1963 AMC 12 Solutions

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1. Which one of the following points is *not* on the graph of $y = \dfrac{x}{x+1}$?

A $(0,0)$

B $\left(-\dfrac{1}{2}, -1\right)$

C $\left(\dfrac{1}{2}, \dfrac{1}{3}\right)$

D $(-1,1)$

E $(-2,2)$

Solution:

The expression $\frac{x}{x+1}$ is undefined when $x = -1$. Therefore no point whose first coordinate is -1 can lie on its graph.

Thus, the correct answer is **D**.

2. Let $n = x - y^{x-y}$. Find n when $x = 2$ and $y = -2$.

- A -14
- B 0
- C 1
- D 18
- E 256

Solution:

Here $x - y = 2 - (-2) = 4$, so

$$n = 2 - (-2)^4 = 2 - 16 = -14.$$

Therefore, the correct answer is **A**.

3. If the reciprocal of $x + 1$ is $x - 1$, then x equals:

- A 0
- B 1
- C -1
- D ± 1
- E none of these

Solution:

The equation is $\frac{1}{x+1} = x - 1$. Hence $1 = x^2 - 1$, so $x = \pm\sqrt{2}$. Neither value is listed.

Thus, the correct answer is **E**.

4. For what value(s) of k does the pair of equations $y = x^2$ and $y = 3x + k$ have two identical solutions?

A $\frac{4}{9}$

B $-\frac{4}{9}$

C $\frac{9}{4}$

D $-\frac{9}{4}$

E $\frac{9}{4}$ or $-\frac{9}{4}$

Solution:

Intersections satisfy $x^2 - 3x - k = 0$. Two identical solutions require

$$(-3)^2 - 4(1)(-k) = 9 + 4k = 0,$$

giving $k = -\frac{9}{4}$.

Therefore, the correct answer is **D**.

5. If x and $\log_{10} x$ are real numbers and $\log_{10} x < 0$, then:

A $x < 0$

B $-1 < x < 1$

C $0 < x \leq 1$

D $-1 < x < 0$

E $0 < x < 1$

Solution:

The logarithm is real only for $x > 0$. Since base 10 is greater than 1, $\log_{10} x < 0 = \log_{10} 1$ implies $x < 1$.

Thus, the correct answer is **E**.

6. Triangle BAD is right-angled at B. On AD there is a point C for which $AC = CD$ and $AB = BC$. The magnitude of angle DAB, in degrees, is:

A $67\frac{1}{2}$

B 60

C 45

D 30

E $22\frac{1}{2}$

Solution:

Since C is the midpoint of hypotenuse AD, $AC = BC = CD$. Given $AB = BC$, triangle ABC is equilateral. Thus $\angle DAB = \angle CAB = 60^\circ$.

Therefore, the correct answer is **B**.

7. Given the four equations:

$$\begin{aligned}
&(1) \quad 3y - 2x = 12, \\
&(2) \quad -2x - 3y = 10, \\
&(3) \quad 3y + 2x = 12, \\
&(4) \quad 2y + 3x = 10.
\end{aligned}$$

The pair representing perpendicular lines is:

- A (1) and (4)
- B (1) and (3)
- C (1) and (2)
- D (2) and (4)
- E (2) and (3)

Solution:

Line (1) has slope $\frac{2}{3}$, and line (4) has slope $-\frac{3}{2}$. Their product is -1, so these two lines are perpendicular.

Thus, the correct answer is **A**.

8. The smallest positive integer x for which $1260x = N^3$, where N is an integer, is:

- A 1050
- B 1260
- C 1260^2
- D 7350
- E $44{,}100$

Solution:

Since $1260 = 2^2 \cdot 3^2 \cdot 5 \cdot 7$, the least multiplier making every exponent divisible by 3 is

$$2 \cdot 3 \cdot 5^2 \cdot 7^2 = 7350.$$

Thus, the correct answer is **D**.

9. In the expansion of $\left(a-\frac{1}{\sqrt{a}}\right)^{7}$ the coefficient of $a^{-\frac{1}{2}}$ is:

A -7

B 7

C -21

D 21

E 35

Solution:

The term using $-a^{-\frac{1}{2}}$ exactly r times has exponent $7-r-\frac{r}{2}=7-\frac{3r}{2}$. Setting this to $-\frac{1}{2}$ gives $r=5$. Its coefficient is $\binom{7}{5}(-1)^{5}=-21$.

Therefore, the correct answer is **C**.

