1958 AMC 12 Solutions

Typeset by: LIVE by Po-Shen Loh
https://live.poshenloh.com/past-contests/amc12/1958/solutions

1. The value of $\left[2-3(2-3)^{-1}\right]^{-1}$ is:

A 5

B -5

C $\frac{1}{5}$

D $-\frac{1}{5}$

E $\frac{5}{3}$

Solution:

Since $2-3=-1$, its reciprocal is also -1. Therefore

$$[2-3(-1)]^{-1}=5^{-1}=\frac{1}{5}.$$

Thus, the correct answer is **C**.

2. If $\frac{1}{x} - \frac{1}{y} = \frac{1}{z}$, then z equals:

A $y - x$

B $x - y$

C $\dfrac{y-x}{xy}$

D $\dfrac{xy}{y-x}$

E $\dfrac{xy}{x-y}$

Solution:

Combining the fractions,

$$\frac{1}{z} = \frac{1}{x} - \frac{1}{y} = \frac{y-x}{xy}.$$

Taking reciprocals gives

$$z = \frac{xy}{y-x}.$$

Therefore, the correct answer is **D**.

3. Of the following expressions, the one equal to $\dfrac{a^{-1}b^{-1}}{a^{-3}-b^{-3}}$ is:

A $\dfrac{a^2b^2}{b^2-a^2}$

B $\dfrac{a^2b^2}{b^3-a^3}$

C $\dfrac{ab}{b^3-a^3}$

D $\dfrac{a^3-b^3}{ab}$

E $\dfrac{a^3b^3}{a-b}$

Solution:

We have

$$a^{-3}-b^{-3}=\frac{b^3-a^3}{a^3b^3}.$$

Hence

$$\frac{\frac{1}{ab}}{\frac{b^3-a^3}{a^3b^3}}=\frac{a^2b^2}{b^3-a^3}.$$

Thus, the correct answer is **B**.

4. In the expression $\dfrac{x+1}{x-1}$, each x is replaced by $\dfrac{x+1}{x-1}$. The resulting expression, evaluated for $x = \dfrac{1}{2}$, equals:

- A 3
- B -3
- C 1
- D -1
- E none of these

Solution:

Let $f(x) = \frac{x+1}{x-1}$. Then

$$f(f(x)) = \frac{\frac{x+1}{x-1} + 1}{\frac{x+1}{x-1} - 1}$$
$$= \frac{\frac{2x}{x-1}}{\frac{2}{x-1}} = x.$$

At $x = \frac{1}{2}$, the value is $\frac{1}{2}$, which is not listed.

Therefore, the correct answer is **E**.

5. The expression $2+\sqrt{2}+\dfrac{1}{2+\sqrt{2}}+\dfrac{1}{\sqrt{2}-2}$ equals:

A 2

B $2-\sqrt{2}$

C $2+\sqrt{2}$

D $2\sqrt{2}$

E $\dfrac{\sqrt{2}}{2}$

Solution:

Rationalizing gives

$$\frac{1}{2+\sqrt{2}}=\frac{2-\sqrt{2}}{2},$$
$$\frac{1}{\sqrt{2}-2}=-\frac{2+\sqrt{2}}{2}.$$

Their sum is $-\sqrt{2}$. Therefore the whole expression is

$$2+\sqrt{2}-\sqrt{2}=2.$$

Thus, the correct answer is **A**.

6. The arithmetic mean between $\dfrac{x+a}{x}$ and $\dfrac{x-a}{x}$, when $x \neq 0$, is:

A 2, if $a \neq 0$

B 1

C 1, only if $a = 0$

D $\dfrac{a}{x}$

E x

Solution:

The arithmetic mean is

$$\frac{1}{2}\left(\frac{x+a}{x} + \frac{x-a}{x}\right)$$
$$= \frac{1}{2}\left(\frac{2x}{x}\right) = 1.$$

Therefore, the correct answer is **B**.

7. A straight line joins the points $(-1,1)$ and $(3,9)$. Its x-intercept is:

A $-\frac{3}{2}$

B $-\frac{2}{3}$

C $\frac{2}{5}$

D 2

E 3

Solution:

The slope is

$$\frac{9-1}{3-(-1)} = 2.$$

Using $(-1,1)$, the line is $y = 2x + 3$. Setting $y = 0$ gives

$$x = -\frac{3}{2}.$$

Thus, the correct answer is **A**.

