1959 AMC 12 Solutions

Typeset by: LIVE by Po-Shen Loh
https://live.poshenloh.com/past-contests/amc12/1959/solutions

1. Each edge of a cube is increased by 50%. The percent of increase of the surface area of the cube is:

   A 50

   B 125

   C 150

   D 300

   E 750

Solution:

If the original edge length is s, the new edge length is $1.5s$. Thus the surface area is multiplied by

$$(1.5)^2 = 2.25.$$

The increase is 1.25 times the original area, or 125%.

Thus, the correct answer is **B**.

2. Through a point P inside triangle ABC a line is drawn parallel to the base $\overline{AB}$, dividing the triangle into two equal areas. If the altitude to $\overline{AB}$ has length 1, then the distance from P to $\overline{AB}$ is:

A $\frac{1}{2}$

B $\frac{1}{4}$

C $2-\sqrt{2}$

D $\frac{2-\sqrt{2}}{2}$

E $\frac{2+\sqrt{2}}{8}$

Solution:

Let x be the requested distance. The triangle above the parallel line is similar to the original triangle, with linear ratio $1-x$. Its area is half the original area, so

$$(1-x)^2=\frac{1}{2}.$$

Since $0<x<1$, we have $1-x=\frac{1}{\sqrt{2}}$, and therefore

$$x=1-\frac{1}{\sqrt{2}}=\frac{2-\sqrt{2}}{2}.$$

Therefore, the correct answer is **D**.

3. If the diagonals of a quadrilateral are perpendicular to each other, the figure would always be included under the general classification:

- A rhombus
- B rectangles
- C square
- D isosceles trapezoid
- E none of these

Solution:

A kite can have perpendicular diagonals without being a rhombus, rectangle, square, or isosceles trapezoid. Therefore perpendicular diagonals alone do not force any of the first four classifications.

Thus, the correct answer is **E**.

4. If 78 is divided into three parts which are proportional to $1, \frac{1}{3}, \frac{1}{6}$, the middle part is:

- A $9\frac{1}{3}$
- B 13
- C $17\frac{1}{3}$
- D $18\frac{1}{3}$
- E 26

Solution:

Let the parts be x, $\frac{x}{3}$, and $\frac{x}{6}$. Then

$$x + \frac{x}{3} + \frac{x}{6} = \frac{3}{2}x = 78,$$

so $x = 52$. The middle part is

$$\frac{x}{3} = \frac{52}{3} = 17\frac{1}{3}.$$

Thus, the correct answer is **C**.

5. The value of $(256)^{0.16} \cdot (256)^{0.09}$ is:

A 4

B 16

C 64

D 256.25

E -16

Solution:

Using the product rule for powers,

$$\begin{aligned} 256^{0.16} \cdot 256^{0.09} &= 256^{0.25} \\ &= 256^{\frac{1}{4}} = 4. \end{aligned}$$

Therefore, the correct answer is **A**.

6. Given the true statement: If a quadrilateral is a square, then it is a rectangle. It follows that, of the converse and the inverse of this true statement:

- A. only the converse is true
- B. only the inverse is true
- C. both are true
- D. neither is true
- E. the inverse is true, but the converse is sometimes true

Solution:

The converse says that every rectangle is a square, which is false. The inverse says that every quadrilateral that is not a square is not a rectangle, which is also false because a nonsquare rectangle is a counterexample.

Thus neither statement is true, and the correct answer is **D**.

7. The sides of a right triangle are $a, a+d$, and $a+2d$, with a and d both positive. The ratio of a to d is:

A $1{:}3$

B $1{:}4$

C $2{:}1$

D $3{:}1$

E $3{:}4$

Solution:

The Pythagorean theorem gives

$$a^2+(a+d)^2=(a+2d)^2.$$

Simplifying,

$$\begin{aligned} a^2-2ad-3d^2&=0,\\ (a-3d)(a+d)&=0. \end{aligned}$$

Positivity excludes $a=-d$, so $a=3d$.

