1960 AMC 12 Solutions

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1. If 2 is a solution (root) of $x^3 + hx + 10 = 0$, then h equals:

A 10

B 9

C 2

D -2

E -9

Solution:

Substituting the given root gives

$$2^3 + 2h + 10 = 0.$$

Thus $2h = -18$, so $h = -9$.

Therefore, the correct answer is **E**.

2. It takes 5 seconds for a clock to strike 6 o'clock beginning at $6{:}00$ o'clock precisely. If the strikings are uniformly spaced, how long, in seconds, does it take to strike 12 o'clock?

A $9\frac{1}{3}$

B 10

C 11

D $14\frac{2}{3}$

E none of these

Solution:

There are 5 equal intervals from the first of 6 strikes to the last, so each interval lasts 1 second. Twelve strikes contain 11 such intervals and therefore take 11 seconds.

Thus, the correct answer is **C**.

3. Applied to a bill for $\$10,000$ the difference between a discount of 40% and two successive discounts of 36% and 4%, expressed in dollars, is:

A 0

B 144

C 256

D 400

E 416

Solution:

A 40% discount is $\$4000$. The successive discounts total

$$3600 + 0.04(6400) = 3856.$$

Their difference is $4000 - 3856 = 144$ dollars.

Therefore, the correct answer is **B**.

4. Each of two angles of a triangle is 60° and the included side is 4 inches. The area of the triangle, in square inches, is:

- A $8\sqrt{3}$
- B 8
- C $4\sqrt{3}$
- D 4
- E $2\sqrt{3}$

Solution:

The third angle is also 60°, so the triangle is equilateral with side 4. Its area is

$$\frac{\sqrt{3}}{4}(4)^2 = 4\sqrt{3}.$$

Thus, the correct answer is **C**.

5. The number of distinct points common to the graphs of $x^2 + y^2 = 9$ and $y^2 = 9$ is:

A infinitely many

B four

C two

D one

E none

Solution:

Substitution gives $x^2 + 9 = 9$, so $x = 0$. Then $y^2 = 9$ gives $y = 3$ or $y = -3$. Hence there are two common points.

Therefore, the correct answer is **C**.

6. The circumference of a circle is 100 inches. The side of a square inscribed in this circle, expressed in inches, is:

A $\frac{25\sqrt{2}}{\pi}$

B $\frac{50\sqrt{2}}{\pi}$

C $\frac{100}{\pi}$

D $\frac{100\sqrt{2}}{\pi}$

E $50\sqrt{2}$

Solution:

The diameter is $\frac{100}{\pi}$. If the square has side s, its diagonal is $s\sqrt{2}$, so

$$s\sqrt{2} = \frac{100}{\pi}.$$

Therefore $s = \frac{50\sqrt{2}}{\pi}$.

Thus, the correct answer is **B**.

7. Circle I passes through the center of, and is tangent to, circle II. The area of circle I is 4 square inches. Then the area of circle II, in square inches, is:

A 8

B $8\sqrt{2}$

C $8\sqrt{\pi}$

D 16

E $16\sqrt{2}$

Solution:

Because circle I passes through the center of circle II, the distance between the centers equals the radius r of circle I. Internal tangency also makes that distance $R - r$, where R is the radius of circle II. Thus $R = 2r$, and the area is multiplied by $2^2 = 4.$ It is therefore $4 \cdot 4 = 16.$

Thus, the correct answer is **D**.

8. The number $2.5252525\ldots$ can be written as a fraction. When reduced to lowest terms the sum of the numerator and denominator of this fraction is:

A 7

B 29

C 141

D 349

E none of these

Solution:

We have

$$2.525252\ldots = 2 + \frac{52}{99} = \frac{250}{99}.$$

This fraction is already in lowest terms, and $250 + 99 = 349$.

Therefore, the correct answer is **D**.

9. The fraction

$$\frac{a^2+b^2-c^2+2ab}{a^2+c^2-b^2+2ac}$$

is (with suitable restrictions on the values of $a, b,$ and c):

A irreducible

B reducible to -1

C reducible to a polynomial of three terms

D reducible to $\frac{a-b+c}{a+b-c}$

E reducible to $\frac{a+b-c}{a-b+c}$

Solution:

Factoring gives

$$\begin{aligned} &a^2+b^2-c^2+2ab \\ &= (a+b-c)(a+b+c), \\ &a^2+c^2-b^2+2ac \\ &= (a-b+c)(a+b+c). \end{aligned}$$

Cancelling the common factor, under the stated suitable restrictions, leaves

$$\frac{a+b-c}{a-b+c}.$$

Thus, the correct answer is **E**.

