1957 AMC 12 Solutions

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1. The number of distinct lines representing the altitudes, medians, and interior angle bisectors of a triangle that is isosceles, but not equilateral, is:

A 9

B 7

C 6

D 5

E 3

Solution:

The altitude, median, and angle bisector from the apex are the same line. At each of the two base vertices, those three lines are distinct. Thus there are

$$1 + 3 + 3 = 7$$

distinct lines.

Therefore, the correct answer is **B**.

2. In the equation $2x^2 - hx + 2k = 0$, the sum of the roots is 4 and the product of the roots is -3. Then h and k have the values, respectively:

A 8 and -6

B 4 and -3

C -3 and 4

D -3 and 8

E 8 and -3

Solution:

By Vieta's formulas, the sum and product of the roots are

$$\frac{h}{2} = 4,$$
$$\frac{2k}{2} = k = -3.$$

Hence $h = 8$ and $k = -3$.

Therefore, the correct answer is **E**.

3. The simplest form of $1-\dfrac{1}{1+\dfrac{a}{1-a}}$ is:

- A a if $a \neq 0$
- B 1
- C a if $a \neq -1$
- D $1-a$ with no restriction on a
- E a if $a \neq 1$

Solution:

The original expression requires $a \neq 1$. For such a,

$$\begin{aligned} 1+\frac{a}{1-a} &= \frac{1-a+a}{1-a} \\ &= \frac{1}{1-a}. \end{aligned}$$

Therefore the expression is $1-(1-a)=a$. The restriction $a \neq 1$ remains.

Thus, the correct answer is **E**.

4. The first step in finding the product $(3x+2)(x-5)$ by use of the distributive property in the form $a(b+c)=ab+ac$ is:

A $3x^2-13x-10$

B $3x(x-5)+2(x-5)$

C $(3x+2)x+(3x+2)(-5)$

D $3x^2-17x-10$

E $3x^2+2x-15x-10$

Solution:

Using the property in precisely the stated form, set $a=3x+2, b=x,$ and $c=-5.$ Then

$$\begin{gathered}(3x+2)(x-5)\\ =(3x+2)x+(3x+2)(-5).\end{gathered}$$

Thus, the correct answer is **C**.

5. Through the use of theorems on logarithms, $\log\frac{a}{b}+\log\frac{b}{c}+\log\frac{c}{d}-\log\frac{ay}{dx}$ can be reduced to:

A $\log\frac{y}{x}$

B $\log\frac{x}{y}$

C 1

D 0

E $\log\frac{a^2y}{d^2x}$

Solution:

Combining the logarithms gives

$$\begin{aligned}&\log\left(\frac{a}{b}\frac{b}{c}\frac{c}{d}\frac{dx}{ay}\right)\\ &\quad=\log\left(\frac{a}{d}\frac{dx}{ay}\right)\\ &\quad=\log\frac{x}{y}.\end{aligned}$$

Therefore, the correct answer is **B**.

6. An open box is constructed by starting with a rectangular sheet of metal 10 in. by 14 in. and cutting a square of side x inches from each corner. The resulting projections are folded up and the seams welded. The volume of the resulting box is:

A $140x - 48x^2 + 4x^3$

B $140x + 48x^2 + 4x^3$

C $140x + 24x^2 + x^3$

D $140x - 24x^2 + x^3$

E none of these

Solution:

The box has height x and base dimensions $10 - 2x$ and $14 - 2x$. Hence

$$\begin{aligned} V &= x(10 - 2x)(14 - 2x) \\ &= 140x - 48x^2 + 4x^3. \end{aligned}$$

Thus, the correct answer is **A**.

7. The area of a circle inscribed in an equilateral triangle is 48π. The perimeter of this triangle is:

A $72\sqrt{3}$

B $48\sqrt{3}$

C 36

D 24

E 72

Solution:

The inradius satisfies $r^2 = 48$, so $r = 4\sqrt{3}$. If s is the side length, then

$$r = \frac{s\sqrt{3}}{6},$$

which gives $s = 24$. The perimeter is $3s = 72$.

Thus, the correct answer is **E**.

8. The numbers x, y, z are proportional to $2, 3, 5.$ The sum of $x, y,$ and z is $100.$ The number y is given by the equation $y = ax - 10.$ Then a is:

A 2

B $\frac{2}{3}$

C 3

D $\frac{5}{2}$

E 4

Solution:

Let $x = 2t, y = 3t,$ and $z = 5t.$ Since $10t = 100,$ we get $t = 10,$ so $x = 20$ and $y = 30.$ Thus

$$30 = 20a - 10,$$

and $a = 2.$

Therefore, the correct answer is **A**.