Thus the ratio is $3:1$, and the correct answer is **D**.

8. The value of $x^2 - 6x + 13$ can never be less than:

A 4

B 4.5

C 5

D 7

E 13

Solution:

Completing the square,

$$x^2 - 6x + 13 = (x-3)^2 + 4.$$

This expression is at least $4,$ with equality at $x = 3.$

Thus, the correct answer is **A**.

9. A farmer divides his herd of n cows among his four sons so that one son gets one-half the herd, a second son one-fourth, a third son one-fifth, and the fourth son 7 cows. Then n is:

A 80

B 100

C 140

D 180

E 240

Solution:

The first three sons receive

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{5}=\frac{19}{20}$$

of the herd. Thus the remaining $\frac{1}{20}$ of the herd is 7, so $n=140.$

Therefore, the correct answer is **C**.

10. In triangle ABC, with $\overline{AB} = \overline{AC} = 3.6$, a point D is taken on $\overline{AB}$ at a distance 1.2 from A. Point D is joined to point E in the prolongation of $\overline{AC}$ so that triangle AED is equal in area to triangle ABC. Then $\overline{AE}$ equals:

A 4.8

B 5.4

C 7.2

D 10.8

E 12.6

Solution:

Triangles AED and ABC use bases on the same line AC. Since

$$\frac{AD}{AB} = \frac{1.2}{3.6} = \frac{1}{3},$$

the perpendicular distance from D to line AC is one-third the corresponding distance from B. Equality of areas therefore requires

$$AE\left(\frac{h}{3}\right) = AC(h),$$

so $AE = 3AC = 10.8.$

Thus, the correct answer is **D**.

11. The logarithm of 0.0625 to the base 2 is:

- A 0.025
- B 0.25
- C 5
- D -4
- E -2

Solution:

Since

$$0.0625 = \frac{1}{16} = 2^{-4},$$

we have $\log_2(0.0625) = -4$.

Therefore, the correct answer is **D**.

12. By adding the same constant to each of $20, 50, 100$ a geometric progression results. The common ratio is:

A $\frac{5}{3}$

B $\frac{4}{3}$

C $\frac{3}{2}$

D $\frac{1}{2}$

E $\frac{1}{3}$

Solution:

The geometric-progression condition is

$$(50+c)^2 = (20+c)(100+c).$$

Expanding and cancelling c^2 gives $500 = 20c$, so $c = 25$. The common ratio is

$$\frac{50+25}{20+25} = \frac{75}{45} = \frac{5}{3}.$$

Thus, the correct answer is **A**.

13. The arithmetic mean (average) of a set of 50 numbers is $38.$ If two numbers, namely, 45 and $55,$ are discarded, the mean of the remaining set of numbers is:

A 36.5

B 37

C 37.2

D 37.5

E 37.52

Solution:

The original sum is $50 \cdot 38 = 1900.$ After removing 45 and $55,$ the remaining sum is $1800,$ so the new mean is

$$\frac{1800}{48} = 37.5.$$

Therefore, the correct answer is **D**.

14. Given the set S whose elements are zero and the even integers, positive and negative. Of the five operations applied to any pair of elements: (1) addition, (2) subtraction, (3) multiplication, (4) division, (5) finding the arithmetic mean (average), those operations that yield only elements of S are:

A all

B $1, 2, 3, 4$

C $1, 2, 3, 5$

D $1, 2, 3$

E $1, 3, 5$

Solution:

Even integers are closed under addition, subtraction, and multiplication. Division fails, since $\frac{2}{4} = \frac{1}{2} \notin S$. Averaging fails, since the mean of 0 and 2 is $1 \notin S$.

Thus only operations $1, 2,$ and 3 always work, and the correct answer is **D**.

