1954 AMC 12 Solutions

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1. The square of $5-\sqrt{y^2-25}$ is:

A $y^2-5\sqrt{y^2-25}$

B $-y^2$

C y^2

D $(5-y)^2$

E $y^2-10\sqrt{y^2-25}$

Solution:

Expanding gives

$$\begin{aligned}&\left(5-\sqrt{y^2-25}\right)^2\\ &\quad=25-10\sqrt{y^2-25}\\ &\qquad+y^2-25\\ &\quad=y^2-10\sqrt{y^2-25}.\end{aligned}$$

Thus, the correct answer is **E**.

2. The equation

$$\frac{2x^2}{x-1} - \frac{2x+7}{3} + \frac{4-6x}{x-1} + 1 = 0$$

can be transformed by eliminating fractions to the equation $x^2 - 5x + 4 = 0$. The roots of the latter equation are 4 and 1. Then the roots of the first equation are:

A 4 and 1

B only 1

C only 4

D neither 4 nor 1

E 4 and some other root

Solution:

The transformed quadratic factors as $(x-1)(x-4) = 0$. However, the original equation has denominator $x-1$, so it is undefined at $x = 1$. Substitution of $x = 4$ is valid and satisfies the original equation.

Thus, the only root is 4, and the correct answer is **C**.

3. If x varies as the cube of y, and y varies as the fifth root of z, then x varies as the nth power of z, where n is:

A $\frac{1}{15}$

B $\frac{5}{3}$

C $\frac{3}{5}$

D 15

E 8

Solution:

Since y is proportional to $z^{\frac{1}{5}}$, its cube is proportional to $z^{\frac{3}{5}}$. Because x is proportional to y^3,

$$x \propto z^{\frac{3}{5}}.$$

Hence $n = \frac{3}{5}$.

Thus, the correct answer is **C**.

4. If the Highest Common Divisor of 6432 and 132 is diminished by 8, it will equal:

- A -6
- B 6
- C -2
- D 3
- E 4

Solution:

The Euclidean algorithm gives

$$\gcd(6432, 132) = 12.$$

Diminishing this by 8 gives $12 - 8 = 4$.

Thus, the correct answer is **E**.

5. A regular hexagon is inscribed in a circle of radius 10 inches. Its area is:

A $150\sqrt{3}$ sq. in.

B 150 sq. in.

C $25\sqrt{3}$ sq. in.

D 600 sq. in.

E $300\sqrt{3}$ sq. in.

Solution:

The hexagon consists of six equilateral triangles of side 10. Its area is

$$6\left(\frac{\sqrt{3}}{4}\cdot 10^2\right) = 150\sqrt{3}$$

square inches.

Thus, the correct answer is **A**.

6. The value of $\left(\frac{1}{16}\right)^{a^0} + \left(\frac{1}{16a}\right)^0 - 64^{-\frac{1}{2}} - (-32)^{-\frac{4}{5}}$ is:

- A $1\frac{13}{16}$
- B $1\frac{3}{16}$
- C 1
- D $\frac{7}{8}$
- E $\frac{1}{16}$

Solution:

For permissible a, the four terms are

$$\frac{1}{16}, \qquad 1, \qquad \frac{1}{8}, \qquad \frac{1}{16}.$$

Therefore the expression is

$$\frac{1}{16} + 1 - \frac{1}{8} - \frac{1}{16} = \frac{7}{8}.$$

Thus, the correct answer is **D**.

7. A housewife saved $\$2.50$ in buying a dress on sale. If she spent $\$25$ for the dress, she saved about:

A 8%

B 9%

C 10%

D 11%

E 12%

Solution:

The original price was $\$27.50$. The fraction saved was

$$\frac{2.50}{27.50} = \frac{1}{11} \approx 0.0909,$$

or about 9%.

Thus, the correct answer is **B**.

8. The base of a triangle is twice as long as a side of a square and their areas are the same. Then the ratio of the altitude of the triangle to the side of the square is:

A $\frac{1}{4}$

B $\frac{1}{2}$

C 1

D 2

E 4

Solution:

If the square side is s, the triangle base is $2s$. Equal areas give

$$s^2 = \frac{1}{2}(2s)h = sh.$$

Thus $h = s$, so the requested ratio is 1.

The correct answer is **C**.

9. A point P is outside a circle and is 13 inches from the center. A secant from P cuts the circle at Q and R so that the external segment of the secant $\overline{PQ}$ is 9 inches and $\overline{QR}$ is 7 inches. The radius of the circle is:

- A 3 in.
- B 4 in.
- C 5 in.
- D 6 in.
- E 7 in.

