1953 AMC 12 Solutions

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1. A boy buys oranges at 3 for 10 cents. He will sell them at 5 for 20 cents. In order to make a profit of $\$1.00$, he must sell:

A 67 oranges

B 150 oranges

C 200 oranges

D an infinite number of oranges

E none of these

Solution:

Each orange costs $\frac{10}{3}$ cents and sells for 4 cents, so the profit is $4 - \frac{10}{3} = \frac{2}{3}$ cent per orange. To earn 100 cents, he must sell

$$\frac{100}{\frac{2}{3}} = 150$$

oranges.

Thus, the correct answer is **B**.

2. A refrigerator is offered for sale at $\$250.00$ less successive discounts of 20% and 15%. The sale price of the refrigerator is:

A 35% less than $\$250.00$

B 65% of $\$250.00$

C 77% of $\$250.00$

D 68% of $\$250.00$

E none of these

Solution:

The two discounts leave

$$(0.80)(0.85) = 0.68$$

of the original price. The sale price is therefore 68% of $\$250.00$, or $\$170.00$.

Thus, the correct answer is **D**.

3. The factors of the expression $x^2 + y^2$ are:

A $(x+y)(x-y)$

B $(x+y)^2$

C $(x^{\frac{2}{3}} + y^{\frac{2}{3}})(x^{\frac{4}{3}} + y^{\frac{4}{3}})$

D $(x+iy)(x-iy)$

E none of these

Solution:

Because $i^2 = -1$,

$$\begin{aligned}(x+iy)(x-iy) &= x^2 - (iy)^2 \\ &= x^2 + y^2.\end{aligned}$$

Thus, the correct answer is **D**.

4. The roots of $x(x^2+8x+16)(4-x)=0$ are:

A 0

B $0, 4$

C $0, 4, -4$

D $0, 4, -4, -4$

E none of these

Solution:

The equation factors as

$$x(x+4)^2(4-x)=0.$$

Its roots, counted with multiplicity, are $0, 4, -4, -4$.

Thus, the correct answer is **D**.

5. If $\log_6 x = 2.5$, the value of x is:

A 90

B 36

C $36\sqrt{6}$

D 0.5

E none of these

Solution:

The equation gives

$$x = 6^{2.5} = 6^2\sqrt{6} = 36\sqrt{6}.$$

Thus, the correct answer is **C**.

6. Charles has $5q + 1$ quarters and Richard has $q + 5$ quarters. The difference in their money in dimes is:

A $10(q - 1)$

B $\frac{2}{5}(4q - 4)$

C $\frac{2}{5}(q - 1)$

D $\frac{5}{2}(q - 1)$

E none of these

Solution:

The difference is

$$(5q + 1) - (q + 5) = 4q - 4$$

quarters. Multiplying by $\frac{5}{2}$ dimes per quarter gives

$$\frac{5}{2}(4q - 4) = 10(q - 1).$$

Thus, the correct answer is **A**.

7. The fraction

$$\frac{\sqrt{a^2+x^2}-\dfrac{x^2-a^2}{\sqrt{a^2+x^2}}}{a^2+x^2}$$

reduces to:

A 0

B $\dfrac{2a^2}{a^2+x^2}$

C $\dfrac{2x^2}{(a^2+x^2)^{\frac{3}{2}}}$

D $\dfrac{2a^2}{(a^2+x^2)^{\frac{3}{2}}}$

E $\dfrac{2x^2}{a^2+x^2}$

Solution:

The numerator is

$$\frac{a^2+x^2-(x^2-a^2)}{\sqrt{a^2+x^2}} = \frac{2a^2}{\sqrt{a^2+x^2}}.$$

Dividing by a^2+x^2 gives

$$\frac{2a^2}{(a^2+x^2)^{\frac{3}{2}}}.$$

Thus, the correct answer is **D**.

8. The value of x at the intersection of $y = \dfrac{8}{x^2+4}$ and $x + y = 2$ is:

A $-2+\sqrt{5}$

B $-2-\sqrt{5}$

C 0

D 2

E none of these

Solution:

Substitution gives

$$(2-x)(x^2+4) = 8.$$

Expanding and simplifying,

$$x(x^2-2x+4) = 0.$$

The quadratic factor has negative discriminant, so the real intersection has $x = 0$.

