + 5 = 9$, giving $x = \frac{1}{2}$.

Therefore, the correct answer is **B**.

7. The sum of the reciprocals of the roots of the equation $ax^2 + bx + c = 0$ is:

A $\frac{1}{a} + \frac{1}{b}$

B $-\frac{c}{b}$

C $\frac{c}{b}$

D $-\frac{a}{b}$

E $-\frac{b}{c}$

Solution:

By Vieta's formulas, $r + s = -\frac{b}{a}$ and $rs = \frac{c}{a}$. Therefore

$$\begin{aligned}\frac{1}{r} + \frac{1}{s} &= \frac{r+s}{rs} \\ &= \frac{-\frac{b}{a}}{\frac{c}{a}} = -\frac{b}{c}.\end{aligned}$$

Thus, the correct answer is **E**.

8. One side of a given triangle is 18 inches. Inside the triangle a line segment is drawn parallel to this side forming a trapezoid whose area is one-third of that of the triangle. The length of this segment, in inches, is:

A $6\sqrt{6}$

B $9\sqrt{2}$

C 12

D $6\sqrt{3}$

E 9

Solution:

The trapezoid occupies one third of the area, so the smaller similar triangle occupies two thirds. Its linear scale factor is $\sqrt{\frac{2}{3}}$. Hence the parallel segment has length

$$18\sqrt{\frac{2}{3}} = 6\sqrt{6}.$$

Therefore, the correct answer is **A**.

9. The vertex of the parabola $y = x^2 - 8x + c$ will be a point on the x-axis if the value of c is:

A -16

B -4

C 4

D 8

E 16

Solution:

Completing the square gives

$$y = (x-4)^2 + c - 16.$$

The vertex is $(4, c-16)$, so it lies on the x-axis when $c = 16$.

Therefore, the correct answer is **E**.

10. The statement $x^2 - x - 6 < 0$ is equivalent to the statement:

- A $-2 < x < 3$
- B $x > -2$
- C $x < 3$
- D $x > 3 \text{ and } x < -2$
- E $x > 3 \text{ or } x < -2$

Solution:

We have $x^2 - x - 6 = (x - 3)(x + 2)$. This product is negative exactly between its roots, so $-2 < x < 3$.

Therefore, the correct answer is **A**.

11. Consider the statements:

I: $(\sqrt{-4})(\sqrt{-16})=\sqrt{(-4)(-16)}$,

II: $\sqrt{(-4)(-16)}=\sqrt{64}$,

and

III: $\sqrt{64}=8$.

Of these the following are *incorrect*:

A none

B I only

C II only

D III only

E I and III only

Solution:

Statement I improperly applies $\sqrt{a}\sqrt{b}=\sqrt{ab}$ to negative radicands; over the reals its left side is undefined, and with principal complex roots it equals -8, not 8. Statements II and III both correctly simplify to 8. Thus only I is incorrect.

Therefore, the correct answer is **B**.

12. A rhombus is inscribed in triangle ABC in such a way that one of its vertices is A and two of its sides lie along AB and AC. If $AC = 6$ inches, $AB = 12$ inches, and $BC = 8$ inches, the side of the rhombus, in inches, is:

A 2

B 3

C $3\frac{1}{2}$

D 4

E 5

Solution:

Use A as the origin and let the vectors to B, C be u, v. If the rhombus side is s, its opposite vertex is $(\frac{s}{12})u + (\frac{s}{6})v$. A point lies on BC when these coefficients sum to 1. Thus

$$\frac{s}{12} + \frac{s}{6} = 1,$$

giving $s = 4$.

Therefore, the correct answer is **D**.

13. Let n be the number of number-pairs (x, y) which satisfy $5y - 3x = 15$ and $x^2 + y^2 \leq 16$. Then n is:

A 0

B 1

C 2

D more than two, but finite

E greater than any finite number

Solution:

The line's distance from the origin is $\frac{15}{\sqrt{(-3)^2+5^2}} = \frac{15}{\sqrt{34}} < 4$. It therefore crosses the interior of the disk $x^2 + y^2 \leq 16$ in a segment containing infinitely many points.

Thus, the correct answer is **E**.

14. The sum of the numerical coefficients in the complete expansion of $(x^2 - 2xy + y^2)^7$ is:

A 0

B 7

C 14

D 128

E 128^2

Solution:

Setting $x = y = 1$ makes every monomial equal 1, so the value is the sum of the numerical coefficients. It is

$$(1 - 2 + 1)^7 = 0.$$

Therefore, the correct answer is **A**.