Solution:

The power of P is

$$PQ \cdot PR = 9(16) = 144.$$

It is also $13^2 - r^2$, so $169 - r^2 = 144$. Hence $r^2 = 25$ and $r = 5$ inches.

Thus, the correct answer is **C**.

10. The sum of the numerical coefficients in the expansion of the binomial $(a+b)^6$ is:

A 32

B 16

C 64

D 48

E 7

Solution:

Setting $a=b=1$ makes every monomial equal to $1,$ so the value of the expansion equals the sum of its coefficients. Thus the sum is

$$(1+1)^6 = 64.$$

The correct answer is **C**.

11. A merchant placed on display some dresses, each with a marked price. He then posted a sign "$\frac{1}{3}$ off on these dresses." The cost of the dresses was $\frac{3}{4}$ of the price at which he actually sold them. Then the ratio of the cost to the marked price was:

A $\frac{1}{2}$

B $\frac{1}{3}$

C $\frac{1}{4}$

D $\frac{2}{3}$

E $\frac{3}{4}$

Solution:

If the marked price is M, then the selling price is $\frac{2M}{3}$. The cost is therefore

$$\frac{3}{4} \cdot \frac{2}{3} M = \frac{1}{2} M.$$

The cost-to-marked-price ratio is $\frac{1}{2}$.

Thus, the correct answer is **A**.

12. The solution of the equations

$$2x - 3y = 7,$$
$$4x - 6y = 20$$

is:

A $x = 18, y = 12$

B $x = 0, y = 0$

C There is no solution

D There are an unlimited number of solutions

E $x = 8, y = 5$

Solution:

The left side of the second equation is exactly twice the left side of the first, but its right side is not twice $7.$ The equations would require the same expression to equal both 14 and $20,$ which is impossible.

Thus, there is no solution, and the correct answer is **C**.

13. A quadrilateral is inscribed in a circle. If angles are inscribed in the four arcs cut off by the sides of the quadrilateral, without intersecting the quadrilateral, their sum will be:

- A 180°
- B 540°
- C 360°
- D 450°
- E 1080°

Solution:

Let the four arcs have measures a, b, c, d, whose sum is 360°. The angle placed in the arc of measure a intercepts the other three arcs, so it measures $\frac{360^\circ - a}{2}$, and similarly for the others. Their sum is

$$\frac{4(360^\circ) - (a+b+c+d)}{2}$$
$$= \frac{1080^\circ}{2}$$
$$= 540^\circ.$$

Thus, the correct answer is **B**.

14. When simplified $\sqrt{1+\left(\frac{x^4-1}{2x^2}\right)^2}$ equals:

A $\frac{x^4+2x^2-1}{2x^2}$

B $\frac{x^4-1}{2x^2}$

C $\frac{\sqrt{x^2+1}}{2}$

D $\frac{x^2}{\sqrt{2}}$

E $\frac{x^2}{2}+\frac{1}{2x^2}$

Solution:

For $x \neq 0,$

$$\begin{aligned} 1+\left(\frac{x^4-1}{2x^2}\right)^2 \\ &= \frac{4x^4+(x^4-1)^2}{4x^4} \\ &= \frac{(x^4+1)^2}{4x^4}. \end{aligned}$$

Because $x^2 > 0,$ its square root is

$$\frac{x^4+1}{2x^2} = \frac{x^2}{2} + \frac{1}{2x^2}.$$

Thus, the correct answer is **E**.

15. $\log 125$ equals:

A $100 \log 1.25$

B $5 \log 3$

C $3 \log 25$

D $3 - 3 \log 2$

E $(\log 25)(\log 5)$

Solution:

Using logarithm laws,

$$\begin{aligned} \log 125 &= \log\left(\frac{1000}{8}\right) \\ &= 3 - \log(2^3) \\ &= 3 - 3\log 2. \end{aligned}$$

Thus, the correct answer is **D**.

16. If $f(x) = 5x^2 - 2x - 1$, then $f(x+h) - f(x)$ equals:

A $5h^2 - 2h$

B $10xh - 4x + 2$

C $10xh - 2x - 2$

D $h(10x + 5h - 2)$

E $3h$

Solution:

Expanding and canceling gives

$$\begin{aligned} & f(x+h) - f(x) \\ &= 5(x+h)^2 - 2(x+h) - 1 \\ &\quad - (5x^2 - 2x - 1) \\ &= 10xh + 5h^2 - 2h \\ &= h(10x + 5h - 2). \end{aligned}$$

Thus, the correct answer is **D**.