Thus, the correct answer is **C**.

9. The number of ounces of water needed to reduce 9 ounces of shaving lotion containing 50% alcohol to a lotion containing 30% alcohol is:

A 3

B 4

C 5

D 6

E 7

Solution:

The lotion initially contains $9(0.50) = 4.5$ ounces of alcohol. If w ounces of water are added, then

$$\frac{4.5}{9+w} = 0.30.$$

Hence $9 + w = 15$ and $w = 6$.

Thus, the correct answer is **D**.

10. The number of revolutions of a wheel, with fixed center and with an outside diameter of 6 feet, required to cause a point on the rim to go one mile is:

A 880

B $\frac{440}{\pi}$

C $\frac{880}{\pi}$

D 440π

E none of these

Solution:

The wheel's circumference is 6π feet, so the required number of revolutions is

$$\frac{5280}{6\pi} = \frac{880}{\pi}.$$

Thus, the correct answer is **C**.

11. A running track is the ring formed by two concentric circles. It is 10 feet wide. The circumferences of the two circles differ by about:

A 10 feet

B 30 feet

C 60 feet

D 100 feet

E none of these

Solution:

If the inner radius is r, the difference is

$$2\pi(r+10) - 2\pi r = 20\pi \approx 62.8$$

feet, which is about 60 feet.

Thus, the correct answer is **C**.

12. The diameters of two circles are 8 inches and 12 inches respectively. The ratio of the area of the smaller to the area of the larger circle is:

A $\frac{2}{3}$

B $\frac{4}{9}$

C $\frac{9}{4}$

D $\frac{1}{2}$

E none of these

Solution:

The area ratio is the square of the diameter ratio:

$$\left(\frac{8}{12}\right)^2 = \left(\frac{2}{3}\right)^2 = \frac{4}{9}.$$

Thus, the correct answer is **B**.

13. A triangle and a trapezoid are equal in area. They also have the same altitude. If the base of the triangle is 18 inches, the median of the trapezoid is:

A 36 inches

B 9 inches

C 18 inches

D not obtainable from these data

E none of these

Solution:

If the common altitude is h, the triangle has area

$$\frac{1}{2}(18)h = 9h.$$

A trapezoid's area is its median times its altitude, so its median must be 9 inches.

Thus, the correct answer is **B**.

14. Given the larger of two circles with center P and radius p and the smaller with center Q and radius q. Draw $\overline{PQ}$. Which of the following statements is false?

A $p - q$ can be equal to $\overline{PQ}$

B $p + q$ can be equal to $\overline{PQ}$

C $p + q$ can be less than $\overline{PQ}$

D $p - q$ can be less than $\overline{PQ}$

E none of these

Solution:

Internal tangency realizes $PQ = p - q$, and external tangency realizes $PQ = p + q$. Disjoint circles can have $PQ > p + q$, and many intersecting or disjoint configurations have $PQ > p - q$. Thus each of A through D can occur, so none of them is false.

Therefore, the correct answer is **E**.

15. A circular piece of metal of maximum size is cut out of a square piece and then a square piece of maximum size is cut out of the circular piece. The total amount of metal wasted is:

A $\frac{1}{4}$ the area of the original square

B $\frac{1}{2}$ the area of the original square

C $\frac{1}{2}$ the area of the circular piece

D $\frac{1}{4}$ the area of the circular piece

E none of these

Solution:

Let the original square have side s. The inscribed circle has diameter s. That diameter is the diagonal of the largest square cut from the circle, so the final square has side $\frac{s}{\sqrt{2}}$ and area $\frac{s^2}{2}$. The material left after both cuts is this final square, so the total waste is $s^2 - \frac{s^2}{2} = \frac{s^2}{2}$.

Thus, the correct answer is **B**.

16. Adams plans a profit of 10% on the selling price of an article and his expenses are 15% of sales. The rate of mark-up on an article that sells for $\$5.00$ is:

A 20%

B 25%

C 30%

D $33\frac{1}{3}\%$

E 35%

Solution:

Profit and expenses are $10\% + 15\% = 25\%$ of sales, so the cost is 75% of the selling price. The markup as a fraction of cost is

$$\frac{25\%}{75\%} = \frac{1}{3} = 33\frac{1}{3}\%.$$

Thus, the correct answer is **D**.