10. Point P is taken interior to a square with side-length a and such that it is equally distant from two consecutive vertices and from the side opposite these vertices. If d represents the common distance, then d equals:

A $\frac{3a}{5}$

B $\frac{5a}{8}$

C $\frac{3a}{8}$

D $\frac{a\sqrt{2}}{2}$

E $\frac{a}{2}$

Solution:

Place the opposite side at height 0 and the two vertices at height a. Then $P = (\frac{a}{2}, d)$. Equality of the vertex distance and d gives

$$\left(\frac{a}{2}\right)^2 + (a-d)^2 = d^2,$$

so $\frac{5a^2}{4} = 2ad$ and $d = \frac{5a}{8}$.

Thus, the correct answer is **B**.

11. The arithmetic mean of a set of 50 numbers is 38. If two numbers of the set, namely 45 and 55, are discarded, the arithmetic mean of the remaining set of numbers is:

- A 38.5
- B 37.5
- C 37
- D 36.5
- E 36

Solution:

The original sum is $50 \cdot 38 = 1900$. After removing $45 + 55 = 100$, the remaining sum is 1800, so the new mean is $\frac{1800}{48} = 37.5$.

Thus, the correct answer is **B**.

12. Three vertices of parallelogram $PQRS$ are $P(-3,-2), Q(1,-5), R(9,1)$ with P and R diagonally opposite. The sum of the coordinates of vertex S is:

A 13

B 12

C 11

D 10

E 9

Solution:

Since $P+R=Q+S$,

$$\begin{aligned} S &= P+R-Q \\ &= (-3,-2)+(9,1)-(1,-5) \\ &= (5,4). \end{aligned}$$

The coordinate sum is $5+4=9.$

Therefore, the correct answer is **E**.

13. If $2^a + 2^b = 3^c + 3^d$, the number of integers a, b, c, d which can possibly be negative is, at most:

A 4

B 3

C 2

D 1

E 0

Solution:

In lowest terms, the left side can have only a power of 2 in its denominator, while the right side can have only a power of 3. Equality therefore forces both sides to be integers. If either c or d were negative, $3^c + 3^d$ would retain a factor 3 in its denominator. Thus $c, d \geq 0$.

Likewise, negative a or b can give an integer only in the exceptional case $a = b = -1$, whose sum is 1. But $3^c + 3^d \geq 2$. Hence $a, b \geq 0$ as well, so none can be negative.

Thus, the correct answer is **E**.

14. Given the equations $x^2 + kx + 6 = 0$ and $x^2 - kx + 6 = 0$. If, when the roots of the equations are suitably listed, each root of the second equation is 5 more than the corresponding root of the first equation, then k equals:

A 5

B -5

C 7

D -7

E none of these

Solution:

The first pair of roots has sum $-k$, while the second has sum k. Since both roots increase by $5, k = -k + 10$, so $k = 5.$

Therefore, the correct answer is **A**.

15. A circle is inscribed in an equilateral triangle, and a square is inscribed in the circle. The ratio of the area of the triangle to the area of the square is:

A $\sqrt{3} : 1$

B $\sqrt{3} : \sqrt{2}$

C $3\sqrt{3} : 2$

D $3 : \sqrt{2}$

E $3 : 2\sqrt{2}$

Solution:

If the circle has radius r, the equilateral triangle has side $2\sqrt{3}r$ and area $3\sqrt{3}r^2$. The square has side $r\sqrt{2}$ and area $2r^2$. The ratio is $3\sqrt{3} : 2$.

Thus, the correct answer is **C**.

16. Three numbers $a, b, c,$ none zero, form an arithmetic progression. Increasing a by 1 or increasing c by 2 results in a geometric progression. Then b equals:

- A 16
- B 14
- C 12
- D 10
- E 8

Solution:

The conditions give

$$2b = a + c,$$
$$b^2 = (a+1)c, \qquad b^2 = a(c+2).$$

Comparing the last two yields $c = 2a$. Thus $b = \frac{3a}{2}$, and $\frac{9a^2}{4} = 2a(a+1)$. Since $a \neq 0, a = 8,$ so $b = 12$.

Therefore, the correct answer is **C**.