17. The graph of the function $f(x) = 2x^3 - 7$ goes:

A up to the right and down to the left

B down to the right and up to the left

C up to the right and up to the left

D down to the right and down to the left

E none of these ways

Solution:

The leading term $2x^3$ controls the end behavior. As x tends to $+\infty$, $f(x)$ tends to $+\infty$; as x tends to $-\infty$, $f(x)$ tends to $-\infty$. Thus the graph goes up to the right and down to the left.

The correct answer is **A**.

18. Of the following sets, the one that includes all values of x which will satisfy $2x - 3 > 7 - x$ is:

A $x > 4$

B $x < \frac{10}{3}$

C $x = \frac{10}{3}$

D $x > \frac{10}{3}$

E $x < 0$

Solution:

Adding $x + 3$ to both sides gives

$$3x > 10,$$

so $x > \frac{10}{3}$.

Thus, the correct answer is **D**.

19. If the three points of contact of a circle inscribed in a triangle are joined, the angles of the resulting triangle:

A are always equal to 60°

B are always one obtuse angle and two unequal acute angles

C are always one obtuse angle and two equal acute angles

D are always acute angles

E are always unequal to each other

Solution:

The radii to two adjacent tangency points are perpendicular to the corresponding sides. The central angle between those radii is $180^\circ - A$. The relevant angle of the contact triangle is half the major intercepted arc, giving

$$90^\circ - \frac{A}{2}.$$

The other two angles are $90^\circ - \frac{B}{2}$ and $90^\circ - \frac{C}{2}$. Since every triangle angle lies strictly between 0° and 180°, all three are acute.

Thus, the correct answer is **D**.

20. The equation $x^3 + 6x^2 + 11x + 6 = 0$ has:

A no negative real roots

B no positive real roots

C no real roots

D 1 positive and 2 negative roots

E 2 positive and 1 negative root

Solution:

The polynomial factors as

$$\begin{aligned} &x^3 + 6x^2 + 11x + 6 \\ &= (x+1)(x+2)(x+3). \end{aligned}$$

Its roots are $-1, -2, -3,$ so it has no positive real roots.

Thus, the correct answer is **B**.

21. The roots of the equation $2\sqrt{x} + 2x^{-\frac{1}{2}} = 5$ can be found by solving:

A $16x^2 - 92x + 1 = 0$

B $4x^2 - 25x + 4 = 0$

C $4x^2 - 17x + 4 = 0$

D $2x^2 - 21x + 2 = 0$

E $4x^2 - 25x - 4 = 0$

Solution:

Multiplying by $\sqrt{x}$ gives

$$2x + 2 = 5\sqrt{x}.$$

Squaring and simplifying,

$$4x^2 + 8x + 4 = 25x,$$

so

$$4x^2 - 17x + 4 = 0.$$

Thus, the correct answer is **C**.

22. The expression $\frac{2x^2 - x}{(x+1)(x-2)} - \frac{4+x}{(x+1)(x-2)}$ cannot be evaluated for $x = -1$ or $x = 2$, since division by zero is not allowed. For other values of x :

- A The expression takes on many different values.
- B The expression has only the value $(2.)$
- C The expression has only the value $(1.)$
- D The expression always has a value between -1 and $2.$
- E The expression has a value greater than 2 or less than $-1.$

Solution:

Combining the numerators gives

$$\begin{aligned} &2x^2 - x - (4+x) \\ &= 2x^2 - 2x - 4 \\ &= 2(x+1)(x-2). \end{aligned}$$

For the allowed values $x \neq -1, 2,$ the factors cancel and the expression equals $2.$

Thus, the correct answer is **B**.

23. If the margin made on an article costing C dollars and selling for S dollars is $M = \frac{1}{n}C$, then the margin is given by:

A $M = \dfrac{1}{n-1}S$

B $M = \dfrac{1}{n}S$

C $M = \dfrac{n}{n+1}S$

D $M = \dfrac{1}{n+1}S$

E $M = \dfrac{n}{n-1}S$

Solution:

Since $S = C + M = C + \frac{C}{n}$,

$$S = \frac{n+1}{n}C,$$
$$C = \frac{n}{n+1}S.$$

Therefore

$$M = \frac{C}{n} = \frac{1}{n+1}S.$$

Thus, the correct answer is **D**.