17. A man has part of $\$4500$ invested at 4% and the rest at 6%. If his annual return on each investment is the same, the average rate of interest which he realizes on the $\$4500$ is:

A 5%

B 4.8%

C 5.2%

D 4.6%

E none of these

Solution:

Let x dollars be invested at 4%. Equal returns give

$$0.04x = 0.06(4500 - x),$$

so $x = 2700$. Each investment returns $\$108$, for a total of $\$216$. The average rate is

$$\frac{216}{4500} = 0.048 = 4.8\%.$$

Thus, the correct answer is **B**.

18. One of the factors of $x^4 + 4$ is:

A $x^2 + 2$

B $x + 1$

C $x^2 - 2x + 2$

D $x^2 - 4$

E none of these

Solution:

Using a difference of squares,

$$\begin{aligned} x^4 + 4 &= (x^2 + 2)^2 - (2x)^2 \\ &= (x^2 - 2x + 2)(x^2 + 2x + 2). \end{aligned}$$

Thus, the correct answer is **C**.

19. In the expression xy^2, the values of x and y are each decreased 25%; the value of the expression is:

A decreased 50%

B decreased 75%

C decreased $\frac{37}{64}$ of its value

D decreased $\frac{27}{64}$ of its value

E none of these

Solution:

The new value is

$$\left(\frac{3}{4}x\right)\left(\frac{3}{4}y\right)^2 = \frac{27}{64}xy^2.$$

Therefore the decrease is $1 - \frac{27}{64} = \frac{37}{64}$ of the original value.

Thus, the correct answer is **C**.

20. If $y = x + \dfrac{1}{x}$, then $x^4 + x^3 - 4x^2 + x + 1 = 0$ becomes:

A $x^2(y^2 + y - 2) = 0$

B $x^2(y^2 + y - 3) = 0$

C $x^2(y^2 + y - 4) = 0$

D $x^2(y^2 + y - 6) = 0$

E none of these

Solution:

Because $x \neq 0$, factor the expression as

$$x^2\left[\left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) - 4\right].$$

Since $x^2 + x^{-2} = y^2 - 2$, this becomes

$$x^2(y^2 + y - 6) = 0.$$

Thus, the correct answer is **D**.

21. If $\log_{10}(x^2 - 3x + 6) = 1$, the value of x is:

A 10 or 2

B 4 or -2

C 3 or -1

D 4 or -1

E none of these

Solution:

The logarithmic equation is equivalent to

$$x^2 - 3x + 6 = 10,$$

so

$$\begin{aligned} x^2 - 3x - 4 &= (x - 4)(x + 1) \\ &= 0. \end{aligned}$$

Thus $x = 4$ or $x = -1$.

The correct answer is **D**.

22. The logarithm of $27\sqrt[4]{9}\sqrt[3]{9}$ to the base 3 is:

A $8\frac{1}{2}$

B $4\frac{1}{6}$

C 5

D 3

E none of these

Solution:

The printed expression is a product of two separate radicals:

$$27\sqrt[4]{9}\sqrt[3]{9} = 3^{3+\frac{1}{2}+\frac{2}{3}} = 3^{\frac{25}{6}}.$$

Its base-3 logarithm is $\frac{25}{6} = 4\frac{1}{6}$.

Thus, the correct answer is **B**.

23. The equation $\sqrt{x+10} - \dfrac{6}{\sqrt{x+10}} = 5$ has:

A an extraneous root between -5 and -1

B an extraneous root between -10 and -6

C a true root between 20 and 25

D two true roots

E two extraneous roots

Solution:

Multiplying by $\sqrt{x+10}$ gives

$$x + 4 = 5\sqrt{x+10}.$$

Squaring produces

$$\begin{aligned} &x^2 - 17x - 234 \\ &= (x - 26)(x + 9) \\ &= 0. \end{aligned}$$

The value $x = 26$ satisfies the original equation, while $x = -9$ makes its left side $1 - 6 = -5$, not 5. Thus -9, which lies between -10 and -6, is extraneous.

The correct answer is **B**.

24. If a, b and c are positive integers less than 10, then $(10a + b)(10a + c)$ equals $100a(a + 1) + bc$ if:

A $b + c = 10$

B $b = c$

C $a + b = 10$

D $a = b$

E $a + b + c = 10$

Solution:

Expanding and canceling $100a^2 + bc$ from both sides leaves

$$10a(b + c) = 100a.$$

Since a is positive, this is equivalent to $b + c = 10$.