17. The expression

$$\frac{\dfrac{a}{a+y}+\dfrac{y}{a-y}}{\dfrac{y}{a+y}-\dfrac{a}{a-y}},$$

a real, $a \neq 0,$ has the value -1 for:

A all but two real values of y

B only two real values of y

C all real values of y

D only one real value of y

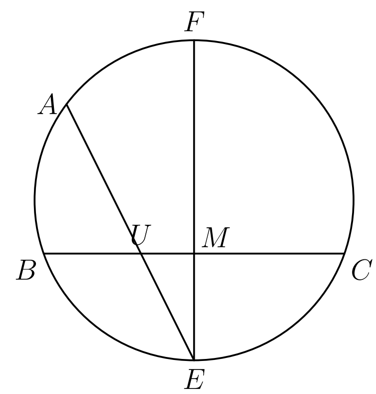
E no real values of y

Solution:

For $y \neq \pm a,$ the numerator simplifies to $\frac{a^2+y^2}{a^2-y^2}$ and the denominator to its negative. Their quotient is therefore $-1.$ The two values $y=a$ and $y=-a$ are undefined.

Thus, the correct answer is **A**.

18. Chord EF is the perpendicular bisector of chord BC, intersecting it in M. Between B and M point U is taken, and EU extended meets the circle in A. Then, for any selection of U, as described, triangle EUM is similar to triangle:



A EFA

B EFC

C ABM

D ABU

E FMC

Solution:

Since EF is a diameter, $\angle EAF = 90°$. Also $EF \perp BC$, so $\angle EMU = 90°$. Because E, U, A are collinear, the two triangles share the same acute angle at E. Hence $\triangle EUM \sim \triangle EFA$.

Therefore, the correct answer is **A**.

19. In counting n colored balls, some red and some black, it was found that 49 of the first 50 counted were red. Thereafter, 7 out of every 8 counted were red. If, in all, 90% or more of the balls counted were red, the maximum value of n is:

A 225

B 210

C 200

D 180

E 175

Solution:

The condition is

$$49 + \frac{7}{8}(n - 50) \geq \frac{9}{10}n.$$

Simplifying gives $\frac{21}{4} \geq \frac{n}{40}$, so $n \leq 210.$ The maximum is $210.$

Thus, the correct answer is **B**.

20. Two men at points R and S, 76 miles apart, set out at the same time to walk towards each other. The man at R walks uniformly at $4\frac{1}{2}$ miles per hour; the man at S walks at $3\frac{1}{4}$ miles per hour for the first hour, at $3\frac{3}{4}$ miles per hour for the second hour, and so on, in arithmetic progression. If the men meet x miles nearer R than S in an integral number of hours, then x is:

A 10

B 8

C 6

D 4

E 2

Solution:

In h hours the first man walks $\frac{9h}{2}$. The second walks an arithmetic-series total

$$\frac{h}{2}\left(2\cdot\frac{13}{4}+(h-1)\frac{1}{2}\right)$$
$$=3h+\frac{h^2}{4}.$$

Their distances sum to 76, giving $h^2+30h-304=0$, whose positive root is $h=8$. They walk 36 and 40 miles, so the meeting point is 4 miles nearer R than S.

Thus, the correct answer is **D**.

21. The expression $x^2 - y^2 - z^2 + 2yz + x + y - z$ has:

A no linear factor with integer coefficients and integer exponents

B the factor $-x + y + z$

C the factor $x - y - z + 1$

D the factor $x + y - z + 1$

E the factor $x - y + z + 1$

Solution:

Let $u = y - z$. The two groups factor as

$$x^2 - u^2 = (x+u)(x-u),$$
$$x + u = (x+u) \cdot 1.$$

Thus the whole expression is $(x+u)(x-u+1)$, and one factor is $x - y + z + 1$.

Therefore, the correct answer is **E**.

22. Acute-angled triangle ABC is inscribed in a circle with center at O; $\overset{\frown}{AB} = 120^\circ$ and $\overset{\frown}{BC} = 72^\circ$. A point E is taken in minor arc AC such that OE is perpendicular to AC. Then the ratio of the magnitudes of angles OBE and BAC is:

A $\frac{5}{18}$

B $\frac{2}{9}$

C $\frac{1}{4}$

D $\frac{1}{3}$

E $\frac{4}{9}$

Solution:

Minor arc AC has measure $360^\circ - 120^\circ - 72^\circ = 168^\circ$. Since $OE \perp AC, E$ bisects that arc, so $\angle BOE = 72^\circ + 84^\circ = 156^\circ$. Isosceles triangle OBE gives $\angle OBE = 12^\circ$, while $\angle BAC = \frac{72^\circ}{2} = 36^\circ$. Their ratio is $\frac{1}{3}$.

Thus, the correct answer is **D**.