24. The values of k for which the equation $2x^2 - kx + x + 8 = 0$ will have real and equal roots are:

A 9 and -7

B only -7

C 9 and 7

D -9 and -7

E only 9

Solution:

A repeated real root requires

$$(1-k)^2 - 4(2)(8) = 0.$$

Thus $(1-k)^2 = 64$, so $1 - k = \pm 8$. The two values are $k = 9$ and $k = -7$.

Thus, the correct answer is **A**.

25. The two roots of the equation $a(b-c)x^2+b(c-a)x+c(a-b)=0$ are 1 and:

A $\dfrac{b(c-a)}{a(b-c)}$

B $\dfrac{a(b-c)}{c(a-b)}$

C $\dfrac{a(b-c)}{b(c-a)}$

D $\dfrac{c(a-b)}{a(b-c)}$

E $\dfrac{c(a-b)}{b(c-a)}$

Solution:

By Vieta's formulas, the product of the two roots is

$$\frac{c(a-b)}{a(b-c)}.$$

Since one root is 1, the other root equals this product.

Thus, the correct answer is **D**.

26. The straight line AB is divided at C so that $\overline{AC} = 3\overline{CB}$. Circles are described on $\overline{AC}$ and $\overline{CB}$ as diameters and a common tangent meets $\overline{AB}$ produced at D. Then $\overline{BD}$ equals:

- A the diameter of the smaller circle
- B the radius of the smaller circle
- C the radius of the larger circle
- D $\overline{CB}\sqrt{3}$
- E the difference of the two radii

Solution:

Let $\overline{CB} = u$, so $\overline{AC} = 3u$ and $\overline{AB} = 4u$. Measured from A, the circle centers are at $\frac{3u}{2}$ and $\frac{7u}{2}$, and their radii are in the ratio $3 : 1$. If $AD = z$, the common external tangent makes D the external center of similitude, so

$$\frac{z - \frac{3}{2}u}{z - \frac{7}{2}u} = 3.$$

Hence $z = \frac{9u}{2}$, and

$$BD = z - 4u = \frac{u}{2},$$

which is the radius of the smaller circle.

Thus, the correct answer is **B**.

27. A right circular cone has for its base a circle having the same radius as a given sphere. The volume of the cone is one-half that of the sphere. The ratio of the altitude of the cone to the radius of its base is:

A $\frac{1}{1}$

B $\frac{1}{2}$

C $\frac{2}{3}$

D $\frac{2}{1}$

E $\sqrt{\frac{5}{4}}$

Solution:

The volume condition is

$$\frac{1}{3}\pi r^2 h = \frac{1}{2}\left(\frac{4}{3}\pi r^3\right).$$

Cancelling the common factors gives $h = 2r$, so the requested ratio is $2 : 1$.

Thus, the correct answer is **D**.

28. If $\dfrac{m}{n} = \dfrac{4}{3}$ and $\dfrac{r}{t} = \dfrac{9}{14}$, the value of $\dfrac{3mr - nt}{4nt - 7mr}$ is:

A $-5\frac{1}{2}$

B $-\dfrac{11}{14}$

C $-1\frac{1}{4}$

D $\dfrac{11}{14}$

E $-\dfrac{2}{3}$

Solution:

Multiplying the two given ratios,

$$\frac{mr}{nt} = \frac{4}{3} \cdot \frac{9}{14} = \frac{6}{7}.$$

Put $mr = 6k$ and $nt = 7k$. Then

$$\begin{aligned}\frac{3mr - nt}{4nt - 7mr} &= \frac{18k - 7k}{28k - 42k} \\ &= -\frac{11}{14}.\end{aligned}$$

Thus, the correct answer is **B**.

29. If the ratio of the legs of a right triangle is $1 : 2$, then the ratio of the corresponding segments of the hypotenuse made by a perpendicular upon it from the vertex is:

A $1:4$

B $1:\sqrt{2}$

C $1:2$

D $1:\sqrt{5}$

E $1:5$

Solution:

If the legs have lengths a and b, their projections on the hypotenuse have lengths $\frac{a^2}{c}$ and $\frac{b^2}{c}$. Their ratio is therefore $a^2 : b^2$. With $a : b = 1 : 2$, this is $1 : 4$.

Thus, the correct answer is **A**.

30. A and B together can do a job in 2 days; B and C can do it in four days; and A and C in $2\frac{2}{5}$ days. The number of days required for A to do the job alone is:

A 1

B 3

C 6

D 12

E 2.8

Solution:

Let a, b, c be the individual daily rates. Then

$$a + b = \frac{1}{2},$$
$$b + c = \frac{1}{4},$$
$$a + c = \frac{5}{12}.$$

Thus

$$\begin{aligned} 2a &= (a+b) + (a+c) \\ &\quad - (b+c) \\ &= \frac{1}{2} + \frac{5}{12} - \frac{1}{4} \\ &= \frac{2}{3}, \end{aligned}$$

so $a = \frac{1}{3}$. Therefore A needs 3 days.

