1952 AMC 12 Solutions

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1. If the radius of a circle is a rational number, its area is given by a number which is:

A Rational

B Irrational

C Integral

D A perfect square

E None of these

Solution:

If the positive rational radius is r, then its area is πr^2. The factor r^2 is a nonzero rational number, and a nonzero rational multiple of the irrational number π is irrational.

Thus, the correct answer is **B**.

2. Two high school classes took the same test. One class of 20 students made an average grade of 80%; the other class of 30 students made an average grade of 70%. The average grade for all students in both classes is:

A 75%

B 74%

C 72%

D 77%

E None of these

Solution:

The two classes earned a combined total of

$$20(80) + 30(70) = 3700$$

percentage points among 50 students. Their combined average is $\frac{3700}{50} = 74\%$.

Thus, the correct answer is **B**.

3. The expression $a^3 - a^{-3}$ equals:

A $\left(a - \frac{1}{a}\right)\left(a^2 + 1 + \frac{1}{a^2}\right)$

B $\left(\frac{1}{a} - a\right)\left(a^2 - 1 + \frac{1}{a^2}\right)$

C $\left(a - \frac{1}{a}\right)\left(a^2 - 2 + \frac{1}{a^2}\right)$

D $\left(\frac{1}{a} - a\right)\left(\frac{1}{a^2} + 1 + a^2\right)$

E None of these

Solution:

Apply the difference-of-cubes identity with $u = a$ and $v = \frac{1}{a}$. Since $uv = 1$,

$$a^3 - \frac{1}{a^3} = \left(a - \frac{1}{a}\right) \cdot \left(a^2 + 1 + \frac{1}{a^2}\right).$$

Thus, the correct answer is **A**.

4. The cost C of sending a parcel post package weighing P pounds, P an integer, is 10 cents for the first pound and 3 cents for each additional pound. The formula for the cost is:

A $C = 10 + 3P$

B $C = 10P + 3$

C $C = 10 + 3(P - 1)$

D $C = 9 + 3P$

E $C = 10P - 7$

Solution:

The first pound costs 10 cents. The remaining $P - 1$ pounds cost $3(P - 1)$ cents, so

$$C = 10 + 3(P - 1).$$

Thus, the correct answer is **C**.

5. The points $(6, 12)$ and $(0, -6)$ are connected by a straight line. Another point on this line is:

A $(3, 3)$

B $(2, 1)$

C $(7, 16)$

D $(-1, -4)$

E $(-3, -8)$

Solution:

The slope is

$$\frac{12 - (-6)}{6 - 0} = 3,$$

so the line is $y = 3x - 6$. Substituting $x = 3$ gives $y = 3$.

Thus, the correct answer is **A**.

6. The difference of the roots of $x^2 - 7x - 9 = 0$ is:

A $+7$

B $+\frac{7}{2}$

C $+9$

D $2\sqrt{85}$

E $\sqrt{85}$

Solution:

The roots are $\frac{7+\sqrt{49+36}}{2}$ and $\frac{7-\sqrt{49+36}}{2}$. Their positive difference is therefore $\sqrt{85}$.

Thus, the correct answer is **E**.

7. When simplified, $(x^{-1}+y^{-1})^{-1}$ is equal to:

A $x+y$

B $\dfrac{xy}{x+y}$

C xy

D $\dfrac{1}{xy}$

E $\dfrac{x+y}{xy}$

Solution:

Combining the terms and then taking the reciprocal gives

$$\begin{aligned}\left(\frac{1}{x}+\frac{1}{y}\right)^{-1} &= \left(\frac{x+y}{xy}\right)^{-1}\\ &= \frac{xy}{x+y}.\end{aligned}$$

Thus, the correct answer is **B**.

8. Two equal circles in the same plane cannot have the following number of common tangents:

A 1

B 2

C 3

D 4

E None of these

Solution:

Two distinct equal circles have 2 common tangents when they intersect, 3 when externally tangent, and 4 when disjoint. They cannot be internally tangent, because internal tangency requires the center distance to equal the difference of the radii, which is 0. At center distance 0 the equal circles coincide and have infinitely many common tangents.