23. A gives B as many cents as B has and C as many cents as C has. Similarly, B then gives A and C as many cents as each then has. C, similarly, then gives A and B as many cents as each then has. If each finally has 16 cents, with how many cents does A start?

A 24

B 26

C 28

D 30

E 32

Solution:

Reverse the transactions. Before C gives, the holdings are $(8, 8, 32)$. Before B gives, they are $(4, 28, 16)$. Before A gives, B and C must have had 14 and 8, while A had $4 + 14 + 8 = 26$.

Therefore, the correct answer is **B**.

24. Consider equations of the form $x^2 + bx + c = 0$. How many such equations have real roots and have coefficients b and c selected from the set of integers $\{1, 2, 3, 4, 5, 6\}$?

A 20

B 19

C 18

D 17

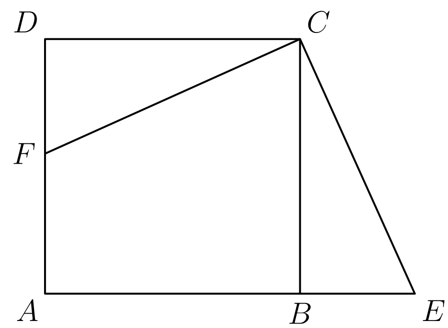
E 16

Solution:

For $c = 1, 2, 3, 4, 5, 6$, the condition $b^2 \geq 4c$ permits respectively $5, 4, 3, 3, 2, 2$ values of b in the given set. Their sum is $5 + 4 + 3 + 3 + 2 + 2 = 19$.

Thus, the correct answer is **B**.

25. Point F is taken in side AD of square $ABCD$. At C a perpendicular is drawn to CF, meeting AB extended at E. The area of $ABCD$ is 256 square inches and the area of triangle CEF is 200 square inches. Then the number of inches in BE is:



A 12

B 14

C 15

D 16

E 20

Solution:

The complementary acute angles and the equal square sides give $\triangle CDF \cong \triangle CBE$, hence $CF = CE$. Since $CF \perp CE$,

$$
\begin{aligned}
200 = [CEF] &= \frac{1}{2} CE \cdot CF \\
&= \frac{1}{2} CE^2,
\end{aligned}
$$

so $CE = 20$. The square side is 16, and right triangle CBE gives $BE = \sqrt{20^2 - 16^2} = 12$.

Therefore, the correct answer is **A**.

26. Form I. Consider the statements

$$\begin{aligned}&(1)\quad p \wedge \neg q \wedge r,\\&(2)\quad \neg p \wedge \neg q \wedge r,\\&(3)\quad p \wedge \neg q \wedge \neg r,\\&(4)\quad \neg p \wedge q \wedge r,\end{aligned}$$

where p, q, r are propositions. How many of these imply the truth of $(p \to q) \to r$?

Form II. Equivalently, test the four corresponding truth assignments against the statement "r is implied by the statement that p implies q."

A 0

B 1

C 2

D 3

E 4

Solution:

In cases (1) and (3), $p \to q$ is false, so the outer implication is true. In cases (2) and (4), $p \to q$ is true, but r is also true, so the outer implication is again true. All four statements imply it.

Thus, the correct answer is **E**.

27. Six straight lines are drawn in a plane with no two parallel and no three concurrent. The number of regions into which they divide the plane is:

- A 16
- B 20
- C 22
- D 24
- E 26

Solution:

Successive lines create $1, 2, \ldots, 6$ new regions. Thus the total is

$$1 + \sum_{k=1}^{6} k = 1 + 21 = 22.$$

Therefore, the correct answer is **C**.

28. Given the equation $3x^2 - 4x + k = 0$ with real roots. The value of k for which the product of the roots of the equation is a maximum is:

A $\frac{16}{9}$

B $\frac{16}{3}$

C $\frac{4}{9}$

D $\frac{4}{3}$

E $-\frac{4}{3}$

Solution:

Real roots require $16 - 12k \geq 0,$ so $k \leq \frac{4}{3}$. The root product is $\frac{k}{3}$, which increases with k. It is therefore largest at $k = \frac{4}{3}$.

Thus, the correct answer is **D**.

29. A particle projected vertically upward reaches, at the end of t seconds, an elevation of s feet where $s = 160t - 16t^2$. The highest elevation is:

A 800

B 640

C 400

D 320

E 160

Solution:

The vertex occurs at $t = -\frac{160}{2 \cdot -16} = 5$. Then

$$s(5) = 160(5) - 16(25) = 400.$$

Thus, the correct answer is **C**.