Thus, the correct answer is **B**.

31. In triangle $ABC, \overline{AB} = \overline{AC}, \angle A = 40^{\circ}$. Point O is within the triangle with $\angle OBC \cong \angle OCA$. The number of degrees in angle BOC is:

A 110

B 35

C 140

D 55

E 70

Solution:

The base angles of the isosceles triangle are

$$\angle ABC = \angle BCA = 70^{\circ}.$$

Put $\angle OBC = \angle OCA = x$. Then $\angle OCB = 70^{\circ} - x$. In triangle BOC,

$$\begin{aligned} \angle BOC &= 180^{\circ} - x \\ &\quad - (70^{\circ} - x) \\ &= 110^{\circ}. \end{aligned}$$

Thus, the correct answer is **A**.

32. The factors of $x^4 + 64$ are:

A $(x^2 + 8)^2$

B $(x^2 + 8)(x^2 - 8)$

C $(x^2 + 2x + 4)(x^2 - 8x + 16)$

D $(x^2 - 4x + 8)(x^2 - 4x - 8)$

E $(x^2 - 4x + 8)(x^2 + 4x + 8)$

Solution:

Using a difference of squares,

$$\begin{aligned} x^4 + 64 &= (x^2 + 8)^2 - 16x^2 \\ &= (x^2 - 4x + 8) \\ &\quad \cdot (x^2 + 4x + 8). \end{aligned}$$

Thus, the correct answer is **E**.

33. A bank charges $\$6$ for a loan of $\$120.$ The borrower receives $\$114$ and repays the loan in 12 installments of $\$10$ a month. The interest rate is approximately:

A 5%

B 6%

C 7%

D 9%

E 15%

Solution:

The balance decreases in equal $\$10$ steps from $\$120$ to $\$10,$ so its average outstanding value is approximately

$$\frac{120+10}{2}=65.$$

The annual charge is therefore about

$$\frac{6}{65}\cdot 100\% \approx 9.2\%.$$

The nearest choice is $9\%.$

Thus, the correct answer is **D**.

34. The fraction $\frac{1}{3}$:

A equals 0.33333333

B is less than 0.33333333 by $\frac{1}{3 \cdot 10^8}$

C is less than 0.33333333 by $\frac{1}{3 \cdot 10^9}$

D is greater than 0.33333333 by $\frac{1}{3 \cdot 10^8}$

E is greater than 0.33333333 by $\frac{1}{3 \cdot 10^9}$

Solution:

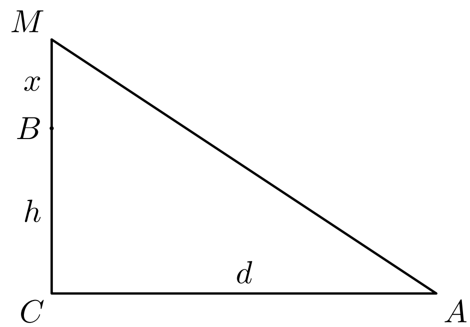
Exactly,

$$\begin{aligned} &\frac{1}{3} - \frac{33333333}{10^8} \\ = &\frac{10^8 - 3(33333333)}{3 \cdot 10^8} \\ = &\frac{1}{3 \cdot 10^8}. \end{aligned}$$

Thus $\frac{1}{3}$ is greater by the stated amount.

Thus, the correct answer is **D**.

35. In the right triangle shown the sum of the distances $\overline{BM}$ and $\overline{MA}$ is equal to the sum of the distances $\overline{BC}$ and $\overline{CA}$. If $\overline{MB} = x, \overline{CB} = h$, and $\overline{CA} = d$, then x equals:



A $\dfrac{hd}{2h+d}$

B $d - h$

C $\dfrac{1}{2}d$

D $h + d - \sqrt{2d}$

E $\sqrt{h^2 + d^2} - h$

Solution:

The condition and the Pythagorean theorem give

$$x + \sqrt{d^2 + (h+x)^2} = h + d.$$

After isolating the radical and squaring,

$$d^2 + (h+x)^2 = (h+d-x)^2.$$

Cancelling common terms leaves

$$4hx + 2dx = 2hd,$$

so $x = \frac{hd}{2h+d}$.