15. In a right triangle the square of the hypotenuse is equal to twice the product of the legs. One of the acute angles of the triangle is:

A 15°

B 30°

C 45°

D 60°

E 75°

Solution:

If the legs are a, b and the hypotenuse is c, then

$$a^2 + b^2 = c^2 = 2ab.$$

Hence $(a - b)^2 = 0$, so the legs are equal. The right triangle is isosceles and its acute angles are 45°.

Thus, the correct answer is **C**.

16. The expression

$$\frac{x^2-3x+2}{x^2-5x+6} \div \frac{x^2-5x+4}{x^2-7x+12},$$

when simplified, is:

A $\frac{(x-1)(x-6)}{(x-3)(x-4)}$

B $\frac{x+3}{x-3}$

C $\frac{x+1}{x-1}$

D 1

E 2

Solution:

Factoring and multiplying by the reciprocal gives

$$\frac{(x-1)(x-2)}{(x-2)(x-3)} \cdot \frac{(x-3)(x-4)}{(x-1)(x-4)} = 1$$

for every value in the domain of the original expression.

Therefore, the correct answer is **D**.

17. If $y = a + \dfrac{b}{x}$, where a and b are constants, and if $y = 1$ when $x = -1$, and $y = 5$ when $x = -5$, then $a + b$ equals:

A -1

B 0

C 1

D 10

E 11

Solution:

The two conditions give

$$a - b = 1, \qquad a - \frac{b}{5} = 5.$$

Subtracting yields $\frac{4b}{5} = 4$, so $b = 5$ and $a = 6$. Therefore $a + b = 11$.

Thus, the correct answer is **E**.

18. The arithmetic mean (average) of the first n positive integers is:

A $\frac{n}{2}$

B $\frac{n^2}{2}$

C n

D $\frac{n-1}{2}$

E $\frac{n+1}{2}$

Solution:

The sum of the first n positive integers is $\frac{n(n+1)}{2}$. Dividing by n gives the mean

$$\frac{n+1}{2}.$$

Therefore, the correct answer is **E**.

19. With the use of three different weights, namely, 1 lb., 3 lb., and 9 lb., how many objects of different weights can be weighed, if the objects to be weighed and the given weights may be placed in either pan of the scale?

A 15

B 13

C 11

D 9

E 7

Solution:

Using coefficients $-1, 0, 1$ for the weights $1, 3, 9$, balanced ternary represents every integer weight from

$$1 \quad \text{through} \quad 1+3+9=13.$$

Thus there are 13 different positive object weights that can be measured.

Therefore, the correct answer is **B**.

20. It is given that x varies directly as y and inversely as the square of z, and that $x = 10$ when $y = 4$ and $z = 14$. Then, when $y = 16$ and $z = 7$, x equals:

- A 180
- B 160
- C 154
- D 140
- E 120

Solution:

Because $x = \frac{ky}{z^2}$,

$$\frac{x_{\text{new}}}{10} = \frac{16}{4}\left(\frac{14}{7}\right)^2 = 4 \cdot 4 = 16.$$

Hence $x_{\text{new}} = 160.$

Thus, the correct answer is **B**.

21. If p is the perimeter of an equilateral triangle inscribed in a circle, the area of the circle is:

A $\frac{\pi p^2}{3}$

B $\frac{\pi p^2}{9}$

C $\frac{\pi p^2}{27}$

D $\frac{\pi p^2}{81}$

E $\frac{\pi p^2 \sqrt{3}}{27}$

Solution:

The side length is $\frac{p}{3}$, so the circumradius is

$$R = \frac{\frac{p}{3}}{\sqrt{3}} = \frac{p}{3\sqrt{3}}.$$

Therefore the circle's area is

$$\pi R^2 = \frac{\pi p^2}{27}.$$

Thus, the correct answer is **C**.

22. The line joining the midpoints of the diagonals of a trapezoid has length $3.$ If the longer base is $97,$ then the shorter base is:

- A 94
- B 92
- C 91
- D 90
- E 89

Solution:

If the shorter base is $b,$ the diagonal-midpoint segment has length half the difference of the bases:

$$\frac{97-b}{2}=3.$$

Thus $97-b=6$ and $b=91.$

Therefore, the correct answer is **C**.

