1956 AMC 12 Solutions

Typeset by: LIVE by Po-Shen Loh
https://live.poshenloh.com/past-contests/amc12/1956/solutions

1. The value of $x + x(x^x)$ when $x = 2$ is:

A 10

B 16

C 18

D 36

E 64

Solution:

Substituting $x = 2$, and evaluating the exponent first, gives

$$\begin{aligned} x + x(x^x) &= 2 + 2(2^2) \\ &= 2 + 8 = 10. \end{aligned}$$

Thus, the correct answer is **A**.

2. Mr. Jones sold two pipes at $\$1.20$ each. Based on the cost his profit on one was 20% and his loss on the other was 20%. On the sale of the pipes, he:

A broke even

B lost $4¢$

C gained $4¢$

D lost $10¢$

E gained $10¢$

Solution:

The pipe sold at a 20% profit cost

$$\frac{\$1.20}{1.20} = \$1.00,$$

while the pipe sold at a 20% loss cost

$$\frac{\$1.20}{0.80} = \$1.50.$$

Their combined cost was $\$2.50$, but they sold for $\$2.40$, so Mr. Jones lost 10 cents.

Thus, the correct answer is **D**.

3. The distance light travels in one year is approximately $5{,}870{,}000{,}000{,}000$ miles. The distance light travels in 100 years is:

- A 587×10^{8} miles
- B 587×10^{10} miles
- C 587×10^{-10} miles
- D 587×10^{12} miles
- E 587×10^{-12} miles

Solution:

The given distance is 587×10^{10} miles. Multiplying by $100 = 10^2$ gives

$$(587 \times 10^{10})10^{2} = 587 \times 10^{12}$$

miles.

Thus, the correct answer is **D**.

4. A man has $\$10{,}000$ to invest. He invests $\$4{,}000$ at 5% and $\$3{,}500$ at 4%. In order to have a yearly income of $\$500$, he must invest the remainder at:

- A 6%
- B 6.1%
- C 6.2%
- D 6.3%
- E 6.4%

Solution:

The amount left to invest is

$$10000 - 4000 - 3500 = 2500.$$

The first two investments earn $0.05(4000) + 0.04(3500) = 340$ dollars, so the remaining investment must earn $500 - 340 = 160$ dollars. Its required rate is

$$\frac{160}{2500} = 0.064 = 6.4\%.$$

Thus, the correct answer is **E**.

5. A nickel is placed on a table. The number of nickels which can be placed around it, each tangent to it and to two others is:

A 4

B 5

C 6

D 8

E 12

Solution:

The centers of the middle nickel and any two neighboring nickels are pairwise two radii apart, so they form an equilateral triangle. Therefore consecutive outer centers subtend 60° at the center of the middle nickel. Exactly

$$\frac{360^\circ}{60^\circ} = 6$$

nickels fit around it.

Thus, the correct answer is **C**.

6. In a group of cows and chickens, the number of legs was 14 more than twice the number of heads. The number of cows was:

A 5

B 7

C 10

D 12

E 14

Solution:

Counting two legs for every head accounts for both legs of each chicken and two of each cow's four legs. Thus every cow contributes exactly two of the 14 extra legs. Hence the number of cows is

$$\frac{14}{2} = 7.$$

Thus, the correct answer is **B**.

7. The roots of the equation $ax^2 + bx + c = 0$ will be reciprocal if:

A $a = b$

B $a = bc$

C $c = a$

D $c = b$

E $c = ab$

Solution:

By Vieta's formulas, the product of the two roots is $\frac{c}{a}$. Reciprocal roots have product 1, so

$$\frac{c}{a} = 1,$$

which requires $c = a$.

Thus, the correct answer is **C**.

8. If $8 \cdot 2^x = 5^{y+8}$, then, when $y = -8, x =$

- A -4
- B -3
- C 0
- D 4
- E 8

Solution:

When $y = -8,$ the right side is $5^0 = 1.$ Therefore

$$8 \cdot 2^x = 2^{x+3} = 1 = 2^0,$$

so $x + 3 = 0$ and $x = -3.$

Thus, the correct answer is **B**.