15. The symbol 25_b represents a two-digit number in the base b. If the number 52_b is double the number 25_b, then b is:

- A 7
- B 8
- C 9
- D 11
- E 12

Solution:

In ordinary notation, $25_b = 2b + 5$ and $52_b = 5b + 2$. Thus

$$5b + 2 = 2(2b + 5),$$

so $b = 8$.

Therefore, the correct answer is **B**.

16. Let line AC be perpendicular to line CE. Connect A to the midpoint D of CE, and connect E to the midpoint B of AC. If AD and EB intersect in point F, and $BC = CD = 15$ inches, then the area of triangle DFE, in square inches, is:

A 50

B $50\sqrt{2}$

C 75

D $\frac{15}{2}\sqrt{105}$

E 100

Solution:

Put $C = (0,0),\ A = (0,30)$, and $E = (30,0)$. Then $B = (0,15)$ and $D = (15,0)$. Lines AD and EB are medians of triangle ACE, so they meet at the centroid $F = (10,10)$. Segment DE has length 15 and F is 10 units above its line. Therefore

$$[DFE] = \frac{1}{2}(15)(10) = 75.$$

Thus, the correct answer is **C**.

17. Given the true statement: The picnic on Sunday will not be held only if the weather is not fair. We can then conclude that:

- A If the picnic is held, Sunday’s weather is undoubtedly fair.
- B If the picnic is not held, Sunday’s weather is possibly unfair.
- C If it is not fair Sunday, the picnic will not be held.
- D If it is fair Sunday, the picnic may be held.
- E If it is fair Sunday, the picnic will be held.

Solution:

The statement says: if the picnic is not held, then the weather is not fair. Its contrapositive is: if the weather is fair, then the picnic will be held.

Therefore, the correct answer is **E**.

18. If $1 - y$ is used as an approximation to the value of $\dfrac{1}{1+y}$, $|y| < 1,$ the ratio of the error made to the correct value is:

A y

B y^2

C $\dfrac{1}{1+y}$

D $\dfrac{y}{1+y}$

E $\dfrac{y^2}{1+y}$

Solution:

The error is

$$\frac{1}{1+y} - (1-y) = \frac{y^2}{1+y}.$$

Dividing by the correct value $\frac{1}{1+y}$ gives the ratio y^2.

Therefore, the correct answer is **B**.

19. If $x^4 + 4x^3 + 6px^2 + 4qx + r$ is exactly divisible by $x^3 + 3x^2 + 9x + 3,$ the value of $(p+q)r$ is:

- A -18
- B 12
- C 15
- D 27
- E 45

Solution:

The leading terms force quotient $x + 1.$ Expanding,

$$\begin{aligned}&(x+1)(x^3+3x^2+9x+3)\\ &= x^4+4x^3+12x^2+12x+3.\end{aligned}$$

Thus $6p = 12,\ 4q = 12,\ r = 3,$ so $p = 2,\ q = 3$ and $(p+q)r = 15.$

Therefore, the correct answer is **C**.

20. For every n the sum S_n of n terms of an arithmetic progression is $2n + 3n^2$. The rth term is:

A $3r^2$

B $3r^2 + 2r$

C $6r - 1$

D $5r + 5$

E $6r + 2$

Solution:

The rth term is the difference of consecutive partial sums. Substitution and simplification give

$$S_r - S_{r-1} = 6r - 1.$$

Therefore, the correct answer is **C**.

21. It is possible to choose $x > \dfrac{2}{3}$ in such a way that the value of

$$\log_{10}(x^2+3) - 2\log_{10} x$$

is:

A negative

B zero

C one

D smaller than any positive number that might be specified

E greater than any positive number that might be specified

Solution:

The expression is

$$\begin{aligned} &\log_{10}\left(\frac{x^2+3}{x^2}\right) \\ &= \log_{10}\left(1+\frac{3}{x^2}\right). \end{aligned}$$

It is always positive, but tends to 0 as x grows. Therefore it can be made smaller than any specified positive number.

Thus, the correct answer is **D**.