23. The set of solutions for the equation $\log_{10}(a^2 - 15a) = 2$ consists of:

A two integers

B one integer and one fraction

C two irrational numbers

D two non-real numbers

E no numbers, that is, the set is empty

Solution:

The equation is equivalent to

$$a^2 - 15a = 100,$$

or $(a - 20)(a + 5) = 0$. Thus $a = 20$ or $a = -5$. In both cases the logarithm's argument is 100, so both solutions are valid integers.

Therefore, the correct answer is **A**.

24. A chemist has m ounces of salt water that is $m\%$ salt. How many ounces of salt must he add to make a solution that is $2m\%$ salt?

A $\dfrac{m}{100+m}$

B $\dfrac{2m}{100-2m}$

C $\dfrac{m^2}{100-2m}$

D $\dfrac{m^2}{100+2m}$

E $\dfrac{2m}{100+m}$

Solution:

Let x ounces of salt be added. The concentration equation is

$$\frac{\frac{m^2}{100}+x}{m+x}=\frac{2m}{100}.$$

Multiplying through and collecting the x-terms gives

$$x(100-2m)=m^2,$$

so $x=\frac{m^2}{100-2m}$.

Therefore, the correct answer is **C**.

25. The symbol $|a|$ means $+a$ if a is greater than or equal to zero, and $-a$ if a is less than or equal to zero; the symbol $<$ means "less than"; the symbol $>$ means "greater than." The set of values x satisfying the inequality $|3-x|<4$ consists of all x such that:

A $x^2 < 49$

B $x^2 > 1$

C $1 < x^2 < 49$

D $-1 < x < 7$

E $-7 < x < 1$

Solution:

The absolute-value inequality is equivalent to

$$-4 < 3 - x < 4.$$

Subtracting 3 and multiplying by -1 reverses the inequalities, giving

$$-1 < x < 7.$$

Thus, the correct answer is **D**.

26. The base of an isosceles triangle is $\sqrt{2}$. The medians to the legs intersect each other at right angles. The area of the triangle is:

A 1.5

B 2

C 2.5

D 3.5

E 4

Solution:

Put the base endpoints at $A = (-\frac{\sqrt{2}}{2}, 0)$ and $B = (\frac{\sqrt{2}}{2}, 0)$, and let the third vertex be $C = (0, h)$. The median from A has direction

$$\left(\frac{3\sqrt{2}}{4}, \frac{h}{2}\right),$$

while the median from B has direction

$$\left(-\frac{3\sqrt{2}}{4}, \frac{h}{2}\right).$$

Their dot product is zero, so

$$-\frac{9}{8} + \frac{h^2}{4} = 0,$$

and $h = \frac{3\sqrt{2}}{2}$. Thus the area is

$$\frac{1}{2} \cdot \sqrt{2} \cdot \frac{3\sqrt{2}}{2} = \frac{3}{2}.$$

Therefore, the correct answer is **A**.

27. Which one of the following statements is not true for the equation

$$ix^2 - x + 2i = 0,$$

where $i = \sqrt{-1}$?

A The sum of the roots is 2

B The discriminant is 9

C The roots are imaginary

D The roots can be found by using the quadratic formula

E The roots can be found by factoring, using imaginary numbers

Solution:

By Vieta's formula, the sum of the roots is

$$-\frac{-1}{i} = \frac{1}{i} = -i,$$

not 2. Also the discriminant is

$$(-1)^2 - 4(i)(2i) = 1 - 8i^2 = 9.$$

The quadratic formula gives the imaginary roots $-i$ and $-2i$, so the remaining statements are true.

Thus, the correct answer is **A**.