9. Simplify $\left[\sqrt[3]{\sqrt[6]{a^9}}\right]^4\left[\sqrt[6]{\sqrt[3]{a^9}}\right]^4$; the result is:

A a^{16}

B a^{12}

C a^8

D a^4

E a^2

Solution:

For each of the two factors, the nested roots and outer fourth power multiply the exponent of a by

$$9 \cdot \frac{1}{6} \cdot \frac{1}{3} \cdot 4 = 2.$$

Thus each factor is a^2, and their product is $a^2a^2 = a^4$.

Therefore, the correct answer is **D**.

10. A circle of radius 10 inches has its center at the vertex C of an equilateral triangle ABC and passes through the other two vertices. The side AC extended through C intersects the circle at D. The number of degrees of angle ADB is:

A 15

B 30

C 60

D 90

E 120

Solution:

Since ABC is equilateral, the central angle $\angle ACB$ is 60°. The inscribed angle $\angle ADB$ intercepts the same minor arc AB, so its measure is half the central angle:

$$\angle ADB = \frac{60^\circ}{2} = 30^\circ.$$

Thus, the correct answer is **B**.

11. The expression $1-\frac{1}{1+\sqrt{3}}+\frac{1}{1-\sqrt{3}}$ equals:

A $1-\sqrt{3}$

B 1

C $-\sqrt{3}$

D $\sqrt{3}$

E $1+\sqrt{3}$

Solution:

Combining the fractional terms,

$$\begin{aligned} &-\frac{1}{1+\sqrt{3}}+\frac{1}{1-\sqrt{3}} \\ &=\frac{(1+\sqrt{3})-(1-\sqrt{3})}{(1+\sqrt{3})(1-\sqrt{3})} \\ &=\frac{2\sqrt{3}}{-2} \\ &=-\sqrt{3}. \end{aligned}$$

Adding the initial 1 gives $1-\sqrt{3}$.

Thus, the correct answer is **A**.

12. If $x^{-1} - 1$ is divided by $x - 1$ the quotient is:

A 1

B $\dfrac{1}{x-1}$

C $-\dfrac{1}{x-1}$

D $\dfrac{1}{x}$

E $-\dfrac{1}{x}$

Solution:

Where the quotient is defined,

$$\begin{aligned} \frac{x^{-1}-1}{x-1} &= \frac{\frac{1-x}{x}}{x-1} \\ &= \frac{-(x-1)}{x(x-1)} \\ &= -\frac{1}{x}. \end{aligned}$$

Thus, the correct answer is **E**.

13. Given two positive integers x and y with $x < y$. The percent that x is less than y is:

A $\frac{100(y-x)}{x}$

B $\frac{100(x-y)}{x}$

C $\frac{100(y-x)}{y}$

D $100(y-x)$

E $100(x-y)$

Solution:

The difference is $y - x$. Measured as a fraction of the reference value y, this is $\frac{y-x}{y}$. Multiplying by 100 converts the fraction to a percent:

$$\frac{100(y-x)}{y}.$$

Thus, the correct answer is **C**.

14. The points $A, B,$ and C are on a circle $O.$ The tangent line at A and the secant BC intersect at P, B lying between C and $P.$ If $BC = 20$ and $PA = 10\sqrt{3}$, then PB equals:

- A 5
- B 10
- C $10\sqrt{3}$
- D 20
- E 30

Solution:

Let $PB = t.$ Since B lies between P and C, we have $PC = t + 20.$ The tangent-secant theorem gives

$$(10\sqrt{3})^2 = t(t+20).$$

Thus $t^2 + 20t - 300 = 0$, or $(t-10)(t+30) = 0.$ A length is positive, so $t = 10.$

Therefore, the correct answer is **B**.

15. The root(s) of $\dfrac{15}{x^2-4} - \dfrac{2}{x-2} = 1$ is (are):

A -5 and 3

B ± 2

C 2 only

D -3 and 5

E 3 only

Solution:

For $x \neq \pm 2$, multiplying by $x^2 - 4$ gives

$$15 - 2(x+2) = x^2 - 4.$$

Hence $x^2 + 2x - 15 = 0$, so

$$(x+5)(x-3) = 0.$$

Both $x = -5$ and $x = 3$ satisfy the domain restriction.

