1955 AMC 12 Solutions

Typeset by: LIVE by Po-Shen Loh
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1. Which one of the following is not equivalent to 0.000000375?

   A 3.75×10^{-7}

   B $3\frac{3}{4} \times 10^{-7}$

   C 375×10^{-9}

   D $\frac{3}{8} \times 10^{-7}$

   E $\frac{3}{8} \times 10^{-6}$

Solution:

The given decimal is 3.75×10^{-7}. Choices A and B state this directly, while

$$375 \times 10^{-9} = 3.75 \times 10^{-7}$$

and $\frac{3}{8} \times 10^{-6} = 0.375 \times 10^{-6} = 3.75 \times 10^{-7}$. But

$$\frac{3}{8} \times 10^{-7} = 3.75 \times 10^{-8},$$

which is ten times smaller.

Thus, the correct answer is **D**.

2. The smaller angle between the hands of a clock at $12{:}25$ p.m. is:

A $132^\circ 30'$

B $137^\circ 30'$

C 150°

D $137^\circ 32'$

E 137°

Solution:

At $12{:}25$, the minute hand is $25 \cdot 6^\circ = 150^\circ$ past 12, while the hour hand is $25 \cdot 0.5^\circ = 12.5^\circ$. Their smaller separation is

$$150^\circ - 12.5^\circ = 137.5^\circ = 137^\circ 30'.$$

Thus, the correct answer is **B**.

3. If each number in a set of ten numbers is increased by 20, the arithmetic mean (average) of the original ten numbers:

- A remains the same
- B is increased by 20
- C is increased by 200
- D is increased by 10
- E is increased by 2

Solution:

If the original sum is S, the new sum is $S + 10 \cdot 20$. Hence the new mean is

$$\frac{S+200}{10} = \frac{S}{10} + 20.$$

The mean is increased by 20.

Thus, the correct answer is **B**.

4. The equality $\dfrac{1}{x-1} = \dfrac{2}{x-2}$ is satisfied by:

A no real values of x

B either $x = 1$ or $x = 2$

C only $x = 1$

D only $x = 2$

E only $x = 0$

Solution:

For $x \neq 1, 2,$ cross-multiplication gives

$$x - 2 = 2(x-1) = 2x - 2,$$

so $x = 0.$ This value makes both original denominators nonzero and satisfies the equality.

Thus, the correct answer is **E**.

5. y varies inversely as the square of x. When $y = 16, x = 1$. When $x = 8, y$ equals:

A 2

B 128

C 64

D $\frac{1}{4}$

E 1024

Solution:

Inverse-square variation gives $y = \frac{k}{x^2}$. Since $16 = \frac{k}{1^2}$, we have $k = 16$. At $x = 8$,

$$y = \frac{16}{8^2} = \frac{1}{4}.$$

Thus, the correct answer is **D**.

6. A merchant buys a number of oranges at 3 for $10¢$ and an equal number at 5 for $20¢$. To "break even" he must sell all at:

- A 8 for $30¢$
- B 3 for $11¢$
- C 5 for $18¢$
- D 11 for $40¢$
- E 13 for $50¢$

Solution:

Suppose he buys 15 oranges at each rate. The first 15 cost $50¢$ and the second 15 cost $60¢$, for $110¢$ total. Thus 30 oranges must sell for $110¢$, which is equivalent to 3 for $11¢$.

Therefore, the correct answer is **B**.

7. If a worker receives a 20 percent cut in wages, he may regain his original pay exactly by obtaining a raise of:

A 20 percent

B 25 percent

C $22\frac{1}{2}$ percent

D $\$20$

E $\$25$

Solution:

If the original wage is W, the reduced wage is $0.8W$. The needed increase is $0.2W$, which as a fraction of the reduced wage is

$$\frac{0.2W}{0.8W} = \frac{1}{4} = 25\%.$$

Thus, the correct answer is **B**.

8. The graph of $x^2 - 4y^2 = 0$:

A is a hyperbola intersecting only the x-axis

B is a hyperbola intersecting only the y-axis

C is a hyperbola intersecting neither axis

D is a pair of straight lines

E does not exist

Solution:

Factoring,

$$x^2 - 4y^2 = (x - 2y)(x + 2y).$$

Thus the graph is the union of the two straight lines $x = 2y$ and $x = -2y$.

The correct answer is **D**.

9. A circle is inscribed in a triangle with sides $8, 15,$ and $17.$ The radius of the circle is:

- A 6
- B 2
- C 5
- D 3
- E 7

Solution:

Because $8^2 + 15^2 = 17^2$, the triangle is right. Its inradius is

$$r = \frac{8 + 15 - 17}{2} = 3.$$

Thus, the correct answer is **D**.