Thus, the correct answer is **A**.

25. In a geometric progression whose terms are positive, any term is equal to the sum of the next two following terms. Then the common ratio is:

A 1

B about $\frac{\sqrt{5}}{2}$

C $\frac{\sqrt{5}-1}{2}$

D $\frac{1-\sqrt{5}}{2}$

E $\frac{2}{\sqrt{5}}$

Solution:

If a term is t, the next two are tr and tr^2. Thus

$$t = tr + tr^2$$

and $r^2 + r - 1 = 0$. Because the terms are positive,

$$r = \frac{\sqrt{5}-1}{2}.$$

Thus, the correct answer is **C**.

26. The base of a triangle is 15 inches. Two lines are drawn parallel to the base, terminating in the other two sides, and dividing the triangle into three equal areas. The length of the parallel closer to the base is:

A $5\sqrt{6}$ inches

B 10 inches

C $4\sqrt{3}$ inches

D 7.5 inches

E none of these

Solution:

The parallel closer to the base bounds a smaller triangle above it whose area is $\frac{2}{3}$ of the whole triangle. If its length is x, similarity gives

$$\frac{x^2}{15^2} = \frac{2}{3}.$$

Hence $x^2 = 150$ and $x = 5\sqrt{6}$.

Thus, the correct answer is **A**.

27. The radius of the first circle is 1 inch, that of the second $\frac{1}{2}$ inch, that of the third $\frac{1}{4}$ inch and so on indefinitely. The sum of the areas of the circles is:

A $\frac{3\pi}{4}$

B 1.3π

C 2π

D $\frac{4\pi}{3}$

E none of these

Solution:

The areas are

$$\pi, \frac{\pi}{4}, \frac{\pi}{16}, \ldots$$

Therefore their sum is

$$\frac{\pi}{1-\frac{1}{4}} = \frac{4\pi}{3}.$$

Thus, the correct answer is **D**.

28. In triangle ABC, sides a, b and c are opposite angles A, B and C respectively. $\overline{AD}$ bisects angle A and meets $\overline{BC}$ at D. Then if $x = CD$ and $y = BD$ the correct proportion is:

A $\quad \frac{x}{a} = \frac{a}{b+c}$

B $\quad \frac{x}{b} = \frac{a}{a+c}$

C $\quad \frac{y}{c} = \frac{c}{b+c}$

D $\quad \frac{y}{c} = \frac{a}{b+c}$

E $\quad \frac{x}{y} = \frac{c}{b}$

Solution:

The angle bisector theorem gives

$$\frac{y}{x} = \frac{c}{b},$$

or $\frac{y}{c} = \frac{x}{b}$. Since $x + y = a$,

$$\frac{y}{c} = \frac{x}{b} = \frac{x+y}{b+c} = \frac{a}{b+c}.$$

Thus, the correct answer is **D**.

29. The number of significant digits in the measurement of the side of a square whose computed area is 1.1025 square inches to the nearest ten-thousandth of a square inch is:

A 2

B 3

C 4

D 5

E 1

Solution:

The reported area means the true area lies in

$$1.10245 \leq A < 1.10255.$$

Taking square roots gives approximately

$$1.049976 \leq \sqrt{A} < 1.050024.$$

Every possible side length therefore rounds to 1.0500 to the nearest ten-thousandth. The trailing zeros after the decimal are significant, so 1.0500 has five significant digits.

Thus, the correct answer is **D**.

30. A house worth $\$9000$ is sold by Mr. A to Mr. B at a 10% loss. Mr. B sells the house back to Mr. A at a 10% gain. The result of the two transactions is:

A Mr. A breaks even

B Mr. B gains $\$900$

C Mr. A loses $\$900$

D Mr. A loses $\$810$

E Mr. B gains $\$1710$

Solution:

Mr. B first pays

$$9000(0.90) = 8100$$

dollars. He then sells the house back for $8100(1.10) = 8910$ dollars. Mr. A receives $\$8100$ and pays $\$8910$, so he loses $\$810$.

Thus, the correct answer is **D**.