10. Given the following six statements:

(1) All women are good drivers.

(2) Some women are good drivers.

(3) No men are good drivers.

(4) All men are bad drivers.

(5) At least one man is a bad driver.

(6) All men are good drivers.

The statement that negates statement (6) is:

A (1)

B (2)

C (3)

D (4)

E (5)

Solution:

The negation of “All men are good drivers” is “At least one man is not a good driver.” In the terminology of the choices, that is statement (5), “At least one man is a bad driver.”

Therefore, the correct answer is **E**.

11. For a given value of $k,$ the product of the roots of

$$x^2 - 3kx + 2k^2 - 1 = 0$$

is $7.$ The roots may be characterized as:

A integral and positive

B integral and negative

C rational, but not integral

D irrational

E imaginary

Solution:

By Vieta's formulas, $2k^2 - 1 = 7,$ so $k = \pm 2.$ The roots are $3 \pm \sqrt{2}$ when $k = 2,$ and $-3 \pm \sqrt{2}$ when $k = -2.$ In either case both roots are irrational.

Thus, the correct answer is **D**.

12. The locus of the centers of all circles of given radius a, in the same plane, passing through a fixed point, is:

A a point

B a straight line

C two straight lines

D a circle

E two circles

Solution:

A circle of radius a passes through the fixed point exactly when its center is distance a from that point. The locus of all such centers is a circle of radius a.

Therefore, the correct answer is **D**.

13. The polygon(s) formed by $y = 3x + 2$, $y = -3x + 2$, and $y = -2$, is (are):

A an equilateral triangle

B an isosceles triangle

C a right triangle

D a triangle and a trapezoid

E a quadrilateral

Solution:

The slanted lines meet at $(0, 2)$, while their intersections with $y = -2$ are $(-\frac{4}{3}, -2)$ and $(\frac{4}{3}, -2)$. The latter two points are symmetric about the y-axis, so the two slanted sides have equal length. The figure is an isosceles triangle.

Thus, the correct answer is **B**.

14. If a and b are real numbers, the equation $3x - 5 + a = bx + 1$ has a unique solution x [the symbol $a \neq 0$ means that a is different from zero]:

- A for all a and b
- B if $a \neq 2b$
- C if $a \neq 6$
- D if $b \neq 0$
- E if $b \neq 3$

Solution:

Rearranging gives

$$(3 - b)x = 6 - a.$$

This has a unique solution exactly when $3 - b \neq 0,$ or $b \neq 3.$

Thus, the correct answer is **E**.

15. Triangle I is equilateral with side A, perimeter P, area K, and circumradius R (radius of the circumscribed circle). Triangle II is equilateral with side a, perimeter p, area k, and circumradius r. If A is different from a, then:

A $P : p = R : r$ only sometimes

B $P : p = R : r$ always

C $P : p = K : k$ only sometimes

D $R : r = K : k$ always

E $R : r = K : k$ only sometimes

Solution:

For equilateral triangles, $P = 3A$ and $R = \frac{A}{\sqrt{3}}$, with analogous formulas for p and r. Therefore

$$P : p = A : a = R : r$$

for every such pair of triangles.

Thus, the correct answer is **B**.

16. In the numeration system with base 5, counting is as follows: $1, 2, 3, 4, 10, 11, 12, 13, 14, 20, \ldots$ The number whose description in the decimal system is 69, when described in the base 5 system, is a number with:

- A two consecutive digits
- B two non-consecutive digits
- C three consecutive digits
- D three non-consecutive digits
- E four digits

Solution:

Since

$$69 = 2 \cdot 25 + 3 \cdot 5 + 4,$$

its base-5 representation is 234_5. Its three digits $2, 3, 4$ are consecutive.

Therefore, the correct answer is **C**.

17. The formula $N = 8 \cdot 10^8 \cdot x^{-\frac{3}{2}}$ gives, for a certain group, the number of individuals whose income exceeds x dollars. The lowest income, in dollars, of the wealthiest 800 individuals is at least:

A 10^4

B 10^6

C 10^8

D 10^{12}

E 10^{16}

Solution:

At the cutoff,

$$800 = 8 \cdot 10^8 x^{-\frac{3}{2}}.$$

Hence $x^{\frac{3}{2}} = 10^6$, and raising both sides to the $\frac{2}{3}$ power gives $x = 10^4$.