10. How many hours does it take a train traveling at an average rate of 40 mph between stops to travel a miles if it makes n stops of m minutes each?

A $\frac{3a+2mn}{120}$

B $3a+2mn$

C $\frac{3a+2mn}{12}$

D $\frac{a+mn}{40}$

E $\frac{a+40mn}{40}$

Solution:

The train moves for $\frac{a}{40}$ hours and is stopped for $\frac{mn}{60}$ hours. Hence the total time is

$$\frac{a}{40}+\frac{mn}{60}=\frac{3a+2mn}{120}.$$

Thus, the correct answer is **A**.

11. The negation of the statement “No slow learners attend this school,” is:

A All slow learners attend this school.

B All slow learners do not attend this school.

C Some slow learners attend this school.

D Some slow learners do not attend this school.

E No slow learners attend this school.

Solution:

The original statement says that every slow learner is absent from the school. Its negation is that at least one slow learner attends the school.

Thus, “Some slow learners attend this school” is correct, so the answer is **C**.

12. The solution of $\sqrt{5x-1}+\sqrt{x-1}=2$ is:

A $x=2, x=1$

B $x=\frac{2}{3}$

C $x=2$

D $x=1$

E $x=0$

Solution:

The domain requires $x \geq 1$. At $x=1$, the left side is $2+0=2$, so $x=1$ works. To see that there is no other solution, note that both radicals are nondecreasing for $x \geq 1$, and $\sqrt{5x-1}$ is strictly increasing. Therefore the sum exceeds 2 for every $x>1$.

Thus, the correct answer is **D**.

13. The fraction $\dfrac{a^{-4}-b^{-4}}{a^{-2}-b^{-2}}$ is equal to:

A $a^{-6}-b^{-6}$

B $a^{-2}-b^{-2}$

C $a^{-2}+b^{-2}$

D $a^{2}+b^{2}$

E $a^{2}-b^{2}$

Solution:

Factoring the numerator,

$$a^{-4}-b^{-4}=(a^{-2}-b^{-2})\cdot(a^{-2}+b^{-2}).$$

Where the original fraction is defined, canceling the common factor leaves $a^{-2}+b^{-2}$.

Thus, the correct answer is **C**.

14. The length of rectangle R is 10 percent more than the side of square S. The width of the rectangle is 10 percent less than the side of the square. The ratio of the areas, $R : S$, is:

- A $99 : 100$
- B $101 : 100$
- C $1 : 1$
- D $199 : 200$
- E $201 : 200$

Solution:

If the square side is s, then the rectangle has dimensions $1.1s$ and $0.9s$. Therefore

$$\begin{aligned}\frac{[R]}{[S]} &= \frac{(1.1s)(0.9s)}{s^2} \\ &= 0.99 = \frac{99}{100}.\end{aligned}$$

Thus, the correct answer is **A**.

15. The ratio of the areas of two concentric circles is $1 : 3$. If the radius of the smaller is r, then the difference between the radii is best approximated by:

A $0.41r$

B 0.73

C 0.75

D $0.73r$

E $0.75r$

Solution:

If the larger radius is R, then

$$\frac{\pi r^2}{\pi R^2} = \frac{1}{3},$$

so $R = r\sqrt{3}$. The difference is

$$R - r = (\sqrt{3} - 1)r \approx 0.732r.$$

Thus, the best approximation is $0.73r$, and the correct answer is **D**.

16. The value of $\dfrac{3}{a+b}$ when $a = 4$ and $b = -4$ is:

A 3

B $\dfrac{3}{8}$

C 0

D any finite number

E meaningless

Solution:

Substitution gives $a + b = 4 + (-4) = 0$. Hence the expression becomes $\frac{3}{0}$, which is undefined.

Thus, the expression is meaningless and the correct answer is **E**.

17. If $\log x - 5\log 3 = -2$, then x equals:

A 1.25

B 0.81

C 2.43

D 0.8

E either 0.8 or 1.25

Solution:

Using common logarithms,

$$\begin{aligned}\log x &= 5\log 3 - 2 \\ &= \log(3^5) - \log 100 \\ &= \log\left(\frac{243}{100}\right).\end{aligned}$$

Therefore $x = 2.43$.

The correct answer is **C**.

18. The discriminant of the equation $x^2 + 2x\sqrt{3} + 3 = 0$ is zero. Hence, its roots are:

- A real and equal
- B rational and equal
- C rational and unequal
- D irrational and unequal
- E imaginary

Solution:

The quadratic formula gives

$$x = \frac{-2\sqrt{3}}{2} = -\sqrt{3}$$

twice because the discriminant is zero. The root is irrational, but the requested description that applies is “real and equal.”