Thus, the correct answer is **A**.

36. A boat has a speed of 15 mph in still water. In a stream that has a current of 5 mph it travels a certain distance downstream and returns. The ratio of the average speed for the round trip to the speed in still water is:

A $\frac{5}{4}$

B $\frac{1}{1}$

C $\frac{8}{9}$

D $\frac{7}{8}$

E $\frac{9}{8}$

Solution:

For a one-way distance d, the total time is

$$\frac{d}{20} + \frac{d}{10} = \frac{3d}{20}.$$

Thus the round-trip average speed is

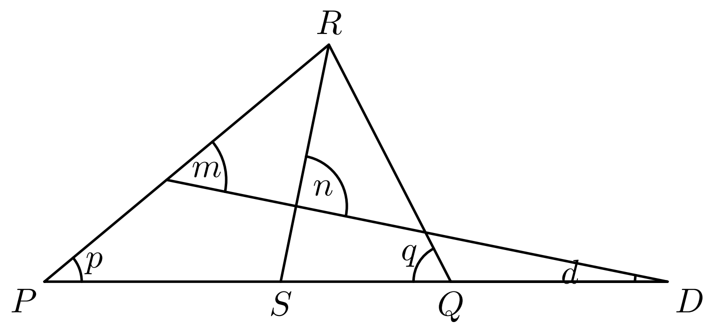
$$\frac{2d}{\frac{3d}{20}} = \frac{40}{3}.$$

Its ratio to 15 is

$$\frac{\frac{40}{3}}{15} = \frac{8}{9}.$$

Thus, the correct answer is **C**.

37. Given triangle PQR with $\overline{RS}$ bisecting $\angle R$, $\overline{PQ}$ extended to D, and $\angle n$ a right angle, then:



A $\angle m = \frac{1}{2}(\angle p - \angle q)$

B $\angle m = \frac{1}{2}(\angle p + \angle q)$

C $\angle d = \frac{1}{2}(\angle q + \angle p)$

D $\angle d = \frac{1}{2}\angle m$

E none of these is correct

Solution:

The angle at R is

$$180^\circ - \angle p - \angle q.$$

Because $\overline{RS}$ bisects this angle, the angle between $\overline{PR}$ and $\overline{RS}$ is

$$90^\circ - \frac{1}{2}(\angle p + \angle q).$$

The line forming $\angle m$ is perpendicular to $\overline{RS}$, so

$$\angle m = \frac{1}{2}(\angle p + \angle q).$$

Thus, the correct answer is **B**.

38. If $\log 2 = 0.3010$ and $\log 3 = 0.4771$, the value of x when $3^{x+3} = 135$ is approximately:

- A 5
- B 1.47
- C 1.67
- D 1.78
- E 1.63

Solution:

The equation reduces to $3^x = 5$, so

$$\begin{aligned} x &= \frac{\log 5}{\log 3} \\ &= \frac{1 - \log 2}{\log 3} \\ &= \frac{0.6990}{0.4771} \\ &\approx 1.47. \end{aligned}$$

Thus, the correct answer is **B**.

39. The locus of the midpoint of a line segment that is drawn from a given external point P to a given circle with center O and radius r, is:

A a straight line perpendicular to $\overline{PO}$

B a straight line parallel to $\overline{PO}$

C a circle with center P and radius r

D a circle with center at the midpoint of $\overline{PO}$ and radius $2r$

E a circle with center at the midpoint of $\overline{PO}$ and radius $\frac{1}{2}r$

Solution:

As the endpoint X moves on the circle centered at O, its midpoint M with the fixed point P is the image of X under a dilation of scale $\frac{1}{2}$ centered at P. Therefore the locus is a circle whose center is the midpoint of $\overline{PO}$ and whose radius is $\frac{r}{2}$.

Thus, the correct answer is **E**.

40. If $\left(a+\frac{1}{a}\right)^2=3,$ then $a^3+\frac{1}{a^3}$ equals:

A $\frac{10\sqrt{3}}{3}$

B $3\sqrt{3}$

C 0

D $7\sqrt{7}$

E $6\sqrt{3}$

Solution:

Let $t=a+\frac{1}{a}$. The standard cubic identity gives

$$a^3+\frac{1}{a^3}=t^3-3t=t(t^2-3).$$

Since $t^2=3,$ this expression is $0.$

Thus, the correct answer is **C**.