9. The value of $x - y^{x-y}$ when $x = 2$ and $y = -2$ is:

A -18

B -14

C 14

D 18

E 256

Solution:

Here $x - y = 2 - (-2) = 4$. Therefore

$$\begin{aligned} x - y^{x-y} &= 2 - (-2)^4 \\ &= 2 - 16 = -14. \end{aligned}$$

Thus, the correct answer is **B**.

10. The graph of $y = 2x^2 + 4x + 3$ has its:

A lowest point at $(-1, 9)$

B lowest point at $(1, 1)$

C lowest point at $(-1, 1)$

D highest point at $(-1, 9)$

E highest point at $(-1, 1)$

Solution:

Completing the square,

$$y = 2(x + 1)^2 + 1.$$

The squared term is nonnegative, so the graph has its lowest point at $(-1, 1)$.

Thus, the correct answer is **C**.

11. The angle formed by the hands of a clock at $2:15$ is:

A 30°

B $27\frac{1}{2}^{\circ}$

C $157\frac{1}{2}^{\circ}$

D $172\frac{1}{2}^{\circ}$

E none of these

Solution:

The minute hand is 90° clockwise from twelve. The hour hand is

$$2(30^{\circ})+\frac{1}{4}(30^{\circ})=67.5^{\circ}$$

from twelve. Their smaller angle is $90^{\circ}-67.5^{\circ}=22.5^{\circ}$, which is not listed.

Therefore, the correct answer is **E**.

12. Comparing the numbers 10^{-49} and $2 \cdot 10^{-50}$, we may say:

- A the first exceeds the second by $8 \cdot 10^{-1}$
- B the first exceeds the second by $2 \cdot 10^{-1}$
- C the first exceeds the second by $8 \cdot 10^{-50}$
- D the second is five times the first
- E the first exceeds the second by 5

Solution:

Using a common power of ten,

$$\begin{aligned} &10^{-49} - 2 \cdot 10^{-50} \\ &= (10 - 2)10^{-50} \\ &= 8 \cdot 10^{-50}. \end{aligned}$$

Thus, the correct answer is **C**.

13. A rational number between $\sqrt{2}$ and $\sqrt{3}$ is:

A $\dfrac{\sqrt{2}+\sqrt{3}}{2}$

B $\dfrac{\sqrt{2}\cdot\sqrt{3}}{2}$

C 1.5

D 1.8

E 1.4

Solution:

Since

$$\begin{aligned}\sqrt{2} \approx 1.414 < 1.5,\\ 1.5 < 1.732 \approx \sqrt{3},\end{aligned}$$

the rational number 1.5 lies between the two radicals.

Therefore, the correct answer is **C**.

14. If $y = \sqrt{x^2 - 2x + 1} + \sqrt{x^2 + 2x + 1}$, then y is:

A $2x$

B $2(x+1)$

C 0

D $|x-1| + |x+1|$

E none of these

Solution:

The two radicands are $(x-1)^2$ and $(x+1)^2$. Hence

$$\begin{aligned} y &= \sqrt{(x-1)^2} \\ &\quad + \sqrt{(x+1)^2} \\ &= |x-1| + |x+1|. \end{aligned}$$

Thus, the correct answer is **D**.

15. The table below shows the distance s in feet a ball rolls down an inclined plane in t seconds.

t	0	1	2	3	4	5
s	0	10	40	90	160	250

The distance s for $t = 2.5$ is:

A 45

B 62.5

C 70

D 75

E 82.5

Solution:

The entries follow $s = 10t^2$. Thus, at $t = 2.5$,

$$s = 10(2.5)^2 = 10(6.25) = 62.5.$$

Therefore, the correct answer is **B**.

16. Goldfish are sold at 15 cents each. The rectangular coordinate graph showing the cost of 1 to 12 goldfish is:

- A a straight line segment
- B a set of horizontal parallel line segments
- C a set of vertical parallel line segments
- D a finite set of distinct points
- E a straight line

Solution:

The rule $c = 15n$ is linear, but the number n of goldfish can only be one of the twelve integers $1, 2, \ldots, 12$. The graph therefore consists of twelve distinct points, not an entire line or segment.

Thus, the correct answer is **D**.

17. A cube is made by soldering twelve 3-inch lengths of wire properly at the vertices of the cube. If a fly alights at one of the vertices and then walks along the edges, the greatest distance it could travel before coming to any vertex a second time, without retracing any distance, is:

A 24 in.

B 12 in.

C 30 in.

D 18 in.

E 36 in.