Thus, the correct answer is **A**.

19. Two numbers whose sum is 6 and the absolute value of whose difference is 8 are roots of the equation:

A $x^2 - 6x + 7 = 0$

B $x^2 - 6x - 7 = 0$

C $x^2 + 6x - 8 = 0$

D $x^2 - 6x + 8 = 0$

E $x^2 + 6x - 7 = 0$

Solution:

Taking the larger number first,

$$u + v = 6, \qquad u - v = 8$$

gives $u = 7$ and $v = -1$. Their product is -7, so the monic equation with these roots is

$$x^2 - 6x - 7 = 0.$$

Thus, the correct answer is **B**.

20. The expression $\sqrt{25-t^2}+5$ equals zero for:

A no real or imaginary values of t

B no real values of t, but for some imaginary values

C no imaginary values of t, but for some real values

D $t=0$

E $t=\pm 5$

Solution:

For the expression to vanish, one would need

$$\sqrt{25-t^2}=-5.$$

The radical sign denotes the principal square root. It is nonnegative for a nonnegative real radicand and, over the complex numbers, is chosen with nonnegative real part; it therefore cannot equal -5. Squaring would introduce the extraneous candidate $t=0$, for which the original expression is 10.

Thus, no real or imaginary value works, and the correct answer is **A**.

21. Represent the hypotenuse of a right triangle by c and the area by A. The altitude on the hypotenuse is:

A $\dfrac{A}{c}$

B $\dfrac{2A}{c}$

C $\dfrac{A}{2c}$

D $\dfrac{A^2}{c}$

E $\dfrac{A}{c^2}$

Solution:

Taking the hypotenuse as the base and writing its altitude as h,

$$A = \frac{1}{2}ch.$$

Solving gives $h = \frac{2A}{c}$.

Thus, the correct answer is **B**.

22. On a $\$10,000$ order a merchant has a choice between three successive discounts of $20\%, 20\%,$ and 10% and three successive discounts of $40\%, 5\%,$ and 5%. By choosing the better offer, he can save:

A nothing at all

B $\$400$

C $\$330$

D $\$345$

E $\$360$

Solution:

The first offer leaves a fraction

$$0.8 \cdot 0.8 \cdot 0.9 = 0.576$$

of the price. The second leaves

$$0.6 \cdot 0.95^2 = 0.5415.$$

The second is cheaper by $0.576 - 0.5415 = 0.0345$ of the order, or

$$0.0345(10000) = 345$$

dollars.

Thus, the correct answer is **D**.

23. In checking the petty cash a clerk counts q quarters, d dimes, n nickels, and c cents. Later he discovers that x of the nickels were counted as quarters and x of the dimes were counted as cents. To correct the total obtained the clerk must:

- A make no correction
- B subtract $11¢$
- C subtract $11x¢$
- D add $11x¢$
- E add $x¢$

Solution:

The x nickels counted as quarters overstate the total by $20x¢$. The x dimes counted as cents understate it by $9x¢$. The net overstatement is

$$20x - 9x = 11x$$

cents, so that amount must be subtracted.

Thus, the correct answer is **C**.

24. The function $4x^2 - 12x - 1$:

A always increases as x increases

B always decreases as x decreases to 1

C cannot equal 0

D has a maximum value when x is negative

E has a minimum value of -10

Solution:

Completing the square,

$$4x^2 - 12x - 1 = 4\left(x - \frac{3}{2}\right)^2 - 10.$$

The squared term is nonnegative, so the minimum value is -10, attained at $x = \frac{3}{2}$.

Thus, the correct answer is **E**.

25. One of the factors of $x^4 + 2x^2 + 9$ is:

A $x^2 + 3$

B $x + 1$

C $x^2 - 3$

D $x^2 - 2x - 3$

E none of these

Solution:

We have

$$\begin{aligned} x^4 + 2x^2 + 9 &= (x^2 + 3)^2 \\ &\quad - (2x)^2 \\ &= (x^2 - 2x + 3) \\ &\quad \cdot (x^2 + 2x + 3). \end{aligned}$$

Neither factor appears among choices A through D.

Thus, the correct answer is **E**.