Thus, the correct answer is **A**.

16. The sum of three numbers is 98. The ratio of the first to the second is $\frac{2}{3}$, and the ratio of the second to the third is $\frac{5}{8}$. The second number is:

- A 15
- B 20
- C 30
- D 32
- E 33

Solution:

The first-to-second ratio $2:3$ and second-to-third ratio $5:8$ combine to

$$\begin{aligned} &\text{first} : \text{second} : \text{third} \\ &= 10 : 15 : 24. \end{aligned}$$

The 49 total parts equal 98, so each part is 2. The second number is $15 \cdot 2 = 30$.

Thus, the correct answer is **C**.

17. The fraction $\dfrac{5x-11}{2x^2+x-6}$ was obtained by adding the two fractions $\dfrac{A}{x+2}$ and $\dfrac{B}{2x-3}$. The values of A and B must be, respectively:

A $5x, -11$

B $-11, 5x$

C $-1, 3$

D $3, -1$

E $5, -11$

Solution:

Combining the proposed partial fractions gives

$$\frac{A(2x-3)+B(x+2)}{(x+2)(2x-3)}.$$

Matching its numerator with $5x-11$ yields

$$\begin{aligned} 2A + B &= 5, \\ -3A + 2B &= -11. \end{aligned}$$

Solving gives $A = 3$ and $B = -1$.

Thus, the correct answer is **D**.

18. If $10^{2y} = 25,$ then 10^{-y} equals:

A $-\frac{1}{5}$

B $\frac{1}{625}$

C $\frac{1}{50}$

D $\frac{1}{25}$

E $\frac{1}{5}$

Solution:

We have

$$(10^y)^2 = 25.$$

Since 10^y is positive, $10^y = 5.$ Taking the reciprocal gives $10^{-y} = \frac{1}{5}.$

Thus, the correct answer is **E**.

19. Two candles of the same height are lighted at the same time. The first is consumed in 4 hours and the second in 3 hours. Assuming that each candle burns at a constant rate, in how many hours after being lighted was the first candle twice the height of the second?

A $\frac{3}{4}$ hr.

B $1\frac{1}{2}$ hr.

C 2 hr.

D $2\frac{2}{5}$ hr.

E $2\frac{1}{2}$ hr.

Solution:

Let each initial height be 1. After t hours the remaining heights are $1-\frac{t}{4}$ and $1-\frac{t}{3}$. The required condition is

$$1-\frac{t}{4}=2\left(1-\frac{t}{3}\right).$$

Multiplying by 12 gives $12-3t=24-8t$, so $5t=12$ and $t=\frac{12}{5}=2\frac{2}{5}$.

Thus, the correct answer is **D**.

20. If $(0.2)^x = 2$ and $\log 2 = 0.3010$, then the value of x to the nearest tenth is:

- A -10.0
- B -0.5
- C -0.4
- D -0.2
- E 10.0

Solution:

Taking common logarithms gives

$$x \log(0.2) = \log 2.$$

Since $\log(0.2) = \log 2 - 1 = -0.6990$,

$$x = \frac{0.3010}{-0.6990} \approx -0.4306.$$

To the nearest tenth, this is -0.4.

Thus, the correct answer is **C**.

21. If each of two intersecting lines intersects a hyperbola and neither line is tangent to the hyperbola, then the possible number of points of intersection with the hyperbola is:

A 2

B 2 or 3

C 2 or 4

D 3 or 4

E $2, 3,$ or 4

Solution:

A line can meet a hyperbola in either one finite point, when its direction is parallel to an asymptote, or two finite points. Since neither line is tangent, the two lines can therefore contribute $1+1=2, 1+2=3,$ or $2+2=4$ intersections with the hyperbola.

Thus, $2, 3,$ or 4 are possible, and the correct answer is **E**.

22. Jones covered a distance of 50 miles on his first trip. On a later trip he traveled 300 miles while going three times as fast. His new time compared with the old time was:

A three times as much

B twice as much

C the same

D half as much

E a third as much

Solution:

If the first speed was v, the old time was $\frac{50}{v}$. The later time was

$$\frac{300}{3v} = \frac{100}{v} = 2\left(\frac{50}{v}\right).$$

Thus the new time was twice the old time.