8. Which of these four numbers, $\sqrt{\pi^2}, \sqrt[3]{0.8}, \sqrt[4]{0.00016}, \sqrt[3]{-1} \cdot \sqrt{(0.09)^{-1}}$, is (are) rational?

- A none
- B all
- C the first and fourth
- D only the fourth
- E only the first

Solution:

The first number is π, which is irrational. The cube root of $\frac{4}{5}$ is irrational. Also

$$\sqrt[4]{0.00016} = \sqrt[4]{16 \cdot 10^{-5}}$$

is irrational. The fourth number is

$$(-1)\sqrt{\frac{100}{9}} = -\frac{10}{3},$$

which is rational.

Therefore, only the fourth is rational, and the correct answer is **D**.

9. A value of x satisfying the equation $x^2 + b^2 = (a - x)^2$ is:

A $\dfrac{b^2 + a^2}{2a}$

B $\dfrac{b^2 - a^2}{2a}$

C $\dfrac{a^2 - b^2}{2a}$

D $\dfrac{a - b}{2}$

E $\dfrac{a^2 - b^2}{2}$

Solution:

Expanding the right side,

$$x^2 + b^2 = a^2 - 2ax + x^2.$$

Cancelling x^2 and solving gives

$$\begin{aligned} 2ax &= a^2 - b^2, \\ x &= \frac{a^2 - b^2}{2a}. \end{aligned}$$

Therefore, the correct answer is **C**.

10. For what real values of k, other than $k = 0$, does the equation $x^2 + kx + k^2 = 0$ have real roots?

A $k \leq 0$

B $k \geq 0$

C $k \geq 1$

D all values of k

E no values of k

Solution:

The discriminant is

$$k^2 - 4k^2 = -3k^2.$$

For every nonzero real k, this is negative, so the quadratic has no real roots.

Thus, the correct answer is **E**.

11. The number of roots satisfying the equation $\sqrt{5-x} = x\sqrt{5-x}$ is:

A unlimited

B 3

C 2

D 1

E 0

Solution:

Factoring gives

$$(1-x)\sqrt{5-x} = 0.$$

Thus either $x = 1$ or $5 - x = 0$, giving $x = 5$. Both values lie in the domain $x \leq 5$ and satisfy the original equation.

Therefore, there are two roots, and the correct answer is **C**.

12. If $P = \dfrac{s}{(1+k)^n}$, then n equals:

A $\dfrac{\log(\frac{s}{P})}{\log(1+k)}$

B $\log \dfrac{s}{P(1+k)}$

C $\log \dfrac{s-P}{1+k}$

D $\log \dfrac{s}{P} + \log(1+k)$

E $\dfrac{\log s}{\log(P(1+k))}$

Solution:

Taking logarithms,

$$\log P = \log s - n\log(1+k).$$

Therefore

$$n = \frac{\log s - \log P}{\log(1+k)} = \frac{\log(\frac{s}{P})}{\log(1+k)}.$$

Thus, the correct answer is **A**.

13. The sum of two numbers is 10; their product is 20. The sum of their reciprocals is:

A $\frac{1}{10}$

B $\frac{1}{2}$

C 1

D 2

E 4

Solution:

If the numbers are x and y, then

$$\frac{1}{x}+\frac{1}{y}=\frac{x+y}{xy}=\frac{10}{20}=\frac{1}{2}.$$

Therefore, the correct answer is **B**.

14. At a dance party a group of boys and girls exchange dances as follows: one boy dances with 5 girls, a second boy dances with 6 girls, and so on, the last boy dancing with all the girls. If b represents the number of boys and g the number of girls, then:

A $b = g$

B $b = \frac{g}{5}$

C $b = g - 4$

D $b = g - 5$

E It is impossible to determine a relation between b and g without knowing the total number of boys and girls

Solution:

The last of the b terms in the sequence $5, 6, 7, \ldots$ is

$$5 + (b - 1) = b + 4.$$

This last number equals the total number g of girls. Hence $g = b + 4$, or $b = g - 4.$

Thus, the correct answer is **C**.

15. A quadrilateral is inscribed in a circle. If an angle is inscribed into each of the four segments outside the quadrilateral, the sum of these four angles, expressed in degrees, is:

A 1080

B 900

C 720

D 540

E 360

Solution:

Let the four minor arcs cut off by the sides have measures $\alpha_1, \ldots, \alpha_4$, whose sum is 360°. The angle in the outside segment corresponding to arc α_i subtends the other arc, so its measure is

$$\frac{360^\circ - \alpha_i}{2}.$$

The sum of the four angles is therefore

$$\frac{4(360^\circ) - 360^\circ}{2} = 540^\circ.$$

Thus, the correct answer is **D**.