26. Mr. A owns a house worth $\$10,000$. He sells it to Mr. B at 10% profit. Mr. B sells the house back to Mr. A at a 10% loss. Then:

A Mr. A comes out even

B Mr. A makes $\$100$

C Mr. A makes $\$1,000$

D Mr. B loses $\$100$

E none of these is correct

Solution:

Mr. A first receives $1.10(10000) = 11000$ dollars. Mr. B then sells at a loss of 10% of his 11000-dollar cost, so Mr. A buys the house back for

$$0.90(11000) = 9900$$

dollars. Mr. A again owns the house and has gained $11000 - 9900 = 1100$ dollars; Mr. B has lost 1100 dollars. None of the stated amounts is correct.

Thus, the correct answer is **E**.

27. If r and s are the roots of $x^2 - px + q = 0$, then $r^2 + s^2$ equals:

A $p^2 + 2q$

B $p^2 - 2q$

C $p^2 + q^2$

D $p^2 - q^2$

E p^2

Solution:

By Vieta's formulas, $r + s = p$ and $rs = q$. Therefore

$$\begin{aligned} r^2 + s^2 &= (r+s)^2 - 2rs \\ &= p^2 - 2q. \end{aligned}$$

Thus, the correct answer is **B**.

28. On the same set of axes are drawn the graph of $y = ax^2 + bx + c$ and the graph of the equation obtained by replacing x by $-x$ in the given equation. If $b \neq 0$ and $c \neq 0$ these two graphs intersect:

- A in two points, one on the x-axis and one on the y-axis
- B in one point located on neither axis
- C only at the origin
- D in one point on the x-axis
- E in one point on the y-axis

Solution:

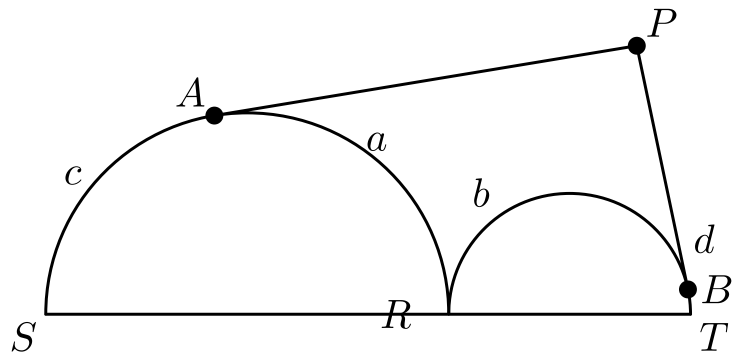
Replacing x by $-x$ gives $y = ax^2 - bx + c$. At an intersection,

$$ax^2 + bx + c = ax^2 - bx + c,$$

so $2bx = 0$. Since $b \neq 0$, we must have $x = 0$, and then $y = c \neq 0$. There is exactly one intersection, $(0, c)$, on the y-axis.

Thus, the correct answer is **E**.

29. In the figure $\overline{PA}$ is tangent to semicircle SAR; $\overline{PB}$ is tangent to semicircle RBT; SRT is a straight line; the arcs are indicated in the figure. Angle APB is measured by:



A $\frac{1}{2}(a-b)$

B $\frac{1}{2}(a+b)$

C $(c-a)-(d-b)$

D $a-b$

E $a+b$

Solution:

Extend $\overline{PR}$ to meet the large circle again at N and the small circle again at M. The equal base angles in the two isosceles triangles formed by the radii show that arcs RN and RM have the same measure; call it u. The tangent-secant theorem gives the two adjacent angles along the reflex route from $\overline{PA}$ to $\overline{PB}$:

$$\angle APR = \frac{2c-u}{2},$$
$$\angle RPB = \frac{2d+u}{2}.$$

Their sum is $c + d$. Hence the other angle between the tangents is

$$\begin{aligned} 360^\circ - (c+d) &= (180^\circ - c) \\ &\quad + (180^\circ - d) \\ &= a + b, \end{aligned}$$

because each upper semicircle has measure 180°.

Thus, the correct answer is **E**.

30. Each of the equations $3x^2 - 2 = 25$, $(2x-1)^2 = (x-1)^2$, $\sqrt{x^2 - 7} = \sqrt{x - 1}$ has:

- A two integral roots
- B no root greater than 3
- C no root zero
- D only one root
- E one negative root and one positive root

Solution:

The first equation gives $x = \pm 3$. Factoring the difference of squares in the second gives

$$\begin{aligned} &[(2x-1) - (x-1)] \\ &\quad \cdot [(2x-1) + (x-1)] \\ &= x(3x-2) = 0, \end{aligned}$$

so $x = 0$ or $\frac{2}{3}$. Squaring the third gives $x^2 - x - 6 = 0$, with candidates 3 and -2; only 3 satisfies the real-domain restrictions. Every root obtained is at most 3.

Thus, each equation has no root greater than 3, and the correct answer is **B**.