28. In triangle $ABC, \overline{AL}$ bisects angle A and $\overline{CM}$ bisects angle C. Points L and M are on $\overline{BC}$ and $\overline{AB}$, respectively. The sides of triangle ABC are a, b, and c. Then $\frac{AM}{MB} = k\frac{CL}{LB}$, where k is:

A 1

B $\frac{bc}{a^2}$

C $\frac{a^2}{bc}$

D $\frac{c}{b}$

E $\frac{c}{a}$

Solution:

By the angle bisector theorem,

$$\frac{AM}{MB} = \frac{AC}{CB} = \frac{b}{a},$$
$$\frac{CL}{LB} = \frac{AC}{AB} = \frac{b}{c}.$$

Therefore

$$k = \frac{\frac{b}{a}}{\frac{b}{c}} = \frac{c}{a}.$$

Thus, the correct answer is **E**.

29. On an examination of n questions a student answers correctly 15 of the first $20.$ Of the remaining questions he answers one third correctly. All the questions have the same credit. If the student's mark is $50\%,$ how many different values of n can there be?

A 4

B 3

C 2

D 1

E the problem cannot be solved

Solution:

The score condition gives

$$15 + \frac{n-20}{3} = \frac{n}{2}.$$

Multiplying by 6 yields $90 + 2n - 40 = 3n,$ so $n = 50.$ This value also makes the number of remaining correct answers an integer. Hence there is exactly one possible value of $n.$

Therefore, the correct answer is **D**.

30. A can run around a circular track in 40 seconds. B, running in the opposite direction, meets A every 15 seconds. What is B's time to run around the track, expressed in seconds?

A $12\frac{1}{2}$

B 24

C 25

D $27\frac{1}{2}$

E 55

Solution:

If B's lap time is t, their relative speed is

$$\frac{1}{40} + \frac{1}{t} = \frac{1}{15}.$$

Thus $\frac{1}{t} = \frac{1}{24}$, so $t = 24$ seconds.

Therefore, the correct answer is **B**.

31. A square, with an area of $40,$ is inscribed in a semicircle. The area of a square that could be inscribed in the entire circle with the same radius is:

- A 80
- B 100
- C 120
- D 160
- E 200

Solution:

Let the semicircle have radius r and the square have side $s,$ where $s^2 = 40.$ A top vertex of the square is $\frac{s}{2}$ horizontally and s vertically from the center, so

$$r^2 = s^2 + \left(\frac{s}{2}\right)^2 = \frac{5}{4}s^2 = 50.$$

A square inscribed in the full circle has diagonal $2r,$ hence area $2r^2 = 100.$

Thus, the correct answer is **B**.

32. The length l of a tangent, drawn from a point A to a circle, is $\frac{4}{3}$ of the radius r. The (shortest) distance from A to the circle is:

A $\frac{1}{2}r$

B r

C $\frac{1}{2}l$

D $\frac{2}{3}l$

E a value between r and l

Solution:

If O is the center and T the point of tangency, then $OT \perp AT$. Hence

$$\begin{aligned} AO &= \sqrt{r^2 + l^2} \\ &= \sqrt{r^2 + \frac{16r^2}{9}} \\ &= \frac{5r}{3}. \end{aligned}$$

The shortest distance from A to the circle is $AO - r = \frac{2r}{3}$. Since $l = \frac{4r}{3}$, this distance equals $\frac{l}{2}$.

Therefore, the correct answer is **C**.

33. A harmonic progression is a sequence of numbers such that their reciprocals are in arithmetic progression. Let S_n represent the sum of the first n terms of the harmonic progression; for example, S_3 represents the sum of the first three terms. If the first three terms of a harmonic progression are $3, 4, 6,$ then:

A $S_4 = 20$

B $S_4 = 25$

C $S_5 = 49$

D $S_6 = 49$

E $S_2 = \frac{1}{2} S_4$

Solution:

The reciprocals begin

$$\frac{1}{3}, \quad \frac{1}{4}, \quad \frac{1}{6},$$

with common difference $-\frac{1}{12}$. The next reciprocal is $\frac{1}{12}$, so the fourth term is $12.$ Therefore

$$S_4 = 3 + 4 + 6 + 12 = 25.$$

Thus, the correct answer is **B**.