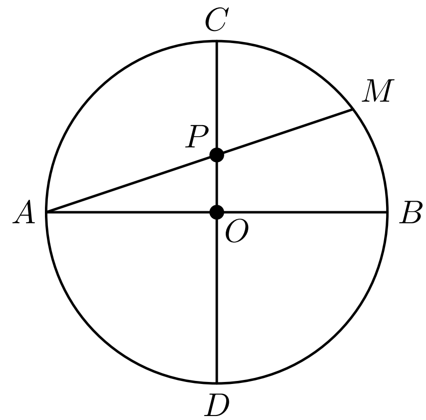
Solution:

The cube has eight vertices. A path can visit all eight once, using seven edges, and then traverse the edge from its last vertex back to its starting vertex; the return is the first repeated vertex. Thus the greatest permitted walk uses eight edges. Its length is

$$8 \cdot 3 = 24 \text{ inches.}$$

Therefore, the correct answer is **A**.

18. Circle O has diameters $\overline{AB}$ and $\overline{CD}$ perpendicular to each other. $\overline{AM}$ is any chord intersecting $\overline{CD}$ at P. Then $AP \cdot AM$ is equal to:



A $AO \cdot OB$

B $AO \cdot AB$

C $CP \cdot CD$

D $CP \cdot PD$

E $CO \cdot OP$

Solution:

Angles APO and ABM are both right angles, and the two triangles share angle A. Therefore triangles APO and ABM are similar. Corresponding sides give

$$\frac{AP}{AB} = \frac{AO}{AM}.$$

Hence $AP \cdot AM = AO \cdot AB$.

Thus, the correct answer is **B**.

19. The base of the decimal number system is ten, meaning, for example, that $123 = 1 \cdot 10^2 + 2 \cdot 10 + 3$. In the binary system, which has base two, the first five positive integers are $1, 10, 11, 100, 101$. The numeral 10011 in the binary system would then be written in the decimal system as:

A 19

B 40

C 10011

D 11

E 7

Solution:

Expanding by binary place value,

$$\begin{aligned}(10011)_2 &= 1 \cdot 2^4 + 1 \cdot 2^1 + 1 \cdot 2^0 \\ &= 16 + 2 + 1 = 19.\end{aligned}$$

Therefore, the correct answer is **A**.

20. A man makes a trip by automobile at an average speed of 50 mph. He returns over the same route at an average speed of 45 mph. His average speed for the entire trip is:

A $47\frac{7}{19}$ mph

B $47\frac{1}{4}$ mph

C $47\frac{1}{2}$ mph

D $47\frac{11}{19}$ mph

E none of these

Solution:

Let each leg have distance d. The average speed is

$$\begin{aligned}\frac{2d}{\frac{d}{50}+\frac{d}{45}} &= \frac{2}{\frac{1}{50}+\frac{1}{45}} \\ &= \frac{900}{19} \\ &= 47\frac{7}{19}\text{ mph.}\end{aligned}$$

Thus, the correct answer is **A**.

21. Start with the theorem "If two angles of a triangle are equal, the triangle is isosceles," and the following four statements:

$1.$ If two angles of a triangle are not equal, the triangle is not isosceles.

$2.$ The base angles of an isosceles triangle are equal.

$3.$ If a triangle is not isosceles, then two of its angles are not equal.

$4.$ A necessary condition that two angles of a triangle be equal is that the triangle be isosceles.

Which combination of statements contains only those which are logically equivalent to the given theorem?

A $1, 2, 3, 4$

B $1, 2, 3$

C $2, 3, 4$

D $1, 2$

E $3, 4$

Solution:

Let P mean that two angles are equal and Q that the triangle is isosceles. Statement 3 is the contrapositive $\neg Q \Rightarrow \neg P$, so it is equivalent to $P \Rightarrow Q$. Statement 4 says that Q is necessary for P, which is another wording of $P \Rightarrow Q$. Statements 1 and 2 are the inverse and converse.

Thus only statements 3 and 4 are equivalent, so the correct answer is **E**.

22. If $\sqrt{x-1}-\sqrt{x+1}+1=0$, then $4x$ equals:

A 5

B $4\sqrt{-1}$

C 0

D $1\frac{1}{4}$

E no real value

Solution:

Rearrange and square:

$$\begin{aligned}\sqrt{x+1} &= \sqrt{x-1}+1,\\ x+1 &= x-1+2\sqrt{x-1}+1.\end{aligned}$$

Thus $2\sqrt{x-1}=1$, so $x-1=\frac{1}{4}$ and $x=\frac{5}{4}$. This value satisfies the original equation. Therefore $4x=5$.

Thus, the correct answer is **A**.

23. The graph of $x^2 + y = 10$ and the graph of $x + y = 10$ meet in two points. The distance between these two points is:

- A less than 1
- B 1
- C $\sqrt{2}$
- D 2
- E more than 2

Solution:

At an intersection,

$$10 - x^2 = 10 - x,$$

so $x = 0$ or $x = 1$. The points are $(0, 10)$ and $(1, 9)$. Their distance is

$$\sqrt{(1-0)^2 + (9-10)^2} = \sqrt{2}.$$

Therefore, the correct answer is **C**.