31. An equilateral triangle whose side is 2 is divided into a triangle and a trapezoid by a line drawn parallel to one of its sides. If the area of the trapezoid equals one-half of the area of the original triangle, the length of the median of the trapezoid is:

A $\frac{\sqrt{6}}{2}$

B $\sqrt{2}$

C $2+\sqrt{2}$

D $\frac{2+\sqrt{2}}{2}$

E $\frac{2\sqrt{3}-\sqrt{6}}{2}$

Solution:

The small triangle also has half the original area. If its side parallel to the original base has length b, similarity gives

$$\left(\frac{b}{2}\right)^2=\frac{1}{2},$$

so $b=\sqrt{2}$. The parallel sides of the trapezoid have lengths 2 and $\sqrt{2}$, so its median has length

$$\frac{2+\sqrt{2}}{2}.$$

Thus, the correct answer is **D**.

32. If the discriminant of $ax^2 + 2bx + c = 0$ is zero, then another true statement about a, b, and c is that:

A they form an arithmetic progression

B they form a geometric progression

C they are unequal

D they are all negative numbers

E only b is negative and a and c are positive

Solution:

A zero discriminant gives

$$(2b)^2 - 4ac = 0,$$

hence $b^2 = ac$. Equivalently, $\frac{a}{b} = \frac{b}{c}$ where the ratios are defined, which is the defining relation for a, b, c to form a geometric progression.

Thus, the correct answer is **B**.

33. Henry starts a trip when the hands of the clock are together between 8 a.m. and 9 a.m. He arrives at his destination between 2 p.m. and 3 p.m. when the hands of the clock are exactly 180° apart. The trip takes:

A 6 hr.

B 6 hr. $43\dfrac{7}{11}$ min.

C 5 hr. $16\dfrac{4}{11}$ min.

D 6 hr. 30 min.

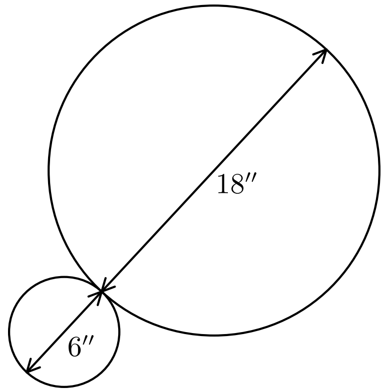
E none of these

Solution:

At $8{:}00$, the hour hand is 240° ahead. The minute hand gains at 5.5° per minute, so the hands coincide $\frac{240}{5.5} = \frac{480}{11}$ minutes after 8. At $2{:}00$, the hour hand is 60° ahead. For the hands to be 180° apart in the relevant direction, the minute hand must gain 240°, again taking $\frac{480}{11}$ minutes. The start and finish therefore have the same minute offset within their hours, exactly six hours apart.

Thus, the correct answer is **A**.

34. A 6-inch and 18-inch diameter pole are placed as in the figure and bound together with wire. The length of the shortest wire that will go around them is:



- A $12\sqrt{3}+16\pi$
- B $12\sqrt{3}+7\pi$
- C $12\sqrt{3}+14\pi$
- D $12+15\pi$
- E 24π

Solution:

The centers are 12 inches apart and their radii differ by 6. Thus each common external tangent segment has length

$$\sqrt{12^2-6^2}=6\sqrt{3},$$

contributing $12\sqrt{3}$ in all. The geometry of the tangent lines leaves a 120° exposed arc on the radius-3 circle and a 240° exposed arc on the radius-9 circle. Their lengths total

$$\begin{aligned}&\frac{120}{360}(2\pi\cdot 3)\\ &\quad+\frac{240}{360}(2\pi\cdot 9)\\ &\quad=2\pi+12\pi\\ &\quad=14\pi.\end{aligned}$$

Hence the wire length is $12\sqrt{3}+14\pi$.

Thus, the correct answer is **C**.

35. Three boys agree to divide a bag of marbles in the following manner. The first boy takes one more than half the marbles. The second takes a third of the number remaining. The third boy finds that he is left with twice as many marbles as the second boy. The original number of marbles:

- A is 8 or 38
- B cannot be determined from the given data
- C is 20 or 26
- D is 14 or 32
- E is none of these

Solution:

The first boy takes $\frac{n}{2}+1$, leaving $\frac{n}{2}-1$. The second takes one third of that remainder, and the third receives the other two thirds, automatically twice the second boy's share. Thus the share condition imposes no unique value of n. It only requires $\frac{n}{2}-1$ to be a nonnegative multiple of 3, so many values such as $8, 14, 20, 26, \ldots$ work.