30. Let

$$F = \log \frac{1+x}{1-x}.$$

Find a new function G by replacing each x in F by

$$\frac{3x + x^3}{1 + 3x^2},$$

and simplify. The simplified expression G is equal to:

A $-F$

B F

C $3F$

D F^3

E $F^3 - F$

Solution:

For $u = \frac{3x+x^3}{1+3x^2}$,

$$\begin{aligned} \frac{1+u}{1-u} &= \frac{1 + 3x + 3x^2 + x^3}{1 - 3x + 3x^2 - x^3} \\ &= \left(\frac{1+x}{1-x}\right)^3. \end{aligned}$$

Therefore $G = \log(\frac{1+x}{1-x})^3$, so $G = 3F$.

Thus, the correct answer is **C**.

31. The number of solutions in positive integers of $2x + 3y = 763$ is:

A 255

B 254

C 128

D 127

E 0

Solution:

We need y odd so that $763 - 3y$ is even, and positivity gives $1 \leq y \leq 253.$ The odd values $1, 3, \ldots, 253$ number $\frac{253+1}{2} = 127.$

Therefore, the correct answer is **D**.

32. The dimensions of a rectangle R are a and b, $a < b$. It is required to obtain a rectangle with dimensions x and y, $x < a, y < a$, so that its perimeter is one-third that of R, and its area is one-third that of R. The number of such (different) rectangles is:

A 0

B 1

C 2

D 4

E infinitely many

Solution:

The conditions give $3(x+y) = a+b$ and $3xy = ab$. Dividing yields

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{a} + \frac{1}{b}.$$

But $x, y < a$ makes the left side greater than $\frac{2}{a}$, while $a < b$ makes the right side less than $\frac{2}{a}$. This is impossible.

Thus, the correct answer is **A**.

33. Given the line $y=\dfrac{3}{4}x+6$ and a line L parallel to the given line and 4 units from it. A possible equation for L is:

A $y=\dfrac{3}{4}x+1$

B $y=\dfrac{3}{4}x$

C $y=\dfrac{3}{4}x-\dfrac{2}{3}$

D $y=\dfrac{3}{4}x-1$

E $y=\dfrac{3}{4}x+2$

Solution:

The given line is $3x-4y+24=0$. A parallel line $y=\frac{3x}{4}+b$ is $3x-4y+4b=0$, so the distance is $\frac{|24-4b|}{5}$. Setting this equal to 4 gives $|6-b|=5$, hence $b=1$ or 11. Choice A is possible.

Thus, the correct answer is **A**.

34. In triangle ABC, side $a = \sqrt{3}$, side $b = \sqrt{3}$, and side $c > 3$. Let x be the largest number such that the magnitude, in degrees, of the angle opposite side c exceeds x. Then x equals:

A 150

B 120

C 105

D 90

E 60

Solution:

The law of cosines gives

$$\cos C = \frac{3+3-c^2}{2\sqrt{3}\sqrt{3}} = \frac{6-c^2}{6}.$$

Since $c > 3, \cos C < -\frac{1}{2}$, so $C > 120^\circ$. Values can approach 120° as c approaches 3 from above, so the largest guaranteed bound is 120.

Therefore, the correct answer is **B**.

35. The lengths of the sides of a triangle are integers, and its area is also an integer. One side is 21 and the perimeter is 48. The shortest side is:

A 8

B 10

C 12

D 14

E 16

Solution:

Let the other sides be x and $27-x$, with $x \le 27-x$. The semiperimeter is 24, so Heron's formula gives

$$\begin{aligned} K^2 &= 24 \cdot 3(24-x)(x-3) \\ &= 72(24-x)(x-3). \end{aligned}$$

The triangle inequalities give $4 \le x \le 13$. Checking these integers, the expression is not a square for $x = 4, \ldots, 9$, while at $x = 10$ it is $72 \cdot 14 \cdot 7 = 7056 = 84^2$. Thus the shortest side is 10.

Therefore, the correct answer is **B**.

36. A person starting with 64 cents and making 6 bets, wins three times and loses three times, the wins and losses occurring in random order. The chance for a win is equal to the chance for a loss. If each wager is for half the money remaining at the time of the bet, then the final result is:

- A a loss of $27¢$
- B a gain of $27¢$
- C a loss of $37¢$
- D neither a gain nor a loss
- E a gain or a loss depending upon the order in which the wins and losses occur

Solution:

After three wins and three losses the amount is

$$\begin{aligned} 64\left(\frac{3}{2}\right)^3\left(\frac{1}{2}\right)^3 &= 64 \cdot \frac{27}{64} \\ &= 27 \text{ cents.} \end{aligned}$$

The loss is $64 - 27 = 37$ cents, regardless of order.