34. Let the roots of $x^2 - 3x + 1 = 0$ be r and s. Then the expression $r^2 + s^2$ is:

A a positive integer

B a positive fraction greater than 1

C a positive fraction less than 1

D an irrational number

E an imaginary number

Solution:

Vieta's formulas give $r + s = 3$ and $rs = 1$. Therefore

$$\begin{aligned} r^2 + s^2 &= (r+s)^2 - 2rs \\ &= 9 - 2 = 7, \end{aligned}$$

which is a positive integer.

Thus, the correct answer is **A**.

35. The symbol $\geq$ means "greater than or equal to"; the symbol $\leq$ means "less than or equal to." In the equation $(x-m)^2-(x-n)^2=(m-n)^2$, m is a fixed positive number, and n is a fixed negative number. The set of values x satisfying the equation is:

A $x \geq 0$

B $x \leq n$

C $x = 0$

D the set of all real numbers

E none of these

Solution:

Factoring the left side gives

$$(n-m)(2x-m-n)$$
$$=(m-n)^2.$$

Since $m \neq n$, division by $n-m$ yields $2x-m-n=n-m$, so $x=n$. This single negative value is not any of choices A through D.

Therefore, the correct answer is **E**.

36. The base of a triangle is 80, and one of the base angles is 60°. The sum of the lengths of the other two sides is 90. The shortest side is:

A 45

B 40

C 36

D 17

E 12

Solution:

Let the side adjacent to the 60° base angle be x, and let the opposite side be $90 - x$. The law of cosines gives

$$(90 - x)^2 = 80^2 + x^2 \\ - 2(80)(x)\cos 60^\circ.$$

Simplifying yields $8100 - 180x = 6400 - 80x$, so $x = 17$. The side lengths are $17,\ 73$, and 80, and the shortest is 17.

Thus, the correct answer is **D**.

37. When simplified the product $\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{5}\right)\cdots\left(1-\frac{1}{n}\right)$ becomes:

A $\frac{1}{n}$

B $\frac{2}{n}$

C $\frac{2(n-1)}{n}$

D $\frac{2}{n(n+1)}$

E $\frac{3}{n(n+1)}$

Solution:

The product telescopes:

$$\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{4}{5}\cdots\frac{n-1}{n}=\frac{2}{n}.$$

Therefore, the correct answer is **B**.

38. If $4x + \sqrt{2x} = 1$, then x :

A is an integer

B is fractional

C is irrational

D is imaginary

E may have two different values

Solution:

Let $u = \sqrt{2x} \geq 0$. Since $x = \frac{u^2}{2}$, the equation becomes

$$2u^2 + u = 1,$$

or $(2u-1)(u+1) = 0$. Thus $u = \frac{1}{2}$ and $x = \frac{u^2}{2} = \frac{1}{8}$. The other quadratic root is inadmissible because $u \geq 0$. Hence x is fractional.

Therefore, the correct answer is **B**.

39. Let S be the sum of the first nine terms of the sequence

$$x+a, \quad x^2+2a, \\ x^3+3a, \quad \ldots.$$

Then S equals:

A $\dfrac{50a+x+x^8}{x+1}$

B $50a-\dfrac{x+x^{10}}{x-1}$

C $\dfrac{x^9-1}{x+1}+45a$

D $\dfrac{x^{10}-x}{x-1}+45a$

E $\dfrac{x^{11}-x}{x-1}+45a$

Solution:

Adding the first nine terms gives

$$S=(x+x^2+\cdots+x^9) \\ +(1+2+\cdots+9)a.$$

Therefore

$$S=\frac{x^{10}-x}{x-1}+45a.$$

The expression is understood by continuity at $x=1$, where both forms equal $9+45a$.

Thus, the correct answer is **D**.