Therefore the original number cannot be determined, and the correct answer is **B**.

36. A cylindrical oil tank, lying horizontally, has an interior length of 10 feet and an interior diameter of 6 feet. If the rectangular surface of the oil has an area of 40 square feet, the depth of the oil is:

A $\sqrt{5}$

B $2\sqrt{5}$

C $3-\sqrt{5}$

D $3+\sqrt{5}$

E either $3-\sqrt{5}$ or $3+\sqrt{5}$

Solution:

The rectangular surface has length $10,$ so its chord width in the circular cross-section is $\frac{40}{10}=4.$ Half the chord is $2,$ and the radius is $3,$ so the distance from the center to the chord is

$$\sqrt{3^2-2^2}=\sqrt{5}.$$

A chord of this length can lie either below or above the center. Measured from the bottom of the tank, the corresponding depths are $3-\sqrt{5}$ and $3+\sqrt{5}.$

Thus, the correct answer is **E**.

37. A three-digit number has, from left to right, the digits $h, t,$ and u with $h > u$. When the number with the digits reversed is subtracted from the original number, the units' digit in the difference is 4. The next two digits, from right to left, are:

A 5 and 9

B 9 and 5

C impossible to tell

D 5 and 4

E 4 and 5

Solution:

The difference is $99(h-u)$. Since $h-u$ is an integer from 1 through 9, and the units digit is 4, we need $9(h-u) \equiv 4 \pmod{10}$. This gives $h-u=6$, so the difference is

$$99 \cdot 6 = 594.$$

Moving from right to left after the units digit 4, the next digits are 9 and 5.

Thus, the correct answer is **B**.

38. Four positive integers are given. Select any three of these integers, find their arithmetic average, and add this result to the fourth integer. Thus the numbers $29, 23,$ 21 and 17 are obtained. One of the original integers is:

A 19

B 21

C 23

D 29

E 17

Solution:

If the original integers sum to S, the result associated with singled-out integer x is

$$x + \frac{S - x}{3} = \frac{S + 2x}{3}.$$

Summing all four reported results counts the total as $2S$, so

$$S = \frac{29 + 23 + 21 + 17}{2} = 45.$$

The result 29 therefore comes from $x = \dfrac{3(29) - 45}{2} = 21.$

Thus, one original integer is 21, and the correct answer is **B**.

39. If $y = x^2 + px + q$, then if the least possible value of y is zero, q is equal to:

A 0

B $\frac{p^2}{4}$

C $\frac{p}{2}$

D $-\frac{p}{2}$

E $\frac{p^2}{4} - q$

Solution:

Completing the square,

$$y = \left(x + \frac{p}{2}\right)^2 + q - \frac{p^2}{4}.$$

Its least value is $q - \frac{p^2}{4}$. Setting this equal to zero gives $q = \frac{p^2}{4}$.

Thus, the correct answer is **B**.

40. If $b \neq d$, the fractions $\dfrac{ax+b}{cx+d}$ and $\dfrac{b}{d}$ are unequal if:

A $a = c = 1$ and $x \neq 0$

B $a = b = 0$

C $a = c = 0$

D $x = 0$

E $ad = bc$

Solution:

Where the fractions are defined, equality would require

$$d(ax+b) = b(cx+d),$$

which simplifies to $x(ad - bc) = 0$. Under choice A, $a = c = 1$ gives $ad - bc = d - b \neq 0$, and $x \neq 0$, so equality is impossible. Each of B through E instead forces equality directly.

Thus, the fractions are unequal under choice **A**.

41. A train traveling from Aytown to Beetown meets with an accident after 1 hr. It is stopped for $\frac{1}{2}$ hr., after which it proceeds at four-fifths of its usual rate, arriving at Beetown 2 hr. late. If the train had covered 80 miles more before the accident, it would have been just 1 hr. late. The usual rate of the train is:

A 20 mph

B 30 mph

C 40 mph

D 50 mph

E 60 mph

Solution:

Let the usual rate be R mph and the total distance be D. After the first hour, the normal time for the remaining distance is $\frac{D}{R} - 1$. Traveling it at $\frac{4R}{5}$ adds one-fourth of that time, so

$$\frac{1}{2} + \frac{1}{4}\left(\frac{D}{R} - 1\right) = 2.$$

If the accident occurs 80 miles later,

$$\frac{1}{2} + \frac{1}{4}\left(\frac{D}{R} - 1 - \frac{80}{R}\right) = 1.$$

Subtracting the second equation from the first gives $\frac{20}{R} = 1$, hence $R = 20$ mph.

Thus, the correct answer is **A**.