Thus, the correct answer is **A**.

18. The pair of equations $3^{x+y} = 81$ and $81^{x-y} = 3$ has:

A no common solution

B the solution $x = 2, y = 2$

C the solution $x = 2\frac{1}{2}, y = 1\frac{1}{2}$

D a common solution in positive and negative integers

E none of these

Solution:

The first equation gives $x + y = 4$. The second gives

$$3^{4(x-y)} = 3,$$

so $x - y = \frac{1}{4}$. Therefore $x = \frac{17}{8}$ and $y = \frac{15}{8}$. This pair is not listed among the first four choices.

Thus, the correct answer is **E**.

19. Consider equation I: $x+y+z=46$, where $x, y,$ and z are positive integers, and equation II: $x+y+z+w=46$, where $x, y, z,$ and w are positive integers. Then:

A I can be solved in consecutive integers

B I can be solved in consecutive even integers

C II can be solved in consecutive integers

D II can be solved in consecutive even integers

E II can be solved in consecutive odd integers

Solution:

The sum of three consecutive integers is divisible by 3, so it cannot be 46. Four consecutive integers beginning with n have sum $4n+6$. Setting $4n+6=46$ gives $n=10$, and indeed

$$10+11+12+13=46.$$

Therefore, the correct answer is **C**.

20. The coefficient of x^7 in the expansion of

$$\left(\frac{x^2}{2} - \frac{2}{x}\right)^8$$

is:

A 56

B -56

C 14

D -14

E 0

Solution:

The term using k factors of $-\frac{2}{x}$ has exponent

$$2(8-k) - k = 16 - 3k.$$

To obtain x^7, we need $k = 3$. Its coefficient is

$$\binom{8}{3}\left(\frac{1}{2}\right)^5(-2)^3 = -14.$$

Thus, the correct answer is **D**.

21. The diagonal of square I is $a + b$. The perimeter of square II with twice the area of I is:

A $(a+b)^2$

B $\sqrt{2}(a+b)^2$

C $2(a+b)$

D $\sqrt{8}(a+b)$

E $4(a+b)$

Solution:

A square with diagonal $a + b$ has area $\frac{(a+b)^2}{2}$. Square II therefore has area $(a+b)^2$, so its side is $a + b$ and its perimeter is $4(a+b)$.

Thus, the correct answer is **E**.

22. The equality $(x+m)^2-(x+n)^2=(m-n)^2$, where m and n are unequal non-zero constants, is satisfied by $x=am+bn$ where:

A $a=0, b$ has a unique non-zero value

B $a=0, b$ has two non-zero values

C $b=0, a$ has a unique non-zero value

D $b=0, a$ has two non-zero values

E a and b each have a unique non-zero value

Solution:

Factoring and using $m \neq n$ gives

$$\begin{aligned}&(m-n)(2x+m+n)\\&=(m-n)^2.\end{aligned}$$

Hence $2x+m+n=m-n$, so $x=-n$. Thus $a=0$ and $b=-1$, a unique nonzero value.

Therefore, the correct answer is **A**.

23. The radius R of a cylindrical box is 8 inches, the height H is 3 inches. The volume $V = \pi R^2 H$ is to be increased by the same fixed positive amount when R is increased by x inches as when H is increased by x inches. This condition is satisfied by:

A no real value of x

B one integral value of x

C one rational, but not integral, value of x

D one irrational value of x

E two real values of x

Solution:

Increasing the radius changes the volume by

$$3\pi\left((8+x)^2 - 8^2\right),$$

while increasing the height changes it by $64\pi x$. Equating these gives

$$48x + 3x^2 = 64x,$$

so $x = 0$ or $x = \frac{16}{3}$. The increase must be positive, leaving one rational but nonintegral value.

Thus, the correct answer is **C**.

24. If $\log_{2x} 216 = x$, where x is real, then x is:

- A a non-square, non-cube integer
- B a non-square, non-cube, non-integral rational number
- C an irrational number
- D a perfect square
- E a perfect cube

Solution:

The equation is equivalent to

$$(2x)^x = 216.$$

Since $216 = 6^3$, $x = 3$ satisfies the equation. To see it is the only admissible real solution, note that the base requires $x > 0$ and $2x \neq 1$; on $0 < x < \frac{1}{2}$, $x \ln(2x) < 0$, while for $x > \frac{1}{2}$ the function $x \ln(2x)$ is strictly increasing. Thus $x = 3$, a non-square, non-cube integer.