31. The rails on a railroad are 30 feet long. As the train passes over the point where the rails are joined, there is an audible click. The speed of the train in miles per hour is approximately the number of clicks heard in:

A 20 seconds

B 2 minutes

C $1\frac{1}{2}$ minutes

D 5 minutes

E none of these

Solution:

A speed of v miles per hour is $\frac{22v}{15}$ feet per second. Since each click represents 30 feet, the click rate is

$$\frac{\frac{22v}{15}}{30} = \frac{11v}{225}$$

clicks per second. In $\frac{225}{11} \approx 20.45$ seconds, the number of clicks is v. The closest listed interval is 20 seconds.

Thus, the correct answer is **A**.

32. Each angle of a rectangle is trisected. The intersections of the pairs of trisectors adjacent to the same side always form:

A a square

B a rectangle

C a parallelogram with unequal sides

D a rhombus

E a quadrilateral with no special properties

Solution:

For each side, the two trisectors adjacent to it meet on that side's perpendicular bisector. Opposite such intersection points are reflections across the center of the rectangle, so the diagonals of the resulting quadrilateral bisect each other. One diagonal lies on the horizontal symmetry axis and the other on the vertical symmetry axis, so they are perpendicular.

A quadrilateral whose diagonals bisect each other is a parallelogram; if those diagonals are perpendicular, its four sides are equal. Hence the quadrilateral is a rhombus. It need not be a square because the two diagonals need not have equal length.

Thus, the correct answer is **D**.

33. The perimeter of an isosceles right triangle is $2p$. Its area is:

A $(2+\sqrt{2})p$

B $(2-\sqrt{2})p$

C $(3-2\sqrt{2})p^2$

D $(1-2\sqrt{2})p^2$

E $(3+2\sqrt{2})p^2$

Solution:

If each leg is s, then

$$s(2+\sqrt{2}) = 2p,$$

so $s = p(2-\sqrt{2})$. Therefore the area is

$$\begin{aligned}\frac{s^2}{2} &= \frac{p^2(2-\sqrt{2})^2}{2} \\ &= (3-2\sqrt{2})p^2.\end{aligned}$$

Thus, the correct answer is **C**.

34. If one side of a triangle is 12 inches and the opposite angle is 30 degrees, then the diameter of the circumscribed circle is:

A 18 inches

B 30 inches

C 24 inches

D 20 inches

E none of these

Solution:

By the extended law of sines, the circumdiameter is

$$d = \frac{12}{\sin 30^\circ} = \frac{12}{\frac{1}{2}} = 24$$

inches.

Thus, the correct answer is **C**.

35. If $f(x) = \dfrac{x(x-1)}{2}$, then $f(x+2)$ equals:

A $f(x) + f(2)$

B $(x+2)f(x)$

C $x(x+2)f(x)$

D $\dfrac{xf(x)}{x+2}$

E $\dfrac{(x+2)f(x+1)}{x}$

Solution:

Directly,

$$f(x+2) = \frac{(x+2)(x+1)}{2}.$$

Also $f(x+1) = \frac{x(x+1)}{2}$, so

$$\frac{(x+2)f(x+1)}{x} = \frac{(x+2)(x+1)}{2}.$$

Thus, the correct answer is **E**.

36. Determine m so that $4x^2 - 6x + m$ is divisible by $x - 3$. The obtained value, m, is an exact divisor of:

A 12

B 20

C 36

D 48

E 64

Solution:

Divisibility by $x - 3$ requires

$$4(3)^2 - 6(3) + m = 0,$$

so $18 + m = 0$ and $m = -18$. Of the listed numbers, only 36 is an exact multiple of -18.

Thus, the correct answer is **C**.

37. The base of an isosceles triangle is 6 inches and one of the equal sides is 12 inches. The radius of the circle through the vertices of the triangle is:

A $\frac{7\sqrt{15}}{5}$

B $4\sqrt{3}$

C $3\sqrt{5}$

D $6\sqrt{3}$

E none of these

Solution:

The altitude is

$$\sqrt{12^2 - 3^2} = 3\sqrt{15},$$

so the area is $K = \frac{1}{2}(6)(3\sqrt{15}) = 9\sqrt{15}$. Hence the circumradius is

$$R = \frac{(12)(12)(6)}{4(9\sqrt{15})} = \frac{8\sqrt{15}}{5}.$$

This value is not listed.

Thus, the correct answer is **E**.