42. If $a, b,$ and c are positive integers, the radicals $\sqrt{a+\frac{b}{c}}$ and $a\sqrt{\frac{b}{c}}$ are equal when and only when:

A $a = b = c = 1$

B $a = b$ and $c = a = 1$

C $c = \dfrac{b(a^2-1)}{a}$

D $a = b$ and c is any value

E $a = b$ and $c = a - 1$

Solution:

Because all quantities are positive, squaring is reversible:

$$a+\frac{b}{c} = a^2\frac{b}{c}.$$

Multiplying by c gives $ac + b = a^2 b$, so

$$c = \frac{b(a^2-1)}{a}.$$

Thus, the correct answer is **C**.

43. The pairs of values of x and y that are the common solutions of the equations $y = (x+1)^2$ and $xy + y = 1$ are:

A 3 real pairs

B 4 real pairs

C 4 imaginary pairs

D 2 real and 2 imaginary pairs

E 1 real and 2 imaginary pairs

Solution:

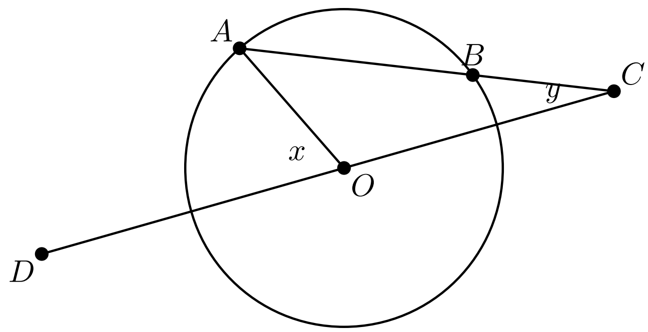
Substitution into $y(x+1) = 1$ gives

$$(x+1)^3 = 1.$$

The three cube roots of 1 consist of one real root and two nonreal roots. Each determines exactly one corresponding value $y = (x+1)^2$. Therefore there is one real pair and two imaginary pairs.

Thus, the correct answer is **E**.

44. In circle O chord $\overline{AB}$ is produced so that $\overline{BC}$ equals a radius of the circle. $\overline{CO}$ is drawn and extended to D. $\overline{AO}$ is drawn. Which of the following expresses the relationship between angles x and y?



A $x = 3y$

B $x = 2y$

C $x = 60^\circ$

D there is no special relationship between x and y

E $x = 2y$ or $x = 3y$, depending upon the length of $\overline{AB}$

Solution:

Because $OB = BC$, triangle OBC is isosceles, so $\angle BOC = \angle BCO = y$. Its exterior angle at B is therefore $\angle ABO = 2y$. Since $OA = OB$, triangle AOB is also isosceles and $\angle OAB = 2y$. In triangle AOC, the angles at A and C are $2y$ and y, so $\angle AOC = 180^\circ - 3y$. Angle $x = \angle AOD$ is supplementary to $\angle AOC$, hence $x = 3y$.

Thus, the correct answer is **A**.

45. Given a geometric sequence with the first term $\neq 0$ and $r \neq 0$ and an arithmetic sequence with the first term $= 0$. A third sequence $1, 1, 2, \ldots$ is formed by adding corresponding terms of the two given sequences. The sum of the first ten terms of the third sequence is:

A 978

B 557

C 467

D 1068

E not possible to determine from the information given

Solution:

The first three termwise sums give

$$\begin{aligned} a &= 1, \\ r + d &= 1, \\ r^2 + 2d &= 2. \end{aligned}$$

Substituting $d = 1 - r$ yields $r(r-2) = 0$. Since $r \neq 0$, we have $r = 2$ and $d = -1$. The first ten geometric terms sum to $2^{10} - 1 = 1023$, while the first ten arithmetic terms sum to $0 - 1 - \cdots - 9 = -45$. Their combined sum is $1023 - 45 = 978$.

Thus, the correct answer is **A**.

46. The graphs of $2x + 3y - 6 = 0, 4x - 3y - 6 = 0, x = 2$, and $y = \dfrac{2}{3}$ intersect in:

A 6 points

B 1 point

C 2 points

D no points

E an unlimited number of points

Solution:

Adding the first two equations gives $6x - 12 = 0$, so $x = 2$. Substitution gives $3y = 2$, hence $y = \frac{2}{3}$. This point also lies on the last two given lines. Therefore all four graphs have the single common point $(2, \frac{2}{3})$.

Thus, the correct answer is **B**.