Therefore, the correct answer is **A**.

25. Let m and n be any two odd numbers, with n less than m. The largest integer which divides all possible numbers of the form $m^2 - n^2$ is:

A 2

B 4

C 6

D 8

E 16

Solution:

We have

$$m^2 - n^2 = (m-n)(m+n).$$

Both factors are even, and one is divisible by 4, so every such difference is divisible by 8. Taking $m = 3, n = 1$ gives $m^2 - n^2 = 8$, proving that no larger integer always divides it.

Thus, the correct answer is **D**.

26. Find the set of x-values satisfying the inequality

$$\left|\frac{5-x}{3}\right| < 2.$$

[The symbol $|a|$ means $+a$ if a is positive, $-a$ if a is negative, 0 if a is zero. The notation $1 < a < 2$ means that a can have any value between 1 and 2, excluding 1 and 2.]

A $1 < x < 11$

B $-1 < x < 11$

C $x < 11$

D x>11

E $|x| < 6$

Solution:

The inequality is equivalent to

$$|5-x| < 6,$$

so $-6 < 5 - x < 6$. Subtracting 5 and reversing signs as needed gives

$$-1 < x < 11.$$

Therefore, the correct answer is **B**.

27. Let S be the sum of the interior angles of a polygon P for which each interior angle is $7\frac{1}{2}$ times the exterior angle at the same vertex. Then:

A $S = 2660^\circ$ and P may be regular

B $S = 2660^\circ$ and P is not regular

C $S = 2700^\circ$ and P is regular

D $S = 2700^\circ$ and P is not regular

E $S = 2700^\circ$ and P may or may not be regular

Solution:

If an exterior angle is e, then

$$e + \frac{15}{2}e = 180^\circ,$$

so $e = \frac{360^\circ}{17}$. Since the exterior angles total 360°, the polygon has 17 vertices and

$$S = (17 - 2)180^\circ = 2700^\circ.$$

All its angles are equal, but its sides need not be equal, so it may or may not be regular.

Thus, the correct answer is **E**.

28. The equation

$$x - \frac{7}{x-3} = 3 - \frac{7}{x-3}$$

has:

A infinitely many integral roots

B no root

C one integral root

D two equal integral roots

E two equal non-integral roots

Solution:

For $x \neq 3,$ subtracting the identical fractions from both sides leaves $x = 3.$ But $x = 3$ makes the original denominators zero, so it is not in the domain. The equation has no root.

Therefore, the correct answer is **B**.

29. Five times A's money added to B's money is more than $\$51.00$. Three times A's money minus B's money is $\$21.00$. If a represents A's money in dollars and b represents B's money in dollars, then:

A a>9, b>6

B a>9, $b < 6$

C a>9, $b = 6$

D a>9, but we can put no bounds on b

E $2a = 3b$

Solution:

From $3a - b = 21$, we have $b = 3a - 21$. Substitution into the inequality gives

$$5a + 3a - 21 > 51,$$

so $a > 9$. It follows that $b = 3a - 21 > 6$.

Thus, the correct answer is **A**.

30. Given the line $3x + 5y = 15$ and a point on this line equidistant from the coordinate axes. Such a point exists in:

A none of the quadrants

B quadrant I only

C quadrants I, II only

D quadrants $\mathrm{I}, \mathrm{II}, \mathrm{III}$ only

E each of the quadrants

Solution:

For $y = x,$ the line gives $8x = 15,$ producing a point in quadrant I. For $y = -x,$ it gives $-2x = 15,$ so $x < 0$ and $y > 0,$ producing a point in quadrant II. There are no other possibilities.

Therefore, the correct answer is **C**.

31. For x^2+2x+5 to be a factor of x^4+px^2+q, the values of p and q must be, respectively:

A $-2, 5$

B $5, 25$

C $10, 20$

D $6, 25$

E $14, 25$

Solution:

Modulo x^2+2x+5, we have $x^2=-2x-5$. Then

$$x^3 = 10 - x, \qquad x^4 = 12x + 5.$$

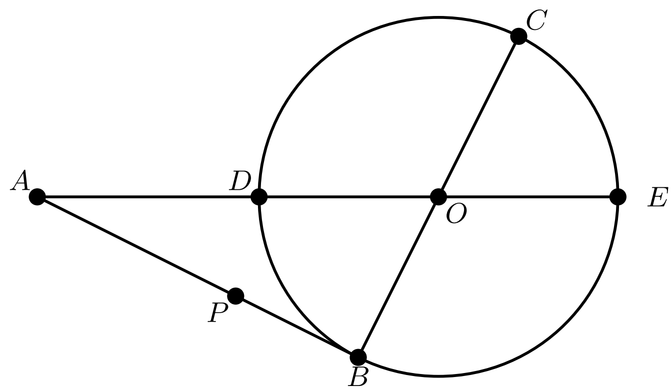
Thus

$$\begin{gathered} x^4+px^2+q \\ \equiv (12-2p)x + (5-5p+q). \end{gathered}$$

Both coefficients vanish when $p=6$ and $q=25$.

Therefore, the correct answer is **D**.

32. In this figure the center of the circle is O. $\overline{AB} \perp \overline{BC}$, $ADOE$ is a straight line, $\overline{AP} = \overline{AD}$, and $\overline{AB}$ has a length twice the radius. Then:



A $\overline{AP}^2 = \overline{PB} \cdot \overline{AB}$

B $\overline{AP} \cdot \overline{DO} = \overline{PB} \cdot \overline{AD}$

C $\overline{AB}^2 = \overline{AD} \cdot \overline{DE}$

D $\overline{AB} \cdot \overline{AD} = \overline{OB} \cdot \overline{AO}$

E none of these

Solution:

Let the radius be r. Since AB is tangent at B,

$$AB^2 = AD \cdot AE.$$

Write $AD = t$. Because the secant passes through the center, $AE = t + 2r$, while $AB = 2r$. Hence

$$t(t + 2r) = 4r^2.$$

Also $AP = AD = t$ and $PB = AB - AP = 2r - t$. The displayed equation rearranges to

$$t^2 = 2r(2r - t) = PB \cdot AB.$$

Therefore $AP^2 = PB \cdot AB$.

Thus, the correct answer is **A**.

33. You are given a sequence of 58 terms; each term has the form $P+n$ where P stands for the product $2 \cdot 3 \cdot 5 \cdots 61$ of all prime numbers (a prime number is a number divisible only by 1 and itself) less than or equal to 61, and n takes, successively, the values $2, 3, 4, \ldots, 59$. Let N be the number of primes appearing in this sequence. Then N is:

A 0

B 16

C 17

D 57

E 58

Solution:

For each n in the given range, choose a prime divisor p of n. Since $p \leq 59$, it is one of the factors of P. Therefore p divides $P+n$. Moreover $P+n > p$, so $P+n$ is composite. No term is prime, and $N = 0$.

Thus, the correct answer is **A**.

34. Two swimmers, at opposite ends of a 90-foot pool, start to swim the length of the pool, one at the rate of 3 feet per second, the other at 2 feet per second. They swim back and forth for 12 minutes. Allowing no loss of time at the turns, find the number of times they pass each other.

- A 24
- B 21
- C 20
- D 19
- E 18

Solution:

Reflecting the pool at its ends turns each back-and-forth path into straight motion. Interior meetings satisfy

$$3t = 90 - 2t + 180k,$$

so $t = 18 + 36k$. For $0 < t < 720$, the values $k = 0, 1, \ldots, 19$ give 20 meetings. The other congruence gives simultaneous arrivals at an endpoint, where the swimmers turn rather than pass each other.

Therefore, the correct answer is **C**.

35. From point P outside a circle, with a circumference of 10 units, a tangent is drawn. Also from P a secant is drawn dividing the circle into unequal arcs with lengths m and n. It is found that t, the length of the tangent, is the mean proportional between m and n. If m and t are integers, then t may have the following number of values:

- A zero
- B one
- C two
- D three
- E infinitely many

Solution:

The arc lengths satisfy $m+n=10$, while the mean-proportional condition gives

$$t^2 = mn = m(10-m).$$

For integral m from 1 through 9, the possible products, up to symmetry, are $9, 16, 21, 24, 25$. The last comes from equal arcs $m=n=5$ and is excluded. The remaining squares give $t=3$ and $t=4$, two values.

Thus, the correct answer is **C**.

36. Let s_1, s_2, s_3 be the respective sums of $n, 2n, 3n$ terms of the same arithmetic progression with a as the first term and d as the common difference. Let $R = s_3 - s_2 - s_1$. Then R is dependent on:

A a and d

B d and n

C a and n

D $a, d,$ and n

E neither a nor d nor n

Solution:

Using the arithmetic-series formula,

$$s_j = \frac{jn}{2}\big(2a + (jn-1)d\big).$$

In $s_3 - s_2 - s_1$, the coefficient of a is $3n - 2n - n = 0$. Simplifying the remaining terms gives

$$R = 2n^2 d.$$

Thus R depends on d and n, but not on a.