38. If $f(a) = a - 2$ and $F(a, b) = b^2 + a$, then $F[3, f(4)]$ is:

A $a^2 - 4a + 7$

B 28

C 7

D 8

E 11

Solution:

First $f(4) = 4 - 2 = 2$. Therefore

$$\begin{aligned} F[3, f(4)] &= F(3, 2) \\ &= 2^2 + 3 = 7. \end{aligned}$$

Thus, the correct answer is **C**.

39. The product, $\log_a b \cdot \log_b a$ is equal to:

A 1

B a

C b

D ab

E none of these

Solution:

For permissible bases and arguments,

$$\begin{aligned}\log_a b \cdot \log_b a &= \frac{\log b}{\log a} \\ &\quad \cdot \frac{\log a}{\log b} \\ &= 1.\end{aligned}$$

Thus, the correct answer is **A**.

40. The negation of the statement “all men are honest,” is:

A no men are honest

B all men are dishonest

C some men are dishonest

D no men are dishonest

E some men are honest

Solution:

The negation of “every man is honest” is “there exists a man who is not honest.” In the language of the choices, some men are dishonest.

Thus, the correct answer is **C**.

41. A girls' camp is located 300 rods from a straight road. On this road, a boys' camp is located 500 rods from the girls' camp. It is desired to build a canteen on the road which shall be exactly the same distance from each camp. The distance of the canteen from each of the camps is:

A 400 rods

B 250 rods

C 87.5 rods

D 200 rods

E none of these

Solution:

Let the foot of the perpendicular from the girls' camp be $(0, 0)$. The camps can be placed at $G = (0, 300)$ and $B = (400, 0)$. If the canteen is $C = (t, 0)$, equidistance gives

$$t^2 + 300^2 = (400 - t)^2,$$

so $t = 87.5$. The common distance is

$$BC = 400 - 87.5 = 312.5$$

rods, which is not listed.

Thus, the correct answer is **E**.

42. The centers of two circles are 41 inches apart. The smaller circle has a radius of 4 inches and the larger one has a radius of 5 inches. The length of the common internal tangent is:

A 41 inches

B 39 inches

C 39.8 inches

D 40.1 inches

E 40 inches

Solution:

The center segment, the tangent segment, and a perpendicular leg of length $4 + 5 = 9$ form a right triangle. Thus the tangent length is

$$\sqrt{41^2 - 9^2} = \sqrt{1600} = 40$$

inches.

Thus, the correct answer is **E**.

43. If the price of an article is increased by per cent p, then the decrease in per cent of sales must not exceed d in order to yield the same income. The value of d is:

A $\dfrac{1}{1+p}$

B $\dfrac{1}{1-p}$

C $\dfrac{p}{1+p}$

D $\dfrac{p}{p-1}$

E $\dfrac{1-p}{1+p}$

Solution:

Writing the percentage rates as fractions of the original quantities, equal income requires

$$(1+p)(1-d)=1.$$

Hence $p-d-pd=0$, so

$$d=\frac{p}{1+p}.$$

Thus, the correct answer is **C**.

44. In solving a problem that reduces to a quadratic equation one student makes a mistake only in the constant term of the equation and obtains 8 and 2 for the roots. Another student makes a mistake only in the coefficient of the first degree term and finds -9 and -1 for the roots. The correct equation was:

A $x^2 - 10x + 9 = 0$

B $x^2 + 10x + 9 = 0$

C $x^2 - 10x + 16 = 0$

D $x^2 - 8x - 9 = 0$

E none of these

Solution:

The first student changed only the constant term, so the correct coefficient of x is the one in the monic quadratic with roots $8, 2$: it is $-(8+2) = -10.$ The second student changed only that linear coefficient, so the correct constant is $(-9)(-1) = 9.$ Therefore the correct equation is

$$x^2 - 10x + 9 = 0.$$

Thus, the correct answer is **A**.

45. The lengths of two line segments are a units and b units respectively. Then the correct relation between them is:

A $\frac{a+b}{2} > \sqrt{ab}$

B $\frac{a+b}{2} < \sqrt{ab}$

C $\frac{a+b}{2} = \sqrt{ab}$

D $\frac{a+b}{2} \leq \sqrt{ab}$

E $\frac{a+b}{2} \geq \sqrt{ab}$

Solution:

Since

$$(a-b)^2 \geq 0,$$

we have $(a+b)^2 \geq 4ab$. Both sides are nonnegative, so

$$\frac{a+b}{2} \geq \sqrt{ab}.$$

Equality occurs when $a = b$.