24. If the square of a number of two digits is decreased by the square of the number formed by reversing the digits, then the result is not always divisible by:

A 9

B the product of the digits

C the sum of the digits

D the difference of the digits

E 11

Solution:

If the digits are a, b, then

$$\begin{aligned}&(10a+b)^2-(10b+a)^2\\&\quad=[9(a-b)][11(a+b)]\\&\quad=99(a-b)(a+b).\end{aligned}$$

This is always divisible by $9, 11,$ the digit sum, and the digit difference. It need not be divisible by the digit product; for example, $21^2-12^2=297$ is not divisible by $2.$

Thus, the correct answer is **B**.

25. The vertices of triangle PQR have coordinates as follows: $P(0,a), Q(b,0), R(c,d)$, where $a, b, c,$ and d are positive. The origin and point R lie on opposite sides of PQ. The area of triangle PQR may be found from the expression:

A $\dfrac{ab+ac+bc+cd}{2}$

B $\dfrac{ac+bd-ab}{2}$

C $\dfrac{ab-ac-bd}{2}$

D $\dfrac{ac+bd+ab}{2}$

E $\dfrac{ac+bd-ab-cd}{2}$

Solution:

The signed doubled area is

$$\begin{vmatrix} 0 & a & 1 \\ b & 0 & 1 \\ c & d & 1 \end{vmatrix} = ab - ac - bd.$$

The line PQ has equation $ax + by = ab$. Since the origin gives a value below ab and R is on the opposite side, $ac + bd > ab$. Thus the area is

$$\frac{ac+bd-ab}{2}.$$

Therefore, the correct answer is **B**.

26. From a point within a triangle, line segments are drawn to the vertices. A necessary and sufficient condition that the three triangles thus formed have equal areas is that the point be:

- A the center of the inscribed circle
- B the center of the circumscribed circle
- C such that the three angles formed at the point each be 120°
- D the intersection of the altitudes of the triangle
- E the intersection of the medians of the triangle

Solution:

The three medians meet at the centroid and divide the triangle into six small triangles of equal area. Each of the three triangles having the centroid and one side of the original triangle consists of two of those small triangles, so their areas are equal. Conversely, equal areas give equal barycentric coordinates, which uniquely locate the centroid.

Thus, the necessary and sufficient point is the intersection of the medians, and the correct answer is **E**.

27. The sum of the reciprocals of the roots of the equation $x^2 + px + q = 0$ is:

A $-\frac{p}{q}$

B $\frac{q}{p}$

C $\frac{p}{q}$

D $-\frac{q}{p}$

E pq

Solution:

If the roots are r, s, then Vieta's formulas give $r + s = -p$ and $rs = q$. Therefore

$$\frac{1}{r} + \frac{1}{s} = \frac{r+s}{rs} = -\frac{p}{q}.$$

Thus, the correct answer is **A**.

28. If a and b are positive and $a \neq 1, b \neq 1,$ then the value of $b^{\log_b a}$ is:

A dependent upon b

B dependent upon a

C dependent upon a and b

D zero

E one

Solution:

By the definition of a logarithm,

$$b^{\log_b a} = a.$$

Thus the value depends on $a,$ not on the choice of admissible base $b.$

Therefore, the correct answer is **B**.

29. The relation $x^2(x^2-1) \geq 0$ is true only for:

Here $x \geq a$ means that x can take on all values greater than a and the value equal to a, while $x \leq a$ has a corresponding meaning with "less than."

A $x \geq 1$

B $-1 \leq x \leq 1$

C $x = 0, x = 1, x = -1$

D $x = 0, x \leq -1, x \geq 1$

E $x \geq 0$

Solution:

At $x = 0$, the product is zero. For $x \neq 0$, the factor x^2 is positive, so the product is nonnegative exactly when

$$x^2 - 1 \geq 0,$$

or $x \leq -1$ or $x \geq 1$. Including the isolated value $x = 0$ gives the set in choice D.

Thus, the correct answer is **D**.

30. The sum of the squares of the first n positive integers is given by the expression $\dfrac{n(n+c)(2n+k)}{6}$, if c and k are, respectively:

A 1 and 2

B 3 and 5

C 2 and 2

D 1 and 1

E 2 and 1

Solution:

The standard formula is

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)}{6} \cdot (2n+1).$$

Therefore $c = 1$ and $k = 1$.

Thus, the correct answer is **D**.