The correct answer is **B**.

23. About the equation $ax^2 - 2x\sqrt{2} + c = 0$, with a and c real constants, we are told that the discriminant is zero. The roots are necessarily:

A equal and integral

B equal and rational

C equal and real

D equal and irrational

E equal and imaginary

Solution:

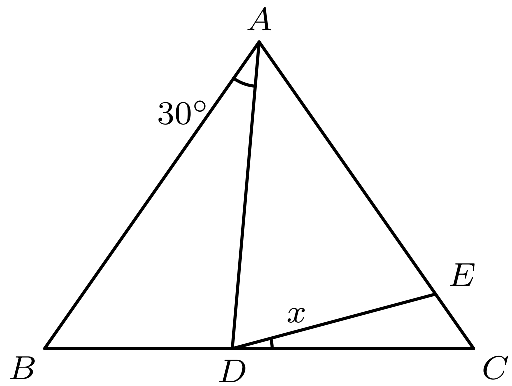
A quadratic with real coefficients and discriminant zero has the repeated root

$$x = \frac{-(-2\sqrt{2})}{2a} = \frac{\sqrt{2}}{a}.$$

Since a is a nonzero real number, this root is real. It need not always be rational or always be irrational.

Thus, the roots are necessarily equal and real, so the correct answer is **C**.

24. In the figure $AB = AC$, angle $BAD = 30^{\circ}$, and $AE = AD$. Then x equals:



A $7\frac{1}{2}^{\circ}$

B 10°

C $12\frac{1}{2}^{\circ}$

D 15°

E 20°

Solution:

Let $\angle DAE = \alpha$. Since $AB = AC$ and $AE = AD$,

$$\angle ACB = 75^{\circ} - \frac{\alpha}{2},$$
$$\angle ADE = 90^{\circ} - \frac{\alpha}{2}.$$

Triangle ACD gives $\angle ADC = 180^{\circ} - \alpha - \angle ACB$, so $\angle ADC = 105^{\circ} - \frac{\alpha}{2}$. Since $\angle ADC = \angle ADE + x$,

$$x = \left(105^{\circ} - \frac{\alpha}{2}\right) - \left(90^{\circ} - \frac{\alpha}{2}\right)$$
$$= 15^{\circ}.$$

Thus, the correct answer is **D**.

25. The sum of all numbers of the form $2k + 1$, where k takes on integral values from 1 to n is:

A n^2

B $n(n+1)$

C $n(n+2)$

D $(n+1)^2$

E $(n+1)(n+2)$

Solution:

We compute

$$\begin{aligned}\sum_{k=1}^{n}(2k+1) &= 2\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 \\ &= n(n+1) + n \\ &= n(n+2).\end{aligned}$$

Thus, the correct answer is **C**.

26. Which one of the following combinations of given parts does not determine the indicated triangle?

- A base angle and vertex angle; isosceles triangle
- B vertex angle and the base; isosceles triangle
- C the radius of the circumscribed circle; equilateral triangle
- D one arm and the radius of the inscribed circle; right triangle
- E two angles and a side opposite one of them; scalene triangle

Solution:

In an isosceles triangle, a base angle and the vertex angle determine all three angles, but no side length is specified. Therefore triangles of every scale have the same given data, so the triangle is not determined. Each other choice includes enough length information, directly or through a radius, to fix the scale as well as the shape.

Thus, the correct answer is **A**.

27. If an angle of a triangle remains unchanged but each of its two including sides is doubled, then the area is multiplied by:

- A 2
- B 3
- C 4
- D 6
- E more than 6

Solution:

If the included sides are a and b, and their unchanged angle is C, the original area is $\frac{1}{2}ab\sin C$. After both sides are doubled, the area is

$$\frac{1}{2}(2a)(2b)\sin C$$
$$= 2ab\sin C$$
$$= 4\left(\frac{1}{2}ab\sin C\right).$$

Thus, the correct answer is **C**.