Thus, the correct answer is **C**.

37. Given points $P_1, P_2, \ldots, P_7$ on a straight line, in the order stated (not necessarily evenly spaced). Let P be an arbitrarily selected point on the line and let s be the sum of the undirected lengths

$$PP_1,\ PP_2,\ \ldots,\ PP_7.$$

Then s is smallest if and only if the point P is:

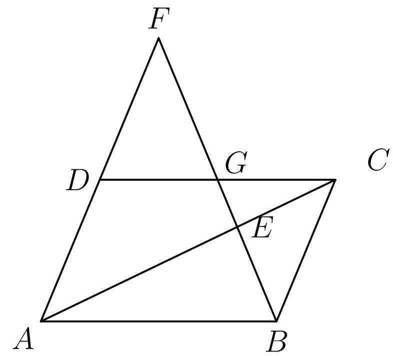
- A midway between P_1 and P_7
- B midway between P_2 and P_6
- C midway between P_3 and P_5
- D at P_4
- E at P_1

Solution:

For any pair P_i, P_{8-i}, the sum $PP_i + PP_{8-i}$ is minimized when P lies between them. The three such intervals all contain P_4. The remaining term PP_4 is uniquely minimized at $P = P_4$.

Thus, the correct answer is **D**.

38. Point F is taken on the extension of side AD of parallelogram $ABCD$. BF intersects diagonal AC at E and side DC at G. If $EF = 32$ and $GF = 24$, then BE equals:



A 4

B 8

C 10

D 12

E 16

Solution:

Use affine coordinates $A = 0, B = u, D = v, C = u + v$, and $F = tv$. A point on FB is $su + t(1 - s)v$. Intersecting DC gives $s_G = \frac{t-1}{t}$, while intersecting AC gives $s_E = \frac{t}{t+1}$. Therefore

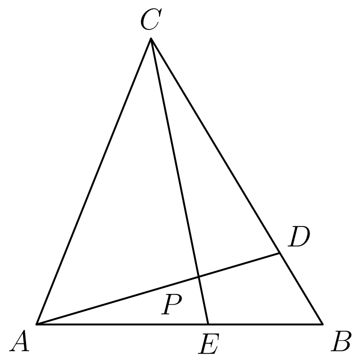
$$\frac{GF}{EF} = \frac{s_G}{s_E} = \frac{t^2 - 1}{t^2}$$
$$= \frac{24}{32} = \frac{3}{4},$$

so $t = 2$ and $s_E = \frac{2}{3}$. Thus $EF = (\frac{2}{3})FB = 32$, making $FB = 48$ and $BE = 16$.

Therefore, the correct answer is **E**.

39. In triangle ABC lines CE and AD are drawn so that $\dfrac{CD}{DB}=\dfrac{3}{1}$ and $\dfrac{AE}{EB}=\dfrac{3}{2}$.

Let $r=\dfrac{CP}{PE}$, where P is the intersection point of CE and AD. Then r equals:



A 3

B $\dfrac{3}{2}$

C 4

D 5

E $\dfrac{5}{2}$

Solution:

The ratio $AE:EB=3:2$ is represented by masses $m_A=2,\ m_B=3$. The ratio $CD:DB=3:1$ then gives $m_C=1$. Point E has mass $m_A+m_B=5$, so along cevian CE,

$$\frac{CP}{PE}=\frac{m_E}{m_C}=\frac{5}{1}=5.$$

Thus, the correct answer is **D**.

40. If x is a number satisfying the equation $\sqrt[3]{x+9} - \sqrt[3]{x-9} = 3$, then x^2 is between:

A 55 and 65

B 65 and 75

C 75 and 85

D 85 and 95

E 95 and 105

Solution:

Let $u = \sqrt[3]{x+9}$ and $v = \sqrt[3]{x-9}$. Then $u - v = 3$ and

$$\begin{aligned} 18 &= u^3 - v^3 \\ &= (u-v)(u^2+uv+v^2), \end{aligned}$$

so $u^2 + uv + v^2 = 6$. Comparing with $(u-v)^2 = 9$ gives $uv = -1$. Hence $(u+v)^2 = 5$. Cubing $2u = 3 \pm \sqrt{5}$ gives $x = \pm 4\sqrt{5}$, and either way $x^2 = 80$, between 75 and 85.

Therefore, the correct answer is **C**.

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