31. A regular octagon is to be formed by cutting equal isosceles right triangles from the corners of a square. If the square has sides of one unit, the leg of each of the triangles has length:

A $\frac{2+\sqrt{2}}{3}$

B $\frac{2-\sqrt{2}}{2}$

C $\frac{1+\sqrt{2}}{2}$

D $\frac{1+\sqrt{2}}{3}$

E $\frac{2-\sqrt{2}}{3}$

Solution:

Let x be a leg of each corner triangle. The horizontal and vertical octagon sides have length $1-2x$, while the four slanted sides have length $x\sqrt{2}$. Thus

$$1-2x=x\sqrt{2},$$

so

$$x=\frac{1}{2+\sqrt{2}}=\frac{2-\sqrt{2}}{2}.$$

Therefore, the correct answer is **B**.

32. The largest of the following integers which divides each of the numbers of the sequence $1^5-1, 2^5-2, 3^5-3, \ldots n^5-n, \ldots$ is:

A 1

B 60

C 15

D 120

E 30

Solution:

Factor

$$n^5-n=n(n-1)(n+1) \cdot (n^2+1).$$

Among three consecutive integers, one is divisible by 3 and at least one is even, so the expression is divisible by 6. Also $n^5 \equiv n \pmod 5$ for every integer n, so it is divisible by 5. Hence every term is divisible by 30. Taking $n=2$ gives $2^5-2=30$, so no larger listed integer can divide every term.

Thus, the correct answer is **E**.

33. If $9^{x+2} = 240 + 9^x$, then the value of x is:

A 0.1

B 0.2

C 0.3

D 0.4

E 0.5

Solution:

Factoring 9^x,

$$81 \cdot 9^x - 9^x = 240,$$

so $80 \cdot 9^x = 240$ and $9^x = 3$. Since $9^{\frac{1}{2}} = 3$, we have $x = \frac{1}{2} = 0.5$.

Therefore, the correct answer is **E**.

34. The points that satisfy the system $x + y = 1, x^2 + y^2 < 25,$ where the symbol “$<$” means “less than,” constitute the following set:

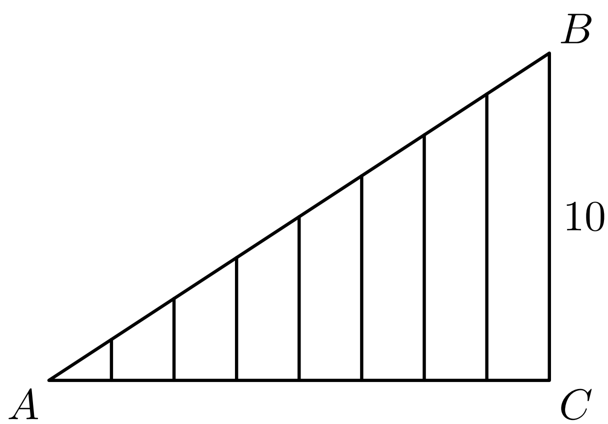
- A only two points
- B an arc of a circle
- C a straight line segment not including the end-points
- D a straight line segment including the end-points
- E a single point

Solution:

The inequality describes the open disk inside the circle of radius 5 centered at the origin. The line $x + y = 1$ crosses the circle in two points. Its portion inside the disk is the segment between those points, but the strict inequality excludes the endpoints.

Thus, the correct answer is **C**.

35. Side $\overline{AC}$ of right triangle ABC is divided into 8 equal parts. Seven line segments parallel to $\overline{BC}$ are drawn to $\overline{AB}$ from the points of division. If $BC = 10$, then the sum of the lengths of the seven line segments:



A) cannot be found from the given information

B) is 33

C) is 34

D) is 35

E) is 45

Solution:

By similarity, the segment at the kth division point has length $\frac{10k}{8}$ for $k = 1, \ldots, 7$. Their sum is

$$\begin{aligned} \frac{10}{8}(1 + 2 + \cdots + 7) &= \frac{10}{8} \cdot 28 \\ &= 35. \end{aligned}$$

Therefore, the correct answer is **D**.

36. If $x + y = 1$, then the largest value of xy is:

- A 1
- B 0.5
- C an irrational number about 0.4
- D 0.25
- E 0

Solution:

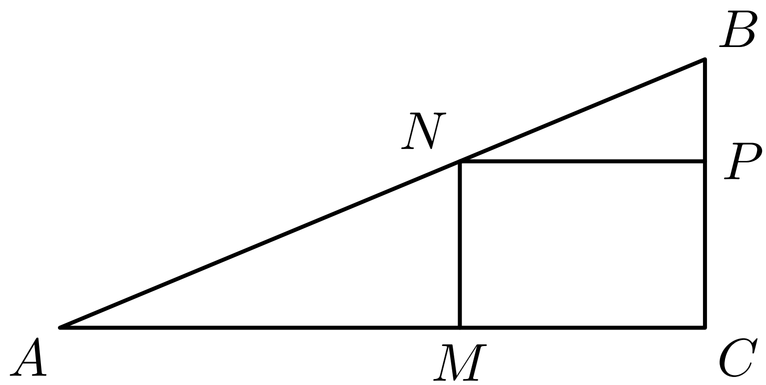
We have

$$xy = x(1-x) = \frac{1}{4} - \left(x - \frac{1}{2}\right)^2.$$

The squared term is nonnegative, so the largest possible product is $\frac{1}{4} = 0.25.$

Thus, the correct answer is **D**.