16. The area of a circle inscribed in a regular hexagon is 100π. The area of the hexagon is:

A 600

B 300

C $200\sqrt{2}$

D $200\sqrt{3}$

E $120\sqrt{5}$

Solution:

The circle has radius $r = 10$, which is the hexagon's apothem. Thus the side length is

$$2r \tan 30^\circ = \frac{20}{\sqrt{3}}.$$

Using one-half the product of perimeter and apothem, the hexagon's area is

$$\frac{1}{2}\left(6 \cdot \frac{20}{\sqrt{3}}\right)(10) = 200\sqrt{3}.$$

Therefore, the correct answer is **D**.

17. If x is positive and $\log x \geq \log 2 + \frac{1}{2}\log x$, then:

A x has no minimum or maximum value

B the maximum value of x is 1

C the minimum value of x is 1

D the maximum value of x is 4

E the minimum value of x is 4

Solution:

The inequality gives

$$\frac{1}{2}\log x \geq \log 2,$$

so $\log x \geq 2\log 2 = \log 4$. Hence $x \geq 4$, and the minimum possible value of x is 4.

Therefore, the correct answer is **E**.

18. The area of a circle is doubled when its radius r is increased by n. Then r equals:

A $n(\sqrt{2}+1)$

B $n(\sqrt{2}-1)$

C n

D $n(2-\sqrt{2})$

E $\dfrac{n\pi}{\sqrt{2}+1}$

Solution:

The doubled-area condition is

$$\pi(r+n)^2 = 2\pi r^2.$$

Since the radii are positive, $r+n=\sqrt{2}\,r$. Thus

$$r = \frac{n}{\sqrt{2}-1} = n(\sqrt{2}+1).$$

Thus, the correct answer is **A**.

19. The sides of a right triangle are a and b and the hypotenuse is c. A perpendicular from the vertex divides c into segments r and s, adjacent respectively to a and b. If $a : b = 1 : 3$, then the ratio of r to s is:

A $1 : 3$

B $1 : 9$

C $1 : 10$

D $3 : 10$

E $1 : \sqrt{10}$

Solution:

Similarity from the altitude-to-hypotenuse construction gives

$$a^2 = cr, \qquad b^2 = cs.$$

Therefore

$$\frac{r}{s} = \frac{a^2}{b^2} = \left(\frac{1}{3}\right)^2 = \frac{1}{9}.$$

Thus, the correct answer is **B**.

20. If $4^x - 4^{x-1} = 24$, then $(2x)^x$ equals:

A $5\sqrt{5}$

B $\sqrt{5}$

C $25\sqrt{5}$

D 125

E 25

Solution:

We have

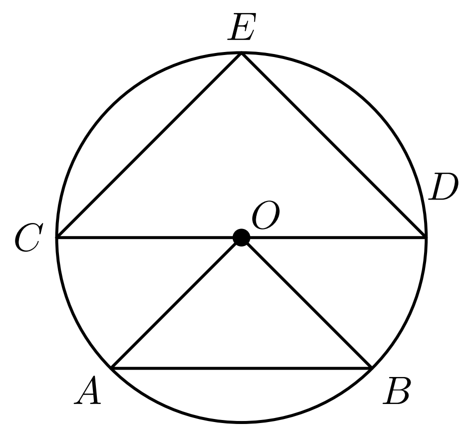
$$4^x - 4^{x-1} = \frac{3}{4}4^x = 24,$$

so $4^x = 32$. Thus $2^{2x} = 2^5$, giving $x = \frac{5}{2}$. Therefore

$$(2x)^x = 5^{\frac{5}{2}} = 25\sqrt{5}.$$

Therefore, the correct answer is **C**.

21. In the accompanying figure $\overline{CE}$ and $\overline{DE}$ are equal chords of a circle with center O. Arc AB is a quarter-circle. Then the ratio of the area of triangle CED to the area of triangle AOB is:



A $\sqrt{2} : 1$

B $\sqrt{3} : 1$

C $4 : 1$

D $3 : 1$

E $2 : 1$

Solution:

Let the circle have radius r. In the pictured configuration, $\overline{CD}$ is a diameter and E is the endpoint of the perpendicular radius, so

$$[CED] = \frac{1}{2}(2r)(r) = r^2.$$

Since arc AB is a quarter-circle, $\angle AOB = 90^\circ$, and

$$[AOB] = \frac{1}{2}r^2.$$

The requested ratio is therefore $2 : 1$.