28. Mr. J left his entire estate to his wife, his daughter, his son, and the cook. His daughter and son got half the estate, sharing in the ratio of 4 to 3. His wife got twice as much as the son. If the cook received a bequest of $\$500$, then the entire estate was:

A $\$3500$

B $\$5500$

C $\$6500$

D $\$7000$

E $\$7500$

Solution:

Let the daughter receive $4x$ and the son $3x$. Together they receive $7x$, which is half the estate. The wife receives $6x$, so the other half gives

$$7x = 6x + 500.$$

Thus $x = 500$, and the whole estate is $2(7x) = 14(500) = 7000$ dollars.

Therefore, the correct answer is **D**.

29. The points of intersection of $xy = 12$ and $x^2 + y^2 = 25$ are joined in succession. The resulting figure is:

A a straight line

B an equilateral triangle

C a parallelogram

D a rectangle

E a square

Solution:

At an intersection,

$$\begin{aligned}(x+y)^2 &= x^2 + y^2 + 2xy \\ &= 25 + 24 = 49,\end{aligned}$$

so $x + y = \pm 7$. Together with $xy = 12$, this gives the four points

$$\begin{aligned}&(3,4),\ (4,3), \\ &(-3,-4),\ (-4,-3).\end{aligned}$$

In their cyclic order around the circle, adjacent side vectors are perpendicular, so the quadrilateral is a rectangle. Its adjacent side lengths are unequal, so it is not a square.

Thus, the correct answer is **D**.

30. If the altitude of an equilateral triangle is $\sqrt{6}$, then the area is:

A $2\sqrt{2}$

B $2\sqrt{3}$

C $3\sqrt{3}$

D $6\sqrt{2}$

E 12

Solution:

Let the side length be s. From

$$\frac{s\sqrt{3}}{2} = \sqrt{6},$$

we get $s = 2\sqrt{2}$. Hence the area is

$$\frac{1}{2}s\sqrt{6} = \frac{1}{2}(2\sqrt{2})(\sqrt{6}) = 2\sqrt{3}.$$

Thus, the correct answer is **B**.

31. In our number system the base is ten. If the base were changed to four you would count as follows: $1, 2, 3, 10, 11, 12, 13, 20, 21, 22, 23, 30, \ldots$ The twentieth number would be:

A 20

B 38

C 44

D 104

E 110

Solution:

The twentieth positive integer has ordinary value 20. Since

$$20 = 1 \cdot 4^2 + 1 \cdot 4 + 0,$$

its base-four representation is 110.

Thus, the correct answer is **E**.

32. George and Henry started a race from opposite ends of the pool. After a minute and a half, they passed each other in the center of the pool. If they lost no time in turning and maintained their respective speeds, how many minutes after starting did they pass each other the second time?

A 3

B $4\frac{1}{2}$

C 6

D $7\frac{1}{2}$

E 9

Solution:

Because they started simultaneously from opposite ends and first met at the center, their speeds are equal. Each reaches the opposite end after 3 minutes. They turn immediately, and each then needs another 1.5 minutes to return to the center, where they meet again. The second meeting occurs after

$$3 + 1.5 = 4.5 = 4\frac{1}{2}$$

minutes.

Thus, the correct answer is **B**.

33. The number $\sqrt{2}$ is equal to:

A a rational fraction

B a finite decimal

C 1.41421

D an infinite repeating decimal

E an infinite non-repeating decimal

Solution:

The number $\sqrt{2}$ is irrational. A rational number has a decimal expansion that either terminates or eventually repeats, whereas an irrational number has an infinite non-repeating decimal expansion. The finite decimal 1.41421 is only an approximation.

Thus, the correct answer is **E**.

34. If n is any whole number, $n^2(n^2-1)$ is always divisible by:

A 12

B 24

C any multiple of 12

D $12-n$

E 12 and 24

Solution:

We have

$$n^2(n^2-1)=n^2(n-1)(n+1).$$

Among $n-1, n, n+1$, one is divisible by 3. If n is even, n^2 is divisible by 4; if n is odd, both $n-1$ and $n+1$ are even, so their product is divisible by 4. Thus the expression is always divisible by 12. It is not always divisible by 24, since $n=2$ gives 12.

Therefore, the correct answer is **A**.