Thus, the correct answer is **E**.

46. Instead of walking along two adjacent sides of a rectangular field, a boy took a short-cut along the diagonal of the field and saved a distance equal to $\frac{1}{2}$ the longer side. The ratio of the shorter side of the rectangle to the longer side was:

A $\frac{1}{2}$

B $\frac{2}{5}$

C $\frac{1}{4}$

D $\frac{3}{4}$

E $\frac{2}{5}$

Solution:

The saving condition gives

$$\sqrt{L^2 + W^2} = \frac{L}{2} + W.$$

Squaring and canceling W^2 yields

$$L^2 = \frac{L^2}{4} + LW,$$

so $\frac{W}{L} = \frac{3}{4}$.

Thus, the correct answer is **D**.

47. If x is greater than zero, then the correct relationship is:

A $\log(1+x) = \dfrac{x}{1+x}$

B $\log(1+x) < \dfrac{x}{1+x}$

C $\log(1+x) > x$

D $\log(1+x) < x$

E none of these

Solution:

For $x > 0$, the standard exponential inequality gives $1 + x < e^x$. Taking natural logarithms yields

$$\ln(1+x) < x.$$

If $\log$ denotes the common logarithm, then

$$\begin{aligned}\log(1+x) &= \frac{\ln(1+x)}{\ln 10} \\ &< \ln(1+x),\end{aligned}$$

so the same listed inequality holds.

Thus, the correct answer is **D**.

48. If the larger base of an isosceles trapezoid equals a diagonal and the smaller base equals the altitude, then the ratio of the smaller base to the larger base is:

A $\frac{1}{2}$

B $\frac{2}{3}$

C $\frac{3}{4}$

D $\frac{3}{5}$

E $\frac{2}{5}$

Solution:

Let the larger base be 1, and let the smaller base and altitude both be r. A diagonal has horizontal projection $\frac{1+r}{2}$ and length 1. Thus

$$1 = r^2 + \left(\frac{1+r}{2}\right)^2.$$

This simplifies to

$$\begin{aligned} 5r^2 + 2r - 3 &= (5r-3)(r+1) \\ &= 0. \end{aligned}$$

The positive ratio is $r = \frac{3}{5}$.

Thus, the correct answer is **D**.

49. The coordinates of A, B and C are $(5,5), (2,1)$ and $(0,k)$ respectively. The value of k that makes $\overline{AC} + \overline{BC}$ as small as possible is:

A 3

B $4\frac{1}{2}$

C $3\frac{6}{7}$

D $4\frac{5}{6}$

E $2\frac{1}{7}$

Solution:

Reflect $B = (2,1)$ across the y-axis to $B' = (-2,1)$. For C on the y-axis, $BC = B'C$, so $AC + BC$ is minimized when A, C, B' are collinear. The line from $A = (5,5)$ to $B' = (-2,1)$ has slope $\frac{4}{7}$. At $x = 0$, its height is

$$5 - \frac{4}{7}(5) = \frac{15}{7} = 2\frac{1}{7}.$$

Hence $k = \frac{15}{7}$.

Thus, the correct answer is **E**.

50. One of the sides of a triangle is divided into segments of 6 and 8 units by the point of tangency of the inscribed circle. If the radius of the circle is 4, then the length of the shortest side of the triangle is:

A 12 units

B 13 units

C 14 units

D 15 units

E 16 units

Solution:

Let the two tangent segments from the third vertex each have length z. Equal tangents from a common vertex make the side lengths

$$14, \qquad 6+z, \qquad 8+z,$$

with semiperimeter $s = 14 + z$. Since the inradius is 4,

$$K = rs = 4(14+z).$$

Heron's formula gives

$$\begin{aligned} K^2 &= (14+z)z \cdot 8 \cdot 6 \\ &= 48z(14+z). \end{aligned}$$

Equating the squares yields $16(14+z) = 48z$, so $z = 7$. The sides are $14, 13, 15$, and the shortest is 13.

Thus, the correct answer is **B**.

Problems: https://live.poshenloh.com/past-contests/amc12/1953