Thus, the correct answer is **E**.

22. A particle is placed on the parabola $y = x^2 - x - 6$ at a point P whose ordinate is $6.$ It is allowed to roll along the parabola until it reaches the nearest point Q whose ordinate is $-6.$ The horizontal distance traveled by the particle (the numerical value of the difference in the abscissas of P and Q) is:

- A 5
- B 4
- C 3
- D 2
- E 1

Solution:

For $y = 6,$

$$x^2 - x - 12 = 0,$$

so $x = 4$ or $-3.$ For $y = -6, x^2 - x = 0,$ so $x = 0$ or $1.$ The nearest lower point to $x = 4$ is $x = 1,$ and the nearest to $x = -3$ is $x = 0.$ Either horizontal distance is $3.$

Therefore, the correct answer is **C**.

23. If, in the expression $x^2 - 3$, x increases or decreases by a positive amount a, the expression changes by an amount:

A $\pm 2ax + a^2$

B $2ax \pm a^2$

C $\pm a^2 - 3$

D $(x+a)^2 - 3$

E $(x-a)^2 - 3$

Solution:

For an increase,

$$\begin{aligned}&(x+a)^2 - 3 - (x^2 - 3)\\&= 2ax + a^2.\end{aligned}$$

For a decrease,

$$\begin{aligned}&(x-a)^2 - 3 - (x^2 - 3)\\&= -2ax + a^2.\end{aligned}$$

Together these changes are $\pm 2ax + a^2$.

Thus, the correct answer is **A**.

24. A man travels m feet due north at 2 minutes per mile. He returns due south to his starting point at 2 miles per minute. The average rate in miles per hour for the entire trip is:

- A 75
- B 48
- C 45
- D 24
- E impossible to determine without knowing the value of m

Solution:

The northbound rate is 30 mph, and the southbound rate is 120 mph. For equal distances, the round-trip average is

$$\frac{2(30)(120)}{30+120} = 48 \text{ mph.}$$

The distance m cancels.

Therefore, the correct answer is **B**.

25. If $\log_k x \cdot \log_5 k = 3$, then x equals:

A k^6

B $5k^3$

C k^3

D 243

E 125

Solution:

By the change-of-base identity,

$$\log_k x \cdot \log_5 k = \log_5 x.$$

Hence $\log_5 x = 3$, so $x = 5^3 = 125.$

Thus, the correct answer is **E**.

26. A set of n numbers has the sum s. Each number of the set is increased by 20, then multiplied by 5, and then decreased by 20. The sum of the numbers in the new set thus obtained is:

A $s + 20n$

B $5s + 80n$

C s

D $5s$

E $5s + 4n$

Solution:

An original number u becomes

$$5(u + 20) - 20 = 5u + 80.$$

Summing over the n original numbers therefore gives

$$5s + 80n.$$

Thus, the correct answer is **B**.

27. The points $(2, -3)$, $(4, 3)$, and $(5, \frac{k}{2})$ are on the same straight line. The value(s) of k is (are):

A 12

B -12

C ± 12

D 12 or 6

E 6 or $6\frac{2}{3}$

Solution:

The slope through the first two points is

$$\frac{3-(-3)}{4-2} = 3.$$

Thus

$$\frac{\frac{k}{2}-3}{5-4} = 3,$$

which gives $\frac{k}{2} = 6$ and $k = 12$.

Therefore, the correct answer is **A**.

28. A 16-quart radiator is filled with water. Four quarts are removed and replaced with pure antifreeze liquid. Then four quarts of the mixture are removed and replaced with pure antifreeze. This is done a third and a fourth time. The fractional part of the final mixture that is water is:

A $\frac{1}{4}$

B $\frac{81}{256}$

C $\frac{27}{64}$

D $\frac{37}{64}$

E $\frac{175}{256}$

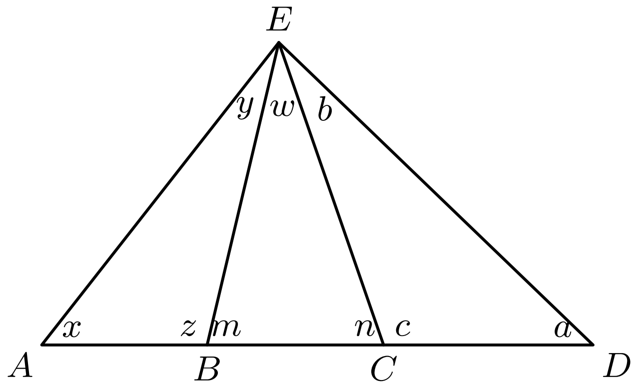
Solution:

Each removal takes one-fourth of the well-mixed radiator contents, so it leaves three-fourths of the water then present. After four replacements, the water fraction is

$$\left(\frac{3}{4}\right)^4 = \frac{81}{256}.$$

Therefore, the correct answer is **B**.