40. In triangle $ABC, \overline{BD}$ is a median. $\overline{CF}$ intersects $\overline{BD}$ at E so that $BE = ED$. Point F is on $\overline{AB}$. Then, if $BF = 5, BA$ equals:

- A 10
- B 12
- C 15
- D 20
- E none of these

Solution:

Use vectors with $A = \mathbf{0},\ B = \mathbf{b}$, and $C = \mathbf{c}$. Since $D = \frac{\mathbf{c}}{2}$ and E is the midpoint of BD,

$$E = \frac{1}{2}\mathbf{b} + \frac{1}{4}\mathbf{c}.$$

A point F on AB has the form $F = t\mathbf{b}$. Since E lies on CF, comparison of the $\mathbf{c}$-coefficient shows that $E = \frac{1}{4}C + \frac{3}{4}F$. Hence

$$\frac{3}{4}t = \frac{1}{2},$$

so $t = \frac{2}{3}$. Thus $\frac{AF}{AB} = \frac{2}{3}$, and $\frac{BF}{BA} = \frac{1}{3}$. Since $BF = 5, BA = 15.$

Therefore, the correct answer is **C**.

41. On the same side of a straight line three circles are drawn as follows: a circle with a radius of 4 inches is tangent to the line, the other two circles are equal, and each is tangent to the line and to the other two circles. The radius of the equal circles is:

- A 24
- B 20
- C 18
- D 16
- E 12

Solution:

Let the equal circles have radius R. Their centers are $2R$ apart, so the center of the radius-4 circle lies midway between them. The horizontal and vertical separations between its center and either large center are R and $R-4$. Tangency gives

$$R^2+(R-4)^2=(R+4)^2.$$

Simplifying yields $R^2-16R=0$, and positivity gives $R=16$.

Thus, the correct answer is **D**.

42. Given three positive integers a, b, and c. Their greatest common divisor is D; their least common multiple is M. Then, which two of the following statements are true?

(1) The product MD cannot be less than abc.

(2) The product MD cannot be greater than abc.

(3) MD equals abc if and only if a, b, c are each prime.

(4) MD equals abc if and only if a, b, c are relatively prime in pairs. (This means: no two have a common factor greater than 1.)

A $1, 2$

B $1, 3$

C $1, 4$

D $2, 3$

E $2, 4$

Solution:

For any prime, let its exponents in a, b, c be $u \leq v \leq w$. Its exponent in MD is $u + w$, while its exponent in abc is $u + v + w$. Thus $MD \leq abc$, proving statement (2). Equality holds exactly when $v = 0$ for every prime, meaning no prime divides two of a, b, c. That is precisely pairwise relative primality, proving statement (4).

Therefore, the correct answer is **E**.

43. The sides of a triangle are $25, 39,$ and $40.$ The diameter of the circumscribed circle is:

A $\frac{133}{3}$

B $\frac{125}{3}$

C 42

D 41

E 40

Solution:

The semiperimeter is $52,$ so Heron's formula gives

$$K = \sqrt{52 \cdot 27 \cdot 13 \cdot 12} = 468.$$

If R is the circumradius, then

$$R = \frac{25 \cdot 39 \cdot 40}{4 \cdot 468} = \frac{125}{6}.$$

Hence the diameter is $2R = \frac{125}{3}.$

Thus, the correct answer is **B**.

44. The roots of $x^2 + bx + c = 0$ are both real and greater than 1. Let $s = b + c + 1$. Then s :

- A may be less than zero
- B may be equal to zero
- C must be greater than zero
- D must be less than zero
- E must be between -1 and 1

Solution:

If the roots are $u, v > 1$, then $b = -(u + v)$ and $c = uv$. Therefore

$$\begin{aligned} s &= 1 - u - v + uv \\ &= (u-1)(v-1) > 0. \end{aligned}$$

Thus s must be greater than zero, and the correct answer is **C**.