22. If $a_2 \neq 0$ and r, s are the roots of $a_0 + a_1x + a_2x^2 = 0$, then the equality

$$a_0 + a_1x + a_2x^2 = a_0\left(1 - \frac{x}{r}\right) \cdot \left(1 - \frac{x}{s}\right)$$

holds:

A for all values of $x, a_0 \neq 0$

B for all values of x

C only when $x = 0$

D only when $x = r$ or $x = s$

E only when $x = r$ or $x = s, a_0 \neq 0$

Solution:

If $a_0 \neq 0$, then $rs = \frac{a_0}{a_2} \neq 0$ and

$$\begin{aligned} &a_0\left(1 - \frac{x}{r}\right)\left(1 - \frac{x}{s}\right) \\ &= \frac{a_0}{rs}(x - r)(x - s) \\ &= a_2(x - r)(x - s), \end{aligned}$$

which is the original polynomial for every x. If $a_0 = 0$, at least one denominator on the right is zero.

Therefore, the correct answer is **A**.

23. If we write $|x^2 - 4| < N$ for all x such that $|x - 2| < 0.01$, the smallest value we can use for N is:

A 0.0301

B 0.0349

C 0.0399

D 0.0401

E 0.0499

Solution:

For $1.99 < x < 2.01$,

$$\begin{aligned} |x^2 - 4| &= |x - 2||x + 2| \\ &< (0.01)(4.01) = 0.0401. \end{aligned}$$

Values with x arbitrarily close to 2.01 make the expression arbitrarily close to 0.0401, so no smaller bound works.

Therefore, the correct answer is **D**.

24. Given the sequence $10^{\frac{1}{11}}, 10^{\frac{2}{11}}, 10^{\frac{3}{11}}, \ldots, 10^{\frac{n}{11}}$, the smallest value of n such that the product of the first n members of this sequence exceeds $100{,}000$ is:

- A 7
- B 8
- C 9
- D 10
- E 11

Solution:

The product is

$$10^{\frac{1+2+\cdots+n}{11}} = 10^{\frac{n(n+1)}{22}}.$$

To exceed $100{,}000 = 10^5$, we need $n(n+1) > 110$. At $n = 10$ equality holds, while $n = 11$ exceeds it.

Therefore, the correct answer is **E**.

25. Let $ABCD$ be a quadrilateral with AB extended to E so that $AB = BE.$ Lines AC and CE are drawn to form angle $ACE.$ For this angle to be a right angle it is necessary that quadrilateral $ABCD$ have:

A all angles equal

B all sides equal

C two pairs of equal sides

D one pair of equal sides

E one pair of equal angles

Solution:

If $\angle ACE = 90^\circ$, then AE is the hypotenuse of right triangle $ACE.$ Its midpoint B is equidistant from $A, C, E.$ Thus $AB = BC,$ so quadrilateral $ABCD$ necessarily has at least one pair of equal sides. No condition involving D follows.

Therefore, the correct answer is **D**.

26. For the numbers a, b, c, d, e define m to be the arithmetic mean of all five numbers; k to be the arithmetic mean of a and b; l to be the arithmetic mean of c, d, and e; and p to be the arithmetic mean of k and l. Then, no matter how a, b, c, d, e are chosen, we shall always have:

A $m = p$

B $m \geq p$

C $m > p$

D $m < p$

E none of these

Solution:

We have

$$m = \frac{2k + 3l}{5}, \qquad p = \frac{k + l}{2},$$

so $m - p = \frac{l - k}{10}$. This difference may be positive, zero, or negative depending on the chosen numbers. None of the first four relations always holds.

Therefore, the correct answer is **E**.

27. When $y^2 + my + 2$ is divided by $y - 1$ the quotient is $f(y)$ and the remainder is R_1. When $y^2 + my + 2$ is divided by $y + 1$ the quotient is $g(y)$ and the remainder is R_2. If $R_1 = R_2$, then m is:

- A 0
- B 1
- C 2
- D -1
- E an undetermined constant

Solution:

The remainder theorem gives $R_1 = 1 + m + 2 = m + 3$ and $R_2 = 1 - m + 2 = 3 - m$. Equality implies $m = 0$.

Therefore, the correct answer is **A**.