35. A rhombus is formed by two radii and two chords of a circle whose radius is 16 feet. The area of the rhombus in square feet is:

A 128

B $128\sqrt{3}$

C 256

D 512

E $512\sqrt{3}$

Solution:

All four sides of the rhombus equal the circle's radius, 16. A chord of length equal to the radius forms an equilateral triangle with the two radii to its endpoints, so the included central angle is 60°. Therefore the rhombus has area

$$\begin{aligned} 16^2 \sin 60^\circ &= 256 \cdot \frac{\sqrt{3}}{2} \\ &= 128\sqrt{3}. \end{aligned}$$

Thus, the correct answer is **B**.

36. If the sum $1 + 2 + 3 + \cdots + K$ is a perfect square N^2 and if N is less than 100, then the possible values for K are:

- A only 1
- B 1 and 8
- C only 8
- D 8 and 49
- E $1, 8,$ and 49

Solution:

The condition is

$$\frac{K(K+1)}{2} = N^2,$$

or

$$(2K+1)^2 - 8N^2 = 1.$$

The positive solutions of this Pell equation are generated from the fundamental solution $3 + \sqrt{8}$. Their first pairs $(2K+1, N)$ are

$$\begin{aligned}&(3,1),\ (17,6),\\ &(99,35),\ (577,204),\ldots\end{aligned}$$

Thus the values with $N < 100$ are $K = 1, 8, 49.$

Therefore, the correct answer is **E**.

37. On a map whose scale is 400 miles to an inch and a half, a certain estate is represented by a rhombus having a 60° angle. The diagonal opposite 60° is $\frac{3}{16}$ in. The area of the estate in square miles is:

A $\frac{2500}{\sqrt{3}}$

B $\frac{1250}{\sqrt{3}}$

C 1250

D $\frac{5625\sqrt{3}}{2}$

E $1250\sqrt{3}$

Solution:

The diagonal opposite the 60° angle divides the rhombus into two equilateral triangles, so its length equals the rhombus side. The map scale is $\frac{800}{3}$ miles per inch, hence the actual side length is

$$\frac{3}{16}\cdot\frac{800}{3}=50$$

miles. Therefore the rhombus area is

$$\begin{aligned}50^2\sin 60^\circ &= 2500\cdot\frac{\sqrt{3}}{2}\\ &= 1250\sqrt{3}.\end{aligned}$$

Thus, the correct answer is **E**.

38. In a right triangle with sides a and b, and hypotenuse c, the altitude drawn on the hypotenuse is x. Then:

A $ab = x^2$

B $\dfrac{1}{a} + \dfrac{1}{b} = \dfrac{1}{x}$

C $a^2 + b^2 = 2x^2$

D $\dfrac{1}{x^2} = \dfrac{1}{a^2} + \dfrac{1}{b^2}$

E $\dfrac{1}{x} = \dfrac{b}{a}$

Solution:

Equating two area formulas gives

$$\frac{1}{2}ab = \frac{1}{2}cx,$$

so $ab = cx$. Squaring and using $c^2 = a^2 + b^2$ yields

$$a^2b^2 = x^2(a^2 + b^2).$$

Dividing by $a^2b^2x^2$ gives

$$\frac{1}{x^2} = \frac{1}{a^2} + \frac{1}{b^2}.$$

Thus, the correct answer is **D**.

39. The hypotenuse c and one arm a of a right triangle are consecutive integers. The square of the second arm is:

- A ca
- B $\frac{c}{a}$
- C $c+a$
- D $c-a$
- E none of these

Solution:

By the Pythagorean theorem,

$$b^2 = c^2 - a^2 = (c-a)(c+a).$$

Since c and a are consecutive and $c > a$, we have $c - a = 1$. Hence $b^2 = c + a$.

Thus, the correct answer is **C**.

40. If $V = gt + V_0$ and $S = \frac{1}{2}gt^2 + V_0t$, then t equals:

A $\frac{2S}{V + V_0}$

B $\frac{2S}{V - V_0}$

C $\frac{2S}{V_0 - V}$

D $\frac{2S}{V}$

E $2S - V$

Solution:

From the first equation, $gt = V - V_0$. Therefore

$$\begin{aligned} S &= \frac{1}{2}(gt)t + V_0t \\ &= \frac{1}{2}(V - V_0)t + V_0t \\ &= \frac{1}{2}(V + V_0)t. \end{aligned}$$

Solving gives $t = \frac{2S}{V+V_0}$.