29. In a general triangle ADE (as shown), lines EB and EC are drawn. Which of the following angle relations is true?



A $x + z = a + b$

B $y + z = a + b$

C $m + x = w + n$

D $x + z + n = w + c + m$

E $x + y + n = a + b + m$

Solution:

Triangle AEC gives

$$x + y + w + n = 180^\circ.$$

Triangle BED gives

$$m + a + b + w = 180^\circ.$$

Equating the left sides and cancelling w yields

$$x + y + n = a + b + m.$$

Thus, the correct answer is **E**.

30. If $xy = b$ and $\frac{1}{x^2} + \frac{1}{y^2} = a$, then $(x+y)^2$ equals:

A $(a+2b)^2$

B $a^2 + b^2$

C $b(ab+2)$

D $ab(b+2)$

E $\frac{1}{a} + 2b$

Solution:

Since $xy = b$,

$$a = \frac{x^2+y^2}{x^2y^2} = \frac{x^2+y^2}{b^2},$$

so $x^2 + y^2 = ab^2$. Therefore

$$\begin{aligned}(x+y)^2 &= x^2 + y^2 + 2xy \\ &= ab^2 + 2b \\ &= b(ab+2).\end{aligned}$$

Therefore, the correct answer is **C**.

31. The altitude drawn to the base of an isosceles triangle is 8, and the perimeter is 32. The area of the triangle is:

A 56

B 48

C 40

D 32

E 24

Solution:

Let each equal side be a and half the base be b. The perimeter gives $a + b = 16$. The altitude bisects the base, so

$$a^2 - b^2 = 8^2.$$

Thus $(a - b)(a + b) = 64$, giving $a - b = 4$. Hence $b = 6$, so the base is 12. The area is

$$\frac{1}{2}(12)(8) = 48.$$

Therefore, the correct answer is **B**.

32. With $\$1000$ a rancher is to buy steers at $\$25$ each and cows at $\$26$ each. If the number of steers s and the number of cows c are both positive integers, then:

A this problem has no solution

B there are two solutions with s exceeding c

C there are two solutions with c exceeding s

D there is one solution with s exceeding c

E there is one solution with c exceeding s

Solution:

The purchase equation is

$$25s + 26c = 1000.$$

Reducing modulo 25 gives $c \equiv 0 \pmod{25}$. Since c is positive and $26c < 1000$, the only possibility is $c = 25$. Then

$$s = \frac{1000 - 26(25)}{25} = 14.$$

There is one solution, and $c > s$.

Thus, the correct answer is **E**.

33. For one root of $ax^2 + bx + c = 0$ to be double the other, the coefficients a, b, c must be related as follows:

A $4b^2 = 9c$

B $2b^2 = 9ac$

C $2b^2 = 9a$

D $b^2 - 8ac = 0$

E $9b^2 = 2ac$

Solution:

Let the roots be q and $2q$. Vieta's formulas give

$$3q = -\frac{b}{a}, \qquad 2q^2 = \frac{c}{a}.$$

Squaring the first relation yields $9q^2 = \frac{b^2}{a^2}$. Dividing this by the second relation gives

$$\frac{9}{2} = \frac{b^2}{ac},$$

or $2b^2 = 9ac$.

Therefore, the correct answer is **B**.

34. The numerator of a fraction is $6x+1$, the denominator is $7-4x$, and x can have any value between -2 and 2, both included. The values of x for which the numerator is greater than the denominator are:

A $\frac{3}{5} < x \leq 2$

B $\frac{3}{5} \leq x \leq 2$

C $0 < x \leq 2$

D $0 \leq x \leq 2$

E $-2 \leq x \leq 2$

Solution:

The required comparison gives

$$6x + 1 > 7 - 4x,$$

so $10x > 6$ and $x > \frac{3}{5}$. Intersecting this with $-2 \leq x \leq 2$ gives

$$\frac{3}{5} < x \leq 2.$$

Thus, the correct answer is **A**.