37. In right triangle $ABC, BC = 5, AC = 12$, and $AM = x; MN \perp AC, NP \perp BC;$ N is on $\overline{AB}$. If $y = MN + NP$, one-half the perimeter of rectangle $MCPN$, then:



A $y = \frac{1}{2}(5 + 12)$

B $y = \frac{5x}{12} + \frac{12}{5}$

C $y = \frac{144 - 7x}{12}$

D $y = 12$

E $y = \frac{5x}{12} + 6$

Solution:

By similarity,

$$MN = \frac{5}{12}AM = \frac{5x}{12}.$$

Also $NP = MC = AC - AM = 12 - x$. Hence

$$\begin{aligned} y = MN + NP &= \frac{5x}{12} + 12 - x \\ &= \frac{144 - 7x}{12}. \end{aligned}$$

Therefore, the correct answer is **C**.

38. From a two-digit number N we subtract the number with the digits reversed and find that the result is a positive perfect cube. Then:

- A N cannot end in 5
- B N can end in any digit other than 5
- C N does not exist
- D there are exactly 7 values for N
- E there are exactly 10 values for N

Solution:

Writing $N = 10a + b$, the positive difference is

$$\begin{aligned}(10a+b)-(10b+a)\\ = 9(a-b).\end{aligned}$$

It is at most 81. Among $1, 8, 27, 64$, only 27 is divisible by 9, so $a - b = 3$. The digit pairs are

$$\begin{aligned}(3,0),(4,1),(5,2),(6,3),\\ (7,4),(8,5),(9,6),\end{aligned}$$

giving exactly seven values of N.

Thus, the correct answer is **D**.

39. Two men set out at the same time to walk towards each other from M and N, 72 miles apart. The first man walks at the rate of 4 mph. The second man walks 2 miles the first hour, $2\frac{1}{2}$ miles the second hour, 3 miles the third hour, and so on in arithmetic progression. Then the men will meet:

A in 7 hours

B in $8\frac{1}{4}$ hours

C nearer M than N

D nearer N than M

E midway between M and N

Solution:

In t hours, the second man's hourly distances form an arithmetic progression with first term 2 and last term $2+\frac{t-1}{2}$. His distance is

$$\frac{t}{2}\left(4+\frac{t-1}{2}\right)=\frac{t(t+7)}{4}.$$

Together the men cover

$$4t+\frac{t(t+7)}{4}=72,$$

or $t^2+23t-288=0$. The positive root is $t=9$. The first man then walks $4(9)=36$ miles, exactly half the original distance.

Therefore, the correct answer is **E**.

40. If the parabola $y = -x^2 + bx - 8$ has its vertex on the x-axis, then b must be:

A a positive integer

B a positive or a negative rational number

C a positive rational number

D a positive or a negative irrational number

E a negative irrational number

Solution:

The vertex lies on the x-axis exactly when the quadratic has one repeated root. Its discriminant must satisfy

$$b^2 - 4(-1)(-8) = b^2 - 32 = 0.$$

Hence $b = \pm\sqrt{32} = \pm 4\sqrt{2}$, a positive or negative irrational number.

Thus, the correct answer is **D**.

41. Given the system of equations

$$\begin{aligned} ax + (a-1)y &= 1, \\ (a+1)x - ay &= 1. \end{aligned}$$

For which one of the following values of a is there no solution for x and y?

A 1

B 0

C -1

D $\pm\frac{\sqrt{2}}{2}$

E $\pm\sqrt{2}$

Solution:

The coefficient determinant is

$$\begin{aligned} & a(-a) - (a-1)(a+1) \\ &= 1 - 2a^2. \end{aligned}$$

It vanishes when $a = \pm\frac{\sqrt{2}}{2}$. For either value, the two coefficient rows are proportional but the two right sides are not in the same ratio, so the system is inconsistent.

Thus, the correct answer is **D**.

42. If $S = i^n + i^{-n}$, where $i = \sqrt{-1}$ and n is an integer, then the total number of possible distinct values for S is:

- A 1
- B 2
- C 3
- D 4
- E more than 4

Solution:

If n is even, i^n is 1 or -1 and equals its reciprocal, so $S = 2$ or -2. If n is odd, the two terms are i and $-i$, in some order, so $S = 0$. Thus the distinct values are

$$-2,\ 0,\ 2,$$

three values.

Therefore, the correct answer is **C**.