47. The expressions $a + bc$ and $(a+b)(a+c)$ are:

A always equal

B never equal

C equal when $a + b + c = 1$

D equal when $a + b + c = 0$

E equal only when $a = b = c = 0$

Solution:

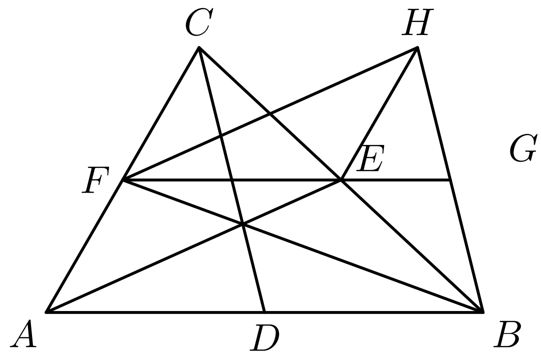
The difference is

$$\begin{aligned}(a+b)(a+c) - (a+bc) \\ = a^2 + ab + ac - a \\ = a(a+b+c-1).\end{aligned}$$

In particular, whenever $a + b + c = 1$, this difference is zero and the expressions are equal.

Thus, the correct answer is **C**.

48. Given triangle ABC with medians $\overline{AE}, \overline{BF}, \overline{CD}$; $\overline{FH}$ parallel and equal in length to $\overline{AE}$; $\overline{BH}$ and $\overline{HE}$ are drawn; $\overline{FE}$ extended meets $\overline{BH}$ in G. Which one of the following statements is not necessarily correct?



A $AEHF$ is a parallelogram

B $\overline{HE} = \overline{HG}$

C $\overline{BH} = \overline{DC}$

D $\overline{FG} = \frac{3}{4}\overline{AB}$

E $\overline{FG}$ is a median of triangle BFH

Solution:

Set $A = (0,0), B = (2,0), C = (u,v)$. Then

$$D = (1,0),$$
$$E = \left(\frac{u+2}{2}, \frac{v}{2}\right),$$
$$F = \left(\frac{u}{2}, \frac{v}{2}\right).$$

Since $\overrightarrow{FH} = \overrightarrow{AE}$, we get $H = (u+1, v)$. The horizontal line FE meets BH halfway from B to H, at $G = (\frac{u+3}{2}, \frac{v}{2})$. These coordinates verify that $AEHF$ is a parallelogram, $\overrightarrow{BH} = \overrightarrow{DC}$, $FG = \frac{3AB}{4}$, and G is the midpoint of BH, making FG a median. But

$$HE = \frac{1}{2}\sqrt{u^2 + v^2},$$
$$HG = \frac{1}{2}\sqrt{(u-1)^2 + v^2},$$

which are not generally equal.

Thus, statement **B** is not necessarily correct.

49. The graphs of $y = \dfrac{x^2 - 4}{x - 2}$ and $y = 2x$ intersect in:

- A one point whose abscissa is 2
- B one point whose abscissa is 0
- C no points
- D two distinct points
- E two identical points

Solution:

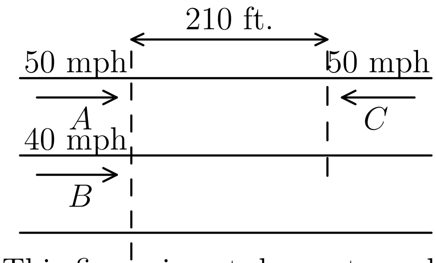
For $x \neq 2$,

$$\frac{x^2 - 4}{x - 2} = x + 2.$$

The lines $y = x + 2$ and $y = 2x$ would meet at $x = 2,\ y = 4$. But $x = 2$ is excluded from the original rational function, so that point is a hole and there is no intersection.

Thus, the correct answer is **C**.

50. In order to pass B going 40 mph on a two-lane highway A, going 50 mph, must gain 30 feet. Meantime, C, 210 feet from A, is headed toward him at 50 mph. If B and C maintain their speeds, then, in order to pass safely, A must increase his speed by:



This figure is not drawn to scale

A. 30 mph

B. 10 mph

C. 5 mph

D. 15 mph

E. 3 mph

Solution:

Let v be A's new speed. Relative to B, A gains at $v - 40$ mph, so the passing time is proportional to $\frac{30}{v-40}$. A and C close their 210-foot separation at $v + 50$ mph, so their meeting time is proportional to $\frac{210}{v+50}$. At the limiting safe time,

$$\frac{30}{v-40} = \frac{210}{v+50}.$$

Thus $30v + 1500 = 210v - 8400$, giving $v = 55$ mph. A must increase his original speed by 5 mph.

Therefore, the correct answer is **C**.

Problems: https://live.poshenloh.com/past-contests/amc12/1955