35. A triangle is formed by joining three points whose coordinates are integers. If the x-unit and the y-unit are each 1 inch, then the area of the triangle, in square inches:

A must be an integer

B may be irrational

C must be irrational

D must be rational

E will be an integer only if the triangle is equilateral

Solution:

After translating one vertex to the origin, write the other two as (a, c) and (b, d), with all coordinates integers. The area is

$$\frac{1}{2}|ad - bc|.$$

Since $ad - bc$ is an integer, the area is an integer or a half-integer, and in either case it is rational.

Therefore, the correct answer is **D**.

36. The sides of a triangle are $30, 70,$ and 80 units. If an altitude is dropped upon the side of length $80,$ the larger segment cut off on this side is:

- A 62
- B 63
- C 64
- D 65
- E 66

Solution:

Let x be the segment adjacent to the side of length $30.$ If the altitude has length $h,$ then

$$\begin{aligned} 30^2 - x^2 &= h^2, \\ h^2 &= 70^2 - (80 - x)^2. \end{aligned}$$

Cancelling x^2 and solving gives $x = 15.$ The other segment is $80 - 15 = 65,$ which is the larger one.

Thus, the correct answer is **D**.

37. The first term of an arithmetic series of consecutive integers is k^2+1. The sum of $2k+1$ terms of this series may be expressed as:

A $k^3+(k+1)^3$

B $(k-1)^3+k^3$

C $(k+1)^3$

D $(k+1)^2$

E $(2k+1)(k+1)^2$

Solution:

The last term is

$$k^2+1+2k=(k+1)^2.$$

The average of the first and last terms is k^2+k+1. Therefore the sum is

$$\begin{aligned} S &= (2k+1)(k^2+k+1) \\ &= k^3+(k+1)^3. \end{aligned}$$

Therefore, the correct answer is **A**.

38. Let r be the distance from the origin to a point P with coordinates x and y. Designate the ratio $\frac{y}{r}$ by s and the ratio $\frac{x}{r}$ by c. Then the values of $s^2 - c^2$ are limited to the numbers:

A less than -1 and greater than $+1$, both excluded

B less than -1 and greater than $+1$, both included

C between -1 and $+1$, both excluded

D between -1 and $+1$, both included

E -1 and $+1$ only

Solution:

Since $r^2 = x^2 + y^2$,

$$s^2 + c^2 = \frac{y^2 + x^2}{r^2} = 1.$$

Thus $s^2 - c^2 = 2s^2 - 1$. Because $0 \leq s^2 \leq 1$, this expression ranges from -1 through 1, with both endpoints included.

Thus, the correct answer is **D**.

39. The symbol $|x|$ means x if x is not negative and $-x$ if x is not positive. We may then say concerning the solution of $|x|^2 + |x| - 6 = 0$ that:

A there is only one root

B the sum of the roots is 1

C the sum of the roots is 0

D the product of the roots is 4

E the product of the roots is -6

Solution:

Let $u = |x| \geq 0$. Then

$$u^2 + u - 6 = 0,$$
$$(u+3)(u-2) = 0.$$

The nonnegative solution is $u = 2$, so $|x| = 2$ and $x = \pm 2$. Their sum is 0.

Therefore, the correct answer is **C**.

40. Given $a_0 = 1, a_1 = 3,$ and the general relation $a_n^2 - a_{n-1}a_{n+1} = (-1)^n$ for $n \geq 1.$ Then a_3 equals:

A $\frac{13}{27}$

B 33

C 21

D 10

E -17

Solution:

For $n = 1,$

$$3^2 - (1)a_2 = -1,$$

so $a_2 = 10.$ For $n = 2,$

$$10^2 - 3a_3 = 1,$$

which gives $a_3 = 33.$

Thus, the correct answer is **B**.

41. The roots of $Ax^2 + Bx + C = 0$ are r and s. For the roots of

$$x^2 + px + q = 0$$

to be r^2 and s^2, p must equal:

A $\dfrac{B^2 - 4AC}{A^2}$

B $\dfrac{B^2 - 2AC}{A^2}$

C $\dfrac{2AC - B^2}{A^2}$

D $B^2 - 2C$

E $2C - B^2$

Solution:

For the original quadratic,

$$r + s = -\frac{B}{A}, \qquad rs = \frac{C}{A}.$$

Hence

$$\begin{aligned} r^2 + s^2 &= (r+s)^2 - 2rs \\ &= \frac{B^2 - 2AC}{A^2}. \end{aligned}$$

The coefficient p is the negative of this sum, so

$$p = \frac{2AC - B^2}{A^2}.$$

Therefore, the correct answer is **C**.