43. We define a lattice point as a point whose coordinates are integers, zero admitted. Then the number of lattice points on the boundary and inside the region bounded by the x-axis, the line $x = 4$, and the parabola $y = x^2$ is:

A 24

B 35

C 34

D 30

E not finite

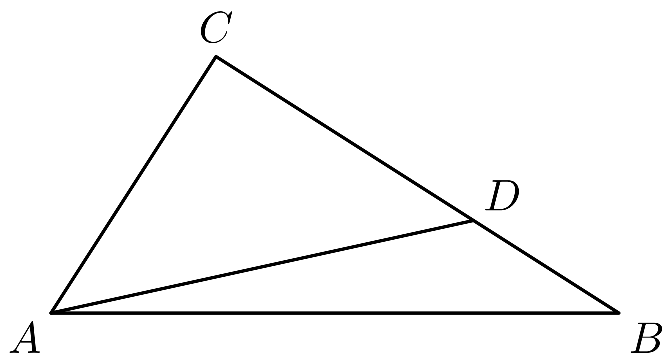
Solution:

For each integer $x = 0, 1, 2, 3, 4$, there are $x^2 + 1$ integer values $y = 0, 1, \ldots, x^2$. Therefore the number of lattice points is

$$\begin{aligned} &(0^2+1)+(1^2+1)+(2^2+1) \\ &\quad +(3^2+1)+(4^2+1) \\ &= 1+2+5+10+17 \\ &= 35. \end{aligned}$$

Thus, the correct answer is **B**.

44. In triangle ABC, $AC = CD$ and $\angle CAB - \angle ABC = 30^\circ$. Then $\angle BAD$ is:



A 30°

B 20°

C $22\frac{1}{2}^\circ$

D 10°

E 15°

Solution:

Let $\alpha = \angle CAB$, $\beta = \angle ABC$, and $x = \angle BAD$. Since D lies on BC,

$$\angle ADC = x + \beta$$

as an exterior angle of triangle ABD. Because $AC = CD$, triangle ACD is isosceles, so $\angle CAD = \angle ADC$. But $\angle CAD = \alpha - x$. Hence

$$\alpha - x = x + \beta,$$

so $2x = \alpha - \beta = 30^\circ$ and $x = 15^\circ$.

Thus, the correct answer is **E**.

45. If two real numbers x and y satisfy the equation $\frac{x}{y} = x - y$, then:

A $x \geq 4$ and $x \leq 0$, where $x \geq a$ means that x can take any value greater than a or equal to a

B y can equal 1

C both x and y must be irrational

D x and y cannot both be integers

E both x and y must be rational

Solution:

Since $y \neq 0$, multiplying by y gives

$$x = xy - y^2,$$

or $y^2 - xy + x = 0$. For a real value of y, its discriminant must satisfy

$$x^2 - 4x = x(x-4) \geq 0.$$

Thus $x \leq 0$ or $x \geq 4$.

Therefore, the correct answer is **A**.

46. Two perpendicular chords intersect in a circle. The segments of one chord are 3 and 4; the segments of the other are 6 and 2. Then the diameter of the circle is:

A $\sqrt{89}$

B $\sqrt{56}$

C $\sqrt{61}$

D $\sqrt{75}$

E $\sqrt{65}$

Solution:

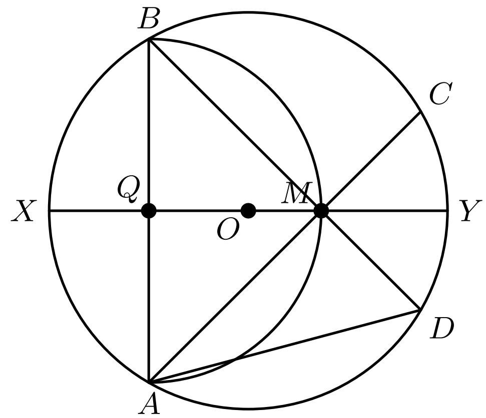
Place the intersection at $(0,0)$ with endpoints $(-3,0),(4,0),(0,-2),(0,6)$. The perpendicular bisectors of the chords are $x=\frac{1}{2}$ and $y=2$, so the center is $(\frac{1}{2},2)$. Using the endpoint $(4,0)$,

$$\begin{aligned} r^2 &= \left(4-\frac{1}{2}\right)^2+(0-2)^2 \\ &= \frac{65}{4}. \end{aligned}$$

Therefore the diameter is $2r=\sqrt{65}$.

Thus, the correct answer is **E**.