28. An escalator (moving staircase) of n uniform steps visible at all times descends at constant speed. Two boys, A and Z, walk down the escalator steadily as it moves, A negotiating twice as many escalator steps per minute as Z. A reaches the bottom after taking 27 steps while Z reaches the bottom after taking 18 steps. Then n is:

A 63

B 54

C 45

D 36

E 30

Solution:

Let Z take v steps per minute, so A takes $2v$, and let the escalator contribute e steps per minute. Their travel times are $\frac{27}{2v}$ and $\frac{18}{v}$. Thus

$$n = 27 + \frac{27e}{2v} = 18 + \frac{18e}{v}.$$

This gives $\frac{e}{v} = 2$, and then $n = 18 + 18(2) = 54.$

Therefore, the correct answer is **B**.

29. Of 28 students taking at least one subject the number taking Mathematics and English only equals the number taking Mathematics only. No student takes English only or History only, and six students take Mathematics and History, but no English. The number taking English and History only is five times the number taking all three subjects. If the number taking all three subjects is even and non-zero, the number taking English and Mathematics only is:

A 5

B 6

C 7

D 8

E 9

Solution:

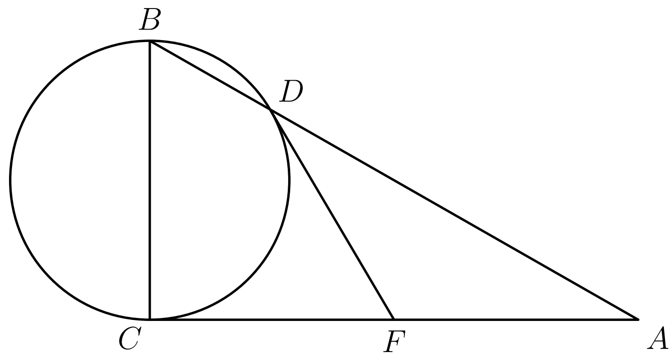
Let t be the number taking all three and x the number taking Mathematics and English only, which also equals the Mathematics-only count. The disjoint-region total is

$$x + x + 6 + 5t + t = 28,$$

so $x = 11 - 3t$. Since t is positive and even, $t = 2$ is the only value giving a nonnegative listed count, and $x = 5$.

Therefore, the correct answer is **A**.

30. Let BC of right triangle ABC be the diameter of a circle intersecting hypotenuse AB in D. At D a tangent is drawn cutting leg CA in F. This information is *not* sufficient to prove that:



- A DF bisects CA
- B DF bisects $\angle CDA$
- C $DF = FA$
- D $\angle A = \angle BCD$
- E $\angle CFD = 2\angle A$

Solution:

Put $C = (0, 0),\ A = (a, 0)$, and $B = (0, b)$. The second intersection of AB with the circle of diameter BC is

$$D = \left(\frac{ab^2}{a^2 + b^2}, \frac{a^2b}{a^2 + b^2}\right).$$

The tangent at D meets CA at $F = (\frac{a}{2}, 0)$. Hence DF bisects CA; also $FC = FD = FA$ by equal tangents from F, which supplies choices C and E, and the circle/right-triangle angles supply choice D.

If DF bisected $\angle CDA$, the angle-bisector theorem in triangle CDA would require $\frac{CF}{FA} = \frac{CD}{DA}$. The left side is 1, while

$$\frac{CD}{DA} = \frac{ab}{a^2} = \frac{b}{a},$$

which need not be 1. Thus choice B is not provable.

Therefore, the correct answer is **B**.

31. The number of real values of x satisfying the equality $(\log_a x)(\log_b x) = \log_a b$, where a and b are positive constants different from 1, is:

- A 0
- B 1
- C 2
- D an integer greater than 2
- E not finite

Solution:

Change of base gives

$$\frac{\ln x}{\ln a}\frac{\ln x}{\ln b} = \frac{\ln b}{\ln a}.$$

Since the denominators are nonzero, $(\ln x)^2 = (\ln b)^2$. Thus $x = b$ or $x = \frac{1}{b}$. These are distinct because $b \neq 1$, so there are two values.

Therefore, the correct answer is **C**.

32. An article costing C dollars is sold for $\$100$ at a loss of x percent of the selling price. It is then resold at a profit of x percent of the new selling price S'. If the difference between S' and C is $1\frac{1}{9}$ dollars, then x is:

A undetermined

B $\frac{80}{9}$

C 10

D $\frac{95}{9}$

E $\frac{100}{9}$

Solution:

A loss of $x\%$ of the $\$100$ selling price means $C = 100 + x$. On resale, $S' - 100 = (\frac{x}{100})S'$, so $S' = \frac{10000}{100-x}$. Therefore

$$\frac{10000}{100-x} - (100+x)$$
$$= \frac{x^2}{100-x} = \frac{10}{9}.$$

Thus $9x^2 + 10x - 1000 = 0$, whose positive root is $x = 10$.