Thus, the correct answer is **A**.

41. The equation $3y^2 + y + 4 = 2(6x^2 + y + 2)$ where $y = 2x$ is satisfied by:

- A. no value of x
- B. all values of x
- C. $x = 0$ only
- D. all integral values of x only
- E. all rational values of x only

Solution:

Substituting $y = 2x$ gives

$$12x^2 + 2x + 4 = 12x^2 + 4x + 4.$$

Cancelling the common quadratic and constant terms leaves $2x = 4x$, so $x = 0$. This value satisfies both equations.

Thus, the correct answer is **C**.

42. The equation $\sqrt{x+4}-\sqrt{x-3}+1=0$ has:

A no root

B one real root

C one real root and one imaginary root

D two imaginary roots

E two real roots

Solution:

For a real solution, $x \geq 3$. Then $x+4>x-3$, so

$$\sqrt{x+4}>\sqrt{x-3}.$$

Consequently $\sqrt{x+4}-\sqrt{x-3}+1>1$, and it cannot equal zero. The original equation is a real radical expression, so nonreal roots are outside its domain.

The correct answer is **A**.

43. The number of scalene triangles having all sides of integral lengths, and perimeter less than 13 is:

A 1

B 2

C 3

D 4

E 18

Solution:

Write the distinct integer sides in increasing order. Checking the small possibilities under the triangle inequality and perimeter bound gives

$$(2,3,4),\ (2,4,5),$$
$$(3,4,5).$$

For $c \geq 6,$ the smallest new scalene candidate satisfying $a+b>c$ is $(3,4,6),$ whose perimeter is already $13,$ and larger choices cannot qualify. Thus there are 3 triangles.

The correct answer is **C**.

44. If $x < a < 0$ means that x and a are numbers such that x is less than a and a is less than zero, then:

- A $x^2 < ax < 0$
- B $x^2 > ax > a^2$
- C $x^2 < a^2 < 0$
- D $x^2 > ax$ but $ax < 0$
- E $x^2 > a^2$ but $a^2 < 0$

Solution:

Since $x < a < 0$, multiplying $x < a$ by the negative number x reverses the inequality and gives $x^2 > ax$. Multiplying the same inequality by the negative number a gives $ax > a^2$. Therefore

$$x^2 > ax > a^2.$$

Thus, the correct answer is **B**.

45. A wheel with a rubber tire has an outside diameter of 25 in. When the radius has been decreased a quarter of an inch, the number of revolutions of the wheel in one mile will:

A be increased about 2%

B be increased about 1%

C be increased about 20%

D be increased $\frac{1}{2}\%$

E remain the same

Solution:

The original radius is 12.5 inches and the new radius is 12.25 inches. For a fixed distance, the number of revolutions varies inversely with radius, so the relative increase is

$$\begin{aligned} \frac{12.5}{12.25} - 1 &= \frac{50}{49} - 1 \\ &= \frac{1}{49} \\ &\approx 2.04\%. \end{aligned}$$

This is about 2%.

Thus, the correct answer is **A**.

46. For the equation $\frac{1+x}{1-x} = \frac{N+1}{N}$ to be true where N is positive, x can have:

A any positive value less than 1

B any value less than 1

C the value zero only

D any non-negative value

E any value

Solution:

Cross-multiplication gives

$$N(1+x) = (N+1)(1-x),$$

so $x(2N+1) = 1$ and

$$x = \frac{1}{2N+1}.$$

Every positive N gives $0 < x < 1$. Conversely, for any $0 < x < 1$,

$$N = \frac{1-x}{2x} > 0,$$

so such an N exists.

Thus, x may be any positive value less than 1, and the correct answer is **A**.

47. An engineer said he could finish a highway section in 3 days with his present supply of a certain type of machine. However, with 3 more of these machines the job could be done in 2 days. If the machines all work at the same rate, how many days would it take to do the job with one machine?