41. The sum of all the roots of $4x^3 - 8x^2 - 63x - 9 = 0$ is:

- A 8
- B 2
- C -8
- D -2
- E 0

Solution:

For $ax^3 + bx^2 + \cdots = 0$, the sum of the roots is $-\frac{b}{a}$. Here it is

$$-\frac{-8}{4} = 2.$$

Thus, the correct answer is **B**.

42. Consider the graphs of $(1)\ y = x^2 - \dfrac{1}{2}x + 2$ and $(2)\ y = x^2 + \dfrac{1}{2}x + 2$ on the same set of axes. These parabolas have exactly the same shape. Then:

- A the graphs coincide.
- B the graph of (1) is lower than the graph of (2).
- C the graph of (1) is to the left of the graph of (2).
- D the graph of (1) is to the right of the graph of (2).
- E the graph of (1) is higher than the graph of (2).

Solution:

The vertex of $y = x^2 + bx + 2$ has x-coordinate $-\frac{b}{2}$. Thus graph (1) has its vertex at $x = \frac{1}{4}$, while graph (2) has its vertex at $x = -\frac{1}{4}$. Their vertex heights are equal, so graph (1) is the same parabola shifted to the right.

Thus, the correct answer is **D**.

43. The hypotenuse of a right triangle is 10 inches and the radius of the inscribed circle is 1 inch. The perimeter of the triangle in inches is:

- A 15
- B 22
- C 24
- D 26
- E 30

Solution:

For a right triangle,

$$r = \frac{a+b-c}{2}.$$

With $r = 1$ and $c = 10$, this gives $a + b = 12$. Hence the perimeter is

$$a + b + c = 12 + 10 = 22.$$

Thus, the correct answer is **B**.

44. A man born in the first half of the nineteenth century was x years old in the year x^2. He was born in:

A 1849

B 1825

C 1812

D 1836

E 1806

Solution:

The year x^2 must lie in the nineteenth century, so $x = 43$ because $43^2 = 1849$ while the neighboring squares fall outside the relevant range. His birth year is therefore

$$x^2 - x = 1849 - 43 = 1806,$$

which is in the first half of the century.

Thus, the correct answer is **E**.

45. In a rhombus $ABCD,$ line segments are drawn within the rhombus, parallel to diagonal $\overline{BD},$ and terminated in the sides of the rhombus. A graph is drawn showing the length of a segment as a function of its distance from vertex $A.$ The graph is:

A a straight line passing through the origin.

B a straight line cutting across the upper right quadrant.

C two line segments forming an upright $(\mathrm{V}.)$

D two line segments forming an inverted $(\mathrm{V}.)$

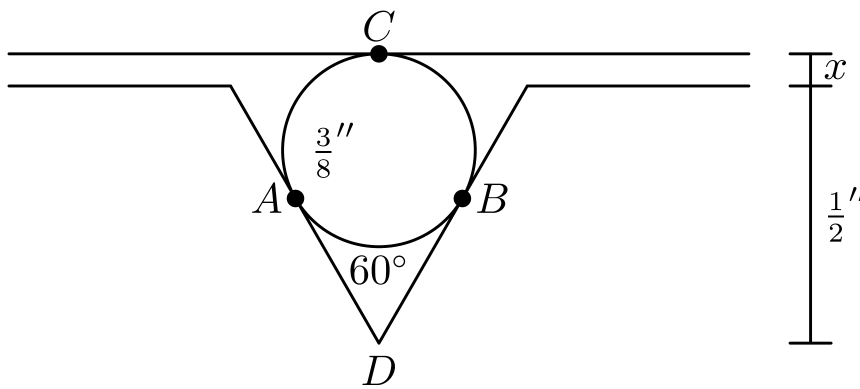
E none of these.

Solution:

Starting at $A,$ the parallel cross section has length 0 and grows linearly until it reaches the diagonal $\overline{BD}.$ Continuing toward the opposite vertex, its length decreases linearly back to $0.$ The graph is therefore made of two line segments forming an inverted $V.$

Thus, the correct answer is **D**.

46. In the diagram, if points $A, B,$ and C are points of tangency, then x equals:



A $\frac{3}{16}$ in.

B $\frac{1}{8}$ in.

C $\frac{1}{32}$ in.

D $\frac{3}{32}$ in.

E $\frac{1}{16}$ in.