47. In circle O, the midpoint of radius $\overline{OX}$ is Q; at Q, $\overline{AB} \perp \overline{XY}$. The semicircle with $\overline{AB}$ as diameter intersects $\overline{XY}$ in M. Line AM intersects circle O in C, and line BM intersects circle O in D. Line AD is drawn. Then, if the radius of circle O is r, AD is:



A $r\sqrt{2}$

B r

C not a side of an inscribed regular polygon

D $\frac{r\sqrt{2}}{2}$

E $r\sqrt{3}$

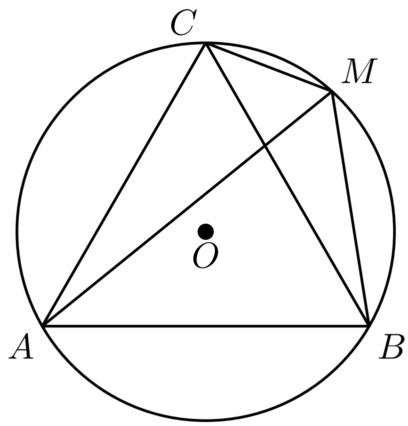
Solution:

Line XY is the perpendicular bisector of AB, so $MA = MB$. Since M lies on the semicircle with diameter AB, $\angle AMB = 90°$. Thus triangle AMB is an isosceles right triangle and $\angle ABM = 45°$. Because B, M, D are collinear, the inscribed angle $\angle ABD = 45°$, so its intercepted arc AD measures $90°$. Therefore chord AD is a side of an inscribed square and has length

$$AD = r\sqrt{2}.$$

Thus, the correct answer is **A**.

48. Let ABC be an equilateral triangle inscribed in circle O. M is a point on arc BC. Lines AM, BM, and CM are drawn. Then AM is:



A equal to $BM + CM$

B less than $BM + CM$

C greater than $BM + CM$

D equal to, less than, or greater than $BM + CM$, depending upon the position of M

E none of these

Solution:

In cyclic quadrilateral $ABMC$, Ptolemy's theorem gives

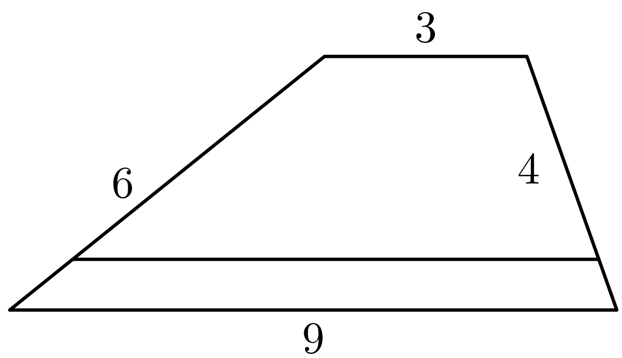
$$AM \cdot BC = AB \cdot CM + AC \cdot BM.$$

Since ABC is equilateral, $AB = BC = AC$. Dividing by this common length yields

$$AM = CM + BM.$$

Therefore, the correct answer is **A**.

49. The parallel sides of a trapezoid are 3 and 9. The non-parallel sides are 4 and 6. A line parallel to the bases divides the trapezoid into two trapezoids of equal perimeters. The ratio in which each of the non-parallel sides is divided is:



A $4:3$

B $3:2$

C $4:1$

D $3:1$

E $6:1$

Solution:

Let the upper pieces of the legs of lengths 4 and 6 be $4t$ and $6t$, respectively. The dividing segment appears once in each perimeter and cancels. Equal perimeters therefore give

$$3+4t+6t=9+4(1-t)+6(1-t).$$

Hence $20t=16$, so $t=\frac{4}{5}$. Each leg is divided in the ratio

$$t:(1-t)=\frac{4}{5}:\frac{1}{5}=4:1.$$

Thus, the correct answer is **C**.

50. In circle O, G is a moving point on diameter $\overline{AB}$. $\overline{AA'}$ is drawn perpendicular to $\overline{AB}$ and equal to AG. $\overline{BB'}$ is drawn perpendicular to $\overline{AB}$, on the same side of diameter $\overline{AB}$ as $\overline{AA'}$, and equal to BG. Let O' be the midpoint of $\overline{A'B'}$. Then, as G moves from A to B, point O' :

A. moves on a straight line parallel to AB

B. remains stationary

C. moves on a straight line perpendicular to AB

D. moves in a small circle intersecting the given circle

E. follows a path which is neither a circle nor a straight line

Solution:

Let $A=(0,0), B=(L,0)$, and $G=(g,0)$. The perpendicular constructions on the same side give

$$A'=(0,g), \qquad B'=(L,L-g).$$

Their midpoint is

$$\begin{aligned} O' &= \left(\frac{L}{2}, \frac{g+L-g}{2}\right) \\ &= \left(\frac{L}{2}, \frac{L}{2}\right), \end{aligned}$$

independent of g. Thus O' remains stationary.

Therefore, the correct answer is **B**.

Problems: https://live.poshenloh.com/past-contests/amc12/1957