A 6

B 12

C 15

D 18

E 36

Solution:

If there are presently m machines, the job requires $3m$ machine-days. With three more machines it requires $2(m+3)$ machine-days, so

$$3m = 2(m+3),$$

giving $m = 6$. The job therefore requires $3m = 18$ machine-days, so one machine would take 18 days.

Thus, the correct answer is **D**.

48. If p is a positive integer, then $\dfrac{3p+25}{2p-5}$ can be a positive integer, if and only if p is:

A at least 3

B equal to $3, 5, 9,$ or 35

C no more than 35

D equal to 35

E equal to 3 or 35

Solution:

Set $q = 2p - 5$. A positive quotient requires $q > 0$, and q is odd. Since $p = \frac{q+5}{2}$,

$$\begin{aligned}\frac{3p+25}{2p-5} &= \frac{3q+65}{2q} \\ &= \frac{3+\frac{65}{q}}{2}.\end{aligned}$$

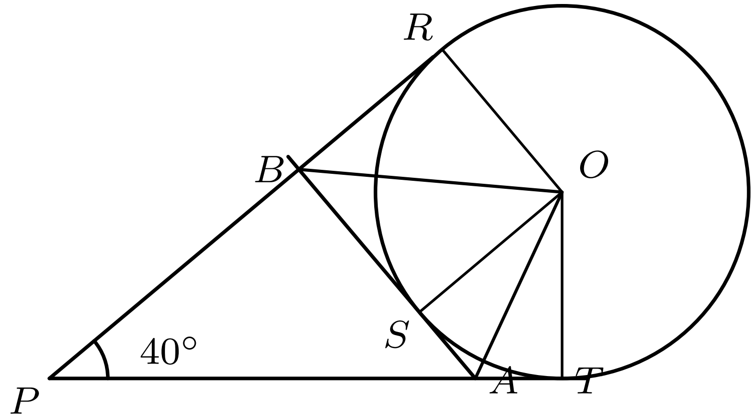
Thus q must be a positive divisor of 65. The possibilities $q = 1, 5, 13, 65$ give

$$p = 3, 5, 9, 35,$$

respectively, and each works. No other positive integer p works.

Therefore the mathematically correct set is $3, 5, 9, 35,$ shown in the adjusted choice **B**.

49. Triangle PAB is formed by three tangents to circle O and $\angle APB = 40°$; then angle AOB equals:



A $45°$

B $50°$

C $55°$

D $60°$

E $70°$

Solution:

The circle lies opposite P across side AB, so O is the excenter opposite P. Thus AO and BO bisect the exterior angles at A and B. If the interior angles at A and B are α and β, then triangle AOB has angles $90° - \frac{\alpha}{2}$ and $90° - \frac{\beta}{2}$ at A and B. Hence

$$\begin{aligned}\angle AOB &= 180° - \left(90° - \frac{\alpha}{2}\right) \\ &\quad - \left(90° - \frac{\beta}{2}\right) \\ &= \frac{\alpha+\beta}{2} = \frac{180° - 40°}{2} \\ &= 70°.\end{aligned}$$

Thus, the correct answer is **E**.

50. In triangle $ABC, CA = CB$. On CB square $BCDE$ is constructed away from the triangle. If x is the number of degrees in angle DAB, then

- A x depends upon triangle ABC
- B x is independent of the triangle
- C x may equal angle CAD
- D x can never equal angle CAB
- E x is greater than 45° but less than 90°

Solution:

Scale the equal sides to 1 and place

$$C = (0,0), \qquad B = (1,0),$$
$$A = (\cos\theta, \sin\theta).$$

Since square $BCDE$ is constructed away from the triangle, $D = (0,-1)$. Put $c = \cos\theta$ and $s = \sin\theta$. Then

$$\overrightarrow{AB} = (1-c, -s),$$
$$\overrightarrow{AD} = (-c, -1-s).$$

Their dot product is $1+s-c$, while the absolute value of their two-dimensional cross product is also $1+s-c$. Therefore

$$\tan\angle DAB = 1,$$

so $\angle DAB = 45^\circ$, independent of θ and hence independent of the triangle's shape.

Thus, the correct answer is **B**.

Problems: https://live.poshenloh.com/past-contests/amc12/1956