42. In a circle with center O, chord $\overline{AB}$ equals chord $\overline{AC}$. Chord $\overline{AD}$ cuts $\overline{BC}$ in E. If $AC = 12$ and $AE = 8$, then AD equals:

A 27

B 24

C 21

D 20

E 18

Solution:

Because E lies on BC and AD, triangles AEC and ACD share angle A. Also $\angle ACE = \angle ACB$ subtends chord AB, while $\angle ADC$ subtends chord AC. Since $AB = AC$, these angles are equal. Thus

$$\triangle AEC \sim \triangle ACD.$$

Corresponding sides give

$$\frac{AE}{AC} = \frac{AC}{AD}.$$

Therefore $\frac{8}{12} = \frac{12}{AD}$, so $AD = 18$.

Thus, the correct answer is **E**.

43. $\overline{AB}$ is the hypotenuse of a right triangle ABC. Median $AD = 7$ and median $BE = 4$. The length of AB is:

A 10

B $5\sqrt{3}$

C $5\sqrt{2}$

D $2\sqrt{13}$

E $2\sqrt{15}$

Solution:

Let $a = BC, b = CA$, and $c = AB$. Since the right angle is at $C, c^2 = a^2 + b^2$. The median formulas give

$$4(7^2) = a^2 + 4b^2,$$
$$4(4^2) = 4a^2 + b^2.$$

Solving yields $a^2 = 4$ and $b^2 = 48$. Therefore

$$\begin{aligned} AB = c &= \sqrt{a^2 + b^2} \\ &= \sqrt{52} = 2\sqrt{13}. \end{aligned}$$

Thus, the correct answer is **D**.

44. Given the true statements:

$1.$ If a is greater than $b,$ then c is greater than $d.$

$2.$ If c is less than $d,$ then e is greater than $f.$

A valid conclusion is:

A If a is less than $b,$ then e is greater than f

B If e is greater than f, then a is less than b

C If e is less than f, then a is greater than b

D If a is greater than $b,$ then e is less than f

E none of these

Solution:

Let P mean $a > b,$ Q mean $c > d,$ and R mean $e > f.$ The premises are

$$P \Rightarrow Q, \qquad \neg Q \Rightarrow R.$$

Their contrapositives are $\neg Q \Rightarrow \neg P$ and $\neg R \Rightarrow Q.$ None of the four proposed implications follows from these relations. For example, knowing P gives $Q,$ not $\neg R;$ and knowing $\neg R$ gives $Q,$ not $P.$

Therefore, the correct answer is **E**.

45. A check is written for x dollars and y cents, x and y both two-digit numbers. In error it is cashed for y dollars and x cents, the incorrect amount exceeding the correct amount by $\$17.82$. Then:

A x cannot exceed 70

B y can equal $2x$

C the amount of the check cannot be a multiple of 5

D the incorrect amount can equal twice the correct amount

E the sum of the digits of the correct amount is divisible by 9

Solution:

In cents, the incorrect amount minus the correct amount is

$$(100y + x) - (100x + y)$$
$$= 99(y - x).$$

Since $\$17.82$ is 1782 cents,

$$99(y - x) = 1782,$$

so $y - x = 18$. The two-digit values $x = 18,\ y = 36$ satisfy this relation and have $y = 2x$. Thus that equality can occur.

Therefore, the correct answer is **B**.

46. For values of x less than 1 but greater than -4, the expression

$$\frac{x^2 - 2x + 2}{2x - 2}$$

has:

A no maximum or minimum value

B a minimum value of 1

C a maximum value of 1

D a minimum value of -1

E a maximum value of -1

Solution:

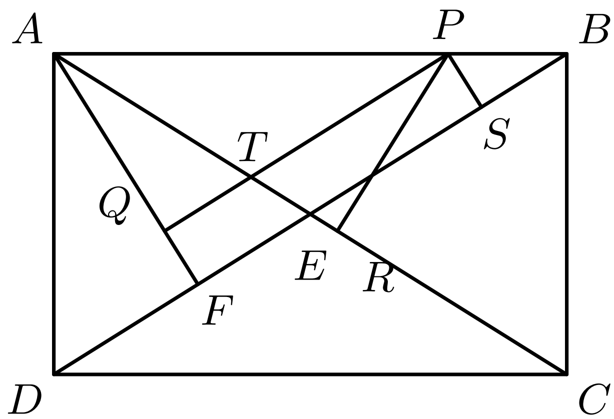
Set $t = x - 1$. Then $-5 < t < 0$, and the expression becomes

$$\frac{t^2 + 1}{2t} = \frac{1}{2}\left(t + \frac{1}{t}\right).$$

For $t < 0$, $t + \frac{1}{t} \leq -2$, with equality when $t = -1$. That value is allowed and corresponds to $x = 0$. Hence the expression is at most -1, and its maximum is -1.