Therefore, the correct answer is **C**.

33. If the number $15!$, that is, $15 \cdot 14 \cdot 13 \cdots 1$, ends with k zeros when given to the base 12 and ends with h zeros when given to the base 10, then $k + h$ equals:

A 5

B 6

C 7

D 8

E 9

Solution:

The prime valuations are

$$v_2(15!) = 11,$$
$$v_3(15!) = 6, \qquad v_5(15!) = 3.$$

Hence $k = \min(\lfloor \frac{11}{2} \rfloor, 6) = 5$ in base 12, and $h = \min(11, 3) = 3$ in base 10. Thus $k + h = 8$.

Therefore, the correct answer is **D**.

34. For $x \geq 0$, the smallest value of $\dfrac{4x^2+8x+13}{6(1+x)}$ is:

A 1

B 2

C $\dfrac{25}{12}$

D $\dfrac{13}{6}$

E $\dfrac{34}{5}$

Solution:

Let $t = x + 1 \geq 1$. Since $4x^2 + 8x + 13 = 4t^2 + 9$, the expression is

$$\frac{2}{3}t + \frac{3}{2t}.$$

By AM-GM this is at least $2\sqrt{(\frac{2}{3})(\frac{3}{2})} = 2$, with equality at $t = \frac{3}{2}$, which is allowed.

Therefore, the correct answer is **B**.

35. The length of a rectangle is 5 inches and its width is less than 4 inches. The rectangle is folded so that two diagonally opposite vertices coincide. If the length of the crease is $\sqrt{6}$, then the width is:

- A $\sqrt{2}$
- B $\sqrt{3}$
- C 2
- D $\sqrt{5}$
- E $\sqrt{\frac{11}{2}}$

Solution:

Place the rectangle at $(0,0)$, $(5,0)$, $(5,w)$, $(0,w)$. The crease sending $(0,0)$ to $(5,w)$ is

$$5x + wy = \frac{25 + w^2}{2}.$$

Because $w < 4$, it meets the two horizontal sides. Between those intersections the vertical change is w and the horizontal change is $\frac{w^2}{5}$. Thus

$$6 = w^2 + \frac{w^4}{25}.$$

Setting $z = w^2$ gives $z^2 + 25z - 150 = 0$, so $z = 5$ and $w = \sqrt{5}$.

Therefore, the correct answer is **D**.

36. Given distinct straight lines OA and OB. From a point in OA a perpendicular is drawn to OB; from the foot of this perpendicular a line is drawn perpendicular to OA. From the foot of this second perpendicular a line is drawn perpendicular to OB; and so on indefinitely. The lengths of the first and second perpendiculars are a and b, respectively. Then the sum of the lengths of the perpendiculars approaches a limit as the number of perpendiculars grows beyond all bounds. This limit is:

A $\dfrac{b}{a-b}$

B $\dfrac{a}{a-b}$

C $\dfrac{ab}{a-b}$

D $\dfrac{b^2}{a-b}$

E $\dfrac{a^2}{a-b}$

Solution:

Each new right triangle has the same acute angle, so the perpendicular lengths form a geometric sequence. Since the first two lengths are a, b, the common ratio is $\frac{b}{a}$. Convergence implies $0 < \frac{b}{a} < 1$, and the sum is

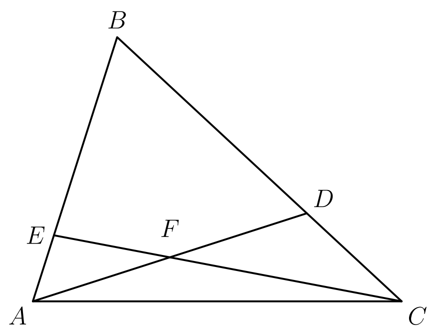
$$\frac{a}{1-\frac{b}{a}} = \frac{a^2}{a-b}.$$

Therefore, the correct answer is **E**.