Therefore, the correct answer is **B**.

37. The base of a triangle is of length b, and the altitude is of length h. A rectangle of height x is inscribed in the triangle with the base of the rectangle in the base of the triangle. The area of the rectangle is:

A $\frac{bx}{h}(h-x)$

B $\frac{hx}{b}(b-x)$

C $\frac{bx}{h}(h-2x)$

D $x(b-x)$

E $x(h-x)$

Solution:

The cross-section parallel to the base at height x has width

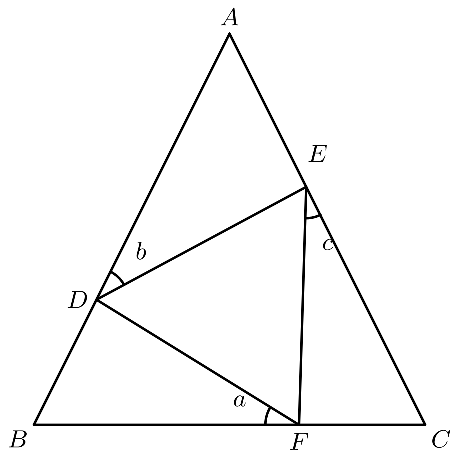
$$b\left(1-\frac{x}{h}\right)=\frac{b(h-x)}{h}$$

by similarity. Multiplying by the rectangle's height x gives

$$\frac{bx}{h}(h-x).$$

Thus, the correct answer is **A**.

38. In this diagram $\overline{AB}$ and $\overline{AC}$ are the equal sides of an isosceles triangle ABC, in which is inscribed equilateral triangle DEF. Designate angle BFD by a, angle ADE by b, and angle FEC by c. Then:



A $b = \dfrac{a+c}{2}$

B $b = \dfrac{a-c}{2}$

C $a = \dfrac{b-c}{2}$

D $a = \dfrac{b+c}{2}$

E none of these

Solution:

Let each base angle of ABC be θ. Measured from the direction BC, the line FD has direction $180^\circ - a$, so the equilateral condition makes DE have direction $60^\circ - a$. At D,

$$b = \theta - (60^\circ - a) = \theta - 60^\circ + a.$$

Similarly, angle chasing at E gives

$$c = 60^\circ + a - \theta.$$

Adding these equations yields $b + c = 2a$.

Therefore $a = \frac{b+c}{2}$, and the correct answer is **D**.

39. To satisfy the equation

$$\frac{a+b}{a} = \frac{b}{a+b},$$

a and b must be:

- A both rational
- B both real but not rational
- C both not real
- D one real, one not real
- E one real, one not real or both not real

Solution:

The denominators require $a \neq 0$ and $a + b \neq 0$. Setting $t = \frac{b}{a}$ and cross-multiplying gives

$$(1+t)^2 = t,$$

or $t^2 + t + 1 = 0$. Its discriminant is -3, so t is not real. Therefore a and b cannot both be real. Depending on the nonzero complex scale chosen for a, one can be real and the other nonreal, or both can be nonreal.

Thus, the correct answer is **E**.

40. Given right triangle ABC with legs $\overline{BC} = 3, \overline{AC} = 4$. Find the length of the shorter angle trisector from C to the hypotenuse:

A $\dfrac{32\sqrt{3} - 24}{13}$

B $\dfrac{12\sqrt{3} - 9}{13}$

C $6\sqrt{3} - 8$

D $\dfrac{5\sqrt{10}}{6}$

E $\dfrac{25}{6}$

Solution:

Place $C = (0,0), B = (3,0), A = (0,4)$. The hypotenuse has equation $\frac{x}{3} + \frac{y}{4} = 1$. The 30° trisector ray is $y = \frac{x}{\sqrt{3}}$. Substitution gives

$$x = \frac{12\sqrt{3}}{4\sqrt{3} + 3}.$$

Its length is $\frac{x}{\cos 30^\circ} = \frac{2x}{\sqrt{3}}$, hence

$$\frac{24}{4\sqrt{3} + 3} = \frac{32\sqrt{3} - 24}{13}.$$

The 60° ray gives the longer trisector.

Thus, the correct answer is **A**.

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