Thus, the correct answer is **E**.

47. $ABCD$ is a rectangle (see the accompanying diagram) with P any point on $\overline{AB}$. $PS \perp BD$ and $PR \perp AC$. $AF \perp BD$ and $PQ \perp AF$. Then $PR + PS$ is equal to:



A PQ

B AE

C $PT + AT$

D AF

E EF

Solution:

Since $AF \perp BD$ and $PQ \perp AF$, we have $PQ \parallel BD$. Also $PS \perp BD$, so $PS \parallel AF$. Thus quadrilateral $QPSF$ is a rectangle, and

$$PS = QF.$$

Let $T = PQ \cap AC$. The diagonals of a rectangle make equal angles with side AB, so $\angle PAT = \angle APT$, giving $AT = PT$. The right triangles ATQ and PTR are similar because they share the angle at T. Since their hypotenuses AT and PT are equal, $AQ = PR$. Therefore

$$PR + PS = AQ + QF = AF.$$

Thus, the correct answer is **D**.

48. Diameter $\overline{AB}$ of a circle with center O is 10 units. C is a point 4 units from A, and on $\overline{AB}$. D is a point 4 units from B, and on $\overline{AB}$. P is any point on the circle. Then the broken-line path from C to P to D :

- A has the same length for all positions of P
- B exceeds 10 units for all positions of P
- C cannot exceed 10 units
- D is shortest when CPD is a right triangle
- E is longest when P is equidistant from C and D

Solution:

Place $O=(0,0), C=(-1,0), D=(1,0)$, and $P=(u,v)$ with $u^2+v^2=25$. Then

$$\begin{aligned} CP^2 &= 26+2u, \\ DP^2 &= 26-2u. \end{aligned}$$

Therefore

$$(CP+DP)^2 = 52 + 2\sqrt{676-4u^2}.$$

This is largest when $u=0$, exactly when $CP=DP$.

Therefore, the correct answer is **E**.

49. In the expansion of $(a+b)^n$ there are $n+1$ dissimilar terms. The number of dissimilar terms in the expansion of $(a+b+c)^{10}$ is:

A 11

B 33

C 55

D 66

E 132

Solution:

Each distinct term is $a^i b^j c^k$ for nonnegative integers satisfying

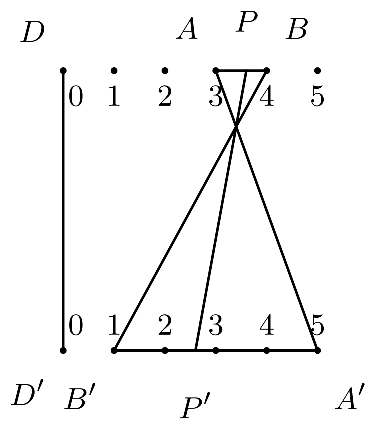
$$i+j+k=10.$$

By stars and bars, the number of such triples is

$$\binom{10+3-1}{3-1}=\binom{12}{2}=66.$$

Thus, the correct answer is **D**.

50. In this diagram a scheme is indicated for associating all the points of segment AB with those of segment $A'B'$, and reciprocally. To describe this association scheme analytically, let x be the distance from a point P on AB to D and let y be the distance from the associated point P' of $A'B'$ to D'. Then for any pair of associated points, if $x = a, x + y$ equals:



A $13a$

B $17a - 51$

C $17 - 3a$

D $\dfrac{17 - 3a}{4}$

E $12a - 34$

Solution:

The two numbered segments are parallel, so projection through the fixed intersection point gives a linear relation between x and y. The endpoint associations shown are $(x, y) = (3, 5)$ and $(x, y) = (4, 1)$. The slope is $\frac{1-5}{4-3} = -4$, so $y = -4x + 17$. If $x = a$, then

$$\begin{aligned} x + y &= a + (-4a + 17) \\ &= 17 - 3a. \end{aligned}$$

Therefore, the correct answer is **C**.

Problems: https://live.poshenloh.com/past-contests/amc12/1958