37. Point E is selected on side AB of triangle ABC in such a way that $AE : EB = 1 : 3$, and point D is selected on side BC so that $CD : DB = 1 : 2$. The point of intersection of AD and CE is F. Then

$$\frac{EF}{FC} + \frac{AF}{FD}$$

is:



A $\frac{4}{5}$

B $\frac{5}{4}$

C $\frac{3}{2}$

D 2

E $\frac{5}{2}$

Solution:

The side ratios are represented by masses $m_A = 3,\ m_B = 1,\ m_C = 2$. Then $m_E = m_A + m_B = 4$ and $m_D = m_B + m_C = 3$. Along CE,

$$\frac{EF}{FC} = \frac{m_C}{m_E} = \frac{1}{2},$$

while along AD,

$$\frac{AF}{FD} = \frac{m_D}{m_A} = 1.$$

Their sum is $\frac{3}{2}$.

Therefore, the correct answer is **C**.

38. A takes m times as long to do a piece of work as B and C together; B takes n times as long as C and A together; and C takes x times as long as A and B together. Then x, in terms of m and n, is:

A $\dfrac{2mn}{m+n}$

B $\dfrac{1}{2(m+n)}$

C $\dfrac{1}{m+n-mn}$

D $\dfrac{1-mn}{m+n+2mn}$

E $\dfrac{m+n+2}{mn-1}$

Solution:

Let the work rates be a, b, c. The time statements give

$$b+c=ma, \qquad c+a=nb,$$
$$a+b=xc.$$

From the first two,

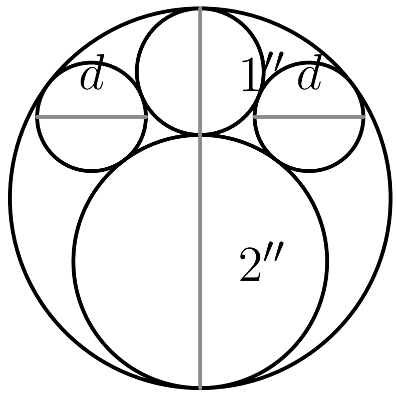
$$\frac{b}{a}=\frac{m+1}{n+1}, \qquad \frac{c}{a}=\frac{mn-1}{n+1}.$$

Hence

$$x=\frac{a+b}{c}=\frac{m+n+2}{mn-1}.$$

Therefore, the correct answer is **E**.

39. A foreman noticed an inspector checking a 3-inch hole with a 2-inch plug and a 1-inch plug and suggested that two more gauges be inserted to be sure that the fit was snug. If the new gauges are alike, then the diameter d of each, to the nearest hundredth of an inch, is:



A 0.87

B 0.86

C 0.83

D 0.75

E 0.71

Solution:

Place the hole's center at the origin. The 2-inch and 1-inch plug centers are $(0, -\frac{1}{2})$ and $(0, 1)$. Let a new gauge have radius r and center (u, v). Tangency to the two plugs gives

$$u^2 + (v + \frac{1}{2})^2 = (1 + r)^2,$$
$$u^2 + (v - 1)^2 = (\frac{1}{2} + r)^2,$$

while internal tangency to the hole gives $u^2 + v^2 = (\frac{3}{2} - r)^2$. Subtracting pairs of equations yields $v = \frac{1}{2} + \frac{r}{3}$ and $v = \frac{3}{2} - 2r$. Thus $r = \frac{3}{7}$, so $d = 2r = \frac{6}{7} \approx 0.86$.

Therefore, the correct answer is **B**.

40. Let n be the number of integer values of x such that $P = x^4 + 6x^3 + 11x^2 + 3x + 31$ is the square of an integer. Then n is:

A 4

B 3

C 2

D 1

E 0

Solution:

Let $N = x^2 + 3x + 1$. Then

$$P = N^2 - 3(x - 10).$$

At $x = 10,\ P = N^2 = 131^2$, so one value works.

If $x > 10$ and $P = y^2$, then $N^2 - y^2 = 3(x - 10)$. Distinct integer squares differing from N^2 differ by at least $2|N| - 1$, which is already greater than $3(x - 10)$, a contradiction. If $x < 10$, the analogous bound

$$\begin{aligned} 3(10 - x) &= y^2 - N^2 \\ &\geq 2|y| - 1 > 2|N| - 1 \end{aligned}$$

can hold only for $x = -6, -5, \ldots, 2$. Direct substitution for these nine integers gives no square. Hence $x = 10$ is the unique solution and $n = 1$.

Therefore, the correct answer is **D**.

Problems: https://live.poshenloh.com/past-contests/amc12/1965