45. If $(\log_3 x)(\log_x 2x)\,(\log_{2x} y) = \log_x x^2$, then y equals:

A $\frac{9}{2}$

B 9

C 18

D 27

E 81

Solution:

The logarithms telescope:

$$(\log_3 x)(\log_x 2x)(\log_{2x} y)$$
$$= \log_3 y.$$

Also $\log_x x^2 = 2$, so $\log_3 y = 2$ and $y = 3^2 = 9.$

Therefore, the correct answer is **B**.

46. A student on vacation for d days observed that

(1) it rained 7 times, morning or afternoon;

(2) when it rained in the afternoon, it was clear in the morning;

(3) there were five clear afternoons;

(4) there were six clear mornings.

Then d equals:

A 7

B 9

C 10

D 11

E 12

Solution:

There are $d-6$ rainy mornings and $d-5$ rainy afternoons. By condition (2), these are distinct rainy occasions, and their total is 7. Hence

$$(d-6)+(d-5)=7,$$

so $2d=18$ and $d=9$.

Thus, the correct answer is **B**.

47. Assume that the following three statements are true:

I. All freshmen are human. II. All students are human. III. Some students think.

Given the following four statements:

(1) All freshmen are students.

(2) Some humans think.

(3) No freshmen think.

(4) Some humans who think are not students.

Those which are logical consequences of I, II, and III are:

A 2

B 4

C $2, 3$

D $2, 4$

E $1, 2$

Solution:

Statement III supplies at least one student who thinks. By statement II, that student is human, so some human thinks and statement (2) follows. Nothing relates freshmen to students, says whether freshmen think, or guarantees a thinking human outside the students. Thus none of (1), (3), or (4) follows.

Therefore, the correct answer is **A**.

48. Given the polynomial $a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$, where n is a positive integer or zero, and a_0 is a positive integer. The remaining a's are integers or zero. Set $h = n + a_0 + |a_1| + |a_2| + \cdots + |a_n|$. [See example 25 for the meaning of $|x|$.] The number of polynomials with $h = 3$ is:

A 3

B 5

C 6

D 7

E 9

Solution:

We count by degree. If $n = 0$, then $a_0 = 3$, giving one polynomial. If $n = 1$, then

$$a_0 + |a_1| = 2.$$

This gives $a_0 = 2, a_1 = 0$ or $a_0 = 1, a_1 = \pm 1$, for three polynomials. If $n = 2$, then $a_0 = 1$ and $a_1 = a_2 = 0$, giving one more. No higher degree is possible. The total is $1 + 3 + 1 = 5$.

Thus, the correct answer is **B**.

49. For the infinite series $1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} - \frac{1}{32} + \frac{1}{64} - \frac{1}{128} - \cdots$, let S be the (limiting) sum. Then S equals:

A 0

B $\frac{2}{7}$

C $\frac{6}{7}$

D $\frac{9}{32}$

E $\frac{27}{32}$

Solution:

Group the series as

$$S = \left(1 - \frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{8} - \frac{1}{16} - \frac{1}{32}\right) + \cdots.$$

The first block is $\frac{1}{4}$, and successive blocks form a geometric series with ratio $\frac{1}{8}$. Hence

$$S = \frac{\frac{1}{4}}{1 - \frac{1}{8}} = \frac{2}{7}.$$

Therefore, the correct answer is **B**.

50. A club with x members is organized into four committees in accordance with these two rules:

(1) Each member belongs to two and only two committees.

(2) Each pair of committees has one and only one member in common.

Then x :

A cannot be determined

B has a single value between 8 and 16

C has two values between 8 and 16

D has a single value between 4 and 8

E has two values between 4 and 8

Solution:

Each member belongs to exactly one unordered pair of committees. Conversely, each pair of committees has exactly one common member. Thus the members are in one-to-one correspondence with the pairs of four committees, so

$$x = \binom{4}{2} = 6.$$

This is a single value between 4 and 8.

Therefore, the correct answer is **D**.

Problems: https://live.poshenloh.com/past-contests/amc12/1959