Solution:

The radius is $\frac{3}{16}$ inch. The center lies on the angle bisector. The perpendicular radius to either sloping side and the segment from D to the center form a right triangle with a $30°$ angle, so the center is

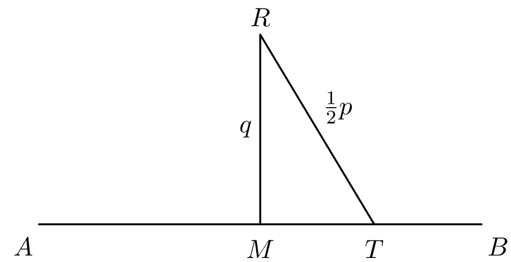
$$\frac{\frac{3}{16}}{\sin 30°} = \frac{3}{8}$$

inch above D. Hence the top tangent is $\frac{3}{8} + \frac{3}{16} = \frac{9}{16}$ inch above D. Since the ledge is $\frac{1}{2}$ inch above D,

$$x = \frac{9}{16} - \frac{1}{2} = \frac{1}{16} \text{ in.}$$

Thus, the correct answer is **E**.

47. At the midpoint of line segment $\overline{AB}$ which is p units long, a perpendicular $\overline{MR}$ is erected with length q units. An arc is described from R with a radius equal to $\frac{1}{2}\overline{AB}$, meeting $\overline{AB}$ at T. Then $\overline{AT}$ and $\overline{TB}$ are the roots of:



A $x^2 + px + q^2 = 0$

B $x^2 - px + q^2 = 0$

C $x^2 + px - q^2 = 0$

D $x^2 - px - q^2 = 0$

E $x^2 - px + q = 0$

Solution:

Let $MT = t$. Since $AM = BM = RT = \frac{p}{2}$, the Pythagorean theorem gives

$$t^2 + q^2 = \frac{p^2}{4}.$$

Also,

$$AT = \frac{p}{2} + t, \qquad TB = \frac{p}{2} - t.$$

Their sum is p, and their product is

$$\frac{p^2}{4} - t^2 = q^2.$$

Therefore they are the roots of $x^2 - px + q^2 = 0$.

Thus, the correct answer is **B**.

48. A train, an hour after starting, meets with an accident which detains it a half hour, after which it proceeds at $\dfrac{3}{4}$ of its former rate and arrives $3\dfrac{1}{2}$ hours late. Had the accident happened 90 miles farther along the line, it would have arrived only 3 hours late. The length of the trip in miles was:

A 400

B 465

C 600

D 640

E 550

Solution:

Let the normal speed be v and the trip length be L. Traveling a distance at $\frac{3v}{4}$ rather than v adds one-third of its normal travel time. In the first case,

$$\frac{1}{2}+\frac{1}{3}\left(\frac{L}{v}-1\right)=\frac{7}{2},$$

so $\frac{L}{v}=10$. If the accident happens 90 miles farther along,

$$\frac{1}{2}+\frac{1}{3}\left(\frac{L}{v}-1-\frac{90}{v}\right)=3.$$

Substituting $\frac{L}{v}=10$ gives $\frac{90}{v}=\frac{3}{2}$, hence $v=60$ and

$$L=10v=600.$$

Thus, the correct answer is **C**.

49. The difference of the squares of two odd numbers is always divisible by 8. If $a > b$, and $2a+1$ and $2b+1$ are the odd numbers, to prove the given statement we put the difference of the squares in the form:

A $(2a+1)^2-(2b+1)^2$

B $4a^2-4b^2+4a-4b$

C $4[a(a+1)-b(b+1)]$

D $4(a-b)(a+b+1)$

E $4(a^2+a-b^2-b)$

Solution:

Expanding and regrouping,

$$(2a+1)^2-(2b+1)^2$$
$$=4[a(a+1)-b(b+1)].$$

Each of $a(a+1)$ and $b(b+1)$ is even, so their difference is even. The displayed expression is consequently divisible by $4 \cdot 2 = 8$.

Thus, the correct answer is **C**.

50. The times between 7 and 8 o'clock, correct to the nearest minute, when the hands of a clock will form an angle of 84 degrees are:

A $7{:}23$ and $7{:}53$

B $7{:}20$ and $7{:}50$

C $7{:}22$ and $7{:}53$

D $7{:}23$ and $7{:}52$

E $7{:}21$ and $7{:}49$

Solution:

At t minutes after $7{:}00$, the hour hand is at $210^\circ + 0.5t^\circ$ and the minute hand is at $6t^\circ$. Thus

$$|210 - 5.5t| = 84.$$

The two solutions are

$$t = \frac{126}{5.5} \approx 22.91,$$
$$t = \frac{294}{5.5} \approx 53.45.$$

To the nearest minute, the times are $7{:}23$ and $7{:}53$.

Thus, the correct answer is **A**.

Problems: https://live.poshenloh.com/past-contests/amc12/1954