Thus exactly 1 common tangent is impossible, so the correct answer is **A**.

9. If $m = \dfrac{cab}{a-b}$, then b equals:

A $\dfrac{m(a-b)}{ca}$

B $\dfrac{cab-ma}{-m}$

C $\dfrac{1}{1+c}$

D $\dfrac{ma}{m+ca}$

E $\dfrac{m+ca}{ma}$

Solution:

Clearing the denominator gives

$$m(a-b) = cab.$$

Hence $ma = b(m+ca)$, so

$$b = \frac{ma}{m+ca}.$$

Thus, the correct answer is **D**.

10. An automobile went up a hill at a speed of 10 miles an hour and down the same distance at a speed of 20 miles an hour. The average speed for the round trip was:

A $12\frac{1}{2}$ mph

B $13\frac{1}{3}$ mph

C $14\frac{1}{2}$ mph

D 15 mph

E None of these

Solution:

For one mile in each direction, the total time is

$$\frac{1}{10}+\frac{1}{20}=\frac{3}{20}$$

hour. Thus the average speed is

$$\frac{2}{\frac{3}{20}}=\frac{40}{3}=13\frac{1}{3}\text{ mph.}$$

Thus, the correct answer is **B**.

11. If $y = f(x) = \dfrac{x+2}{x-1}$, then it is incorrect to say:

A $x = \dfrac{y+2}{y-1}$

B $f(0) = -2$

C $f(1) = 0$

D $f(-2) = 0$

E $f(y) = x$

Solution:

The function is undefined at $x = 1$ because its denominator is zero there, so $f(1) = 0$ is false. The other statements hold: $f(0) = -2$, $f(-2) = 0$, solving $y = \frac{x+2}{x-1}$ gives $x = \frac{y+2}{y-1}$, and this same formula shows that f is its own inverse.

Thus, the correct answer is **C**.

12. The sum to infinity of the terms of an infinite geometric progression is 6. The sum of the first two terms is $4\frac{1}{2}$. The first term of the progression is:

- A 3 or $1\frac{1}{2}$
- B 1
- C $2\frac{1}{2}$
- D 6
- E 9 or 3

Solution:

Let the first term be a and the common ratio be r. The infinite sum gives $a = 6(1-r)$. The first two terms therefore satisfy

$$a(1+r) = 6(1-r^2) = \frac{9}{2}.$$

Thus $r^2 = \frac{1}{4}$, so $r = \frac{1}{2}$ gives $a = 3$, while $r = -\frac{1}{2}$ gives $a = 9$.

Thus, the correct answer is **E**.

13. The function $x^2 + px + q$ with p and q greater than zero has its minimum value when:

A $x = -p$

B $x = \frac{p}{2}$

C $x = -2p$

D $x = \frac{p^2}{4q}$

E $x = -\frac{p}{2}$

Solution:

Completing the square gives

$$x^2 + px + q = \left(x + \frac{p}{2}\right)^2 + q - \frac{p^2}{4}.$$

The square is least when $x = -\frac{p}{2}$.

Thus, the correct answer is **E**.

14. A house and store were sold for $\$12000$ each. The house was sold at a loss of 20% of the cost, and the store at a gain of 20% of the cost. The entire transaction resulted in:

A No loss or gain

B Loss of $\$1000$

C Gain of $\$1000$

D Gain of $\$2000$

E None of these

Solution:

The house cost $\frac{12000}{0.8} = 15000$ dollars, and the store cost $\frac{12000}{1.2} = 10000$ dollars. The combined cost was 25000 dollars, while the combined selling price was 24000 dollars, producing a loss of 1000 dollars.

Thus, the correct answer is **B**.

15. The sides of a triangle are in the ratio $6 : 8 : 9$. Then:

A The triangle is obtuse

B The angles are in the ratio $6 : 8 : 9$

C The triangle is acute

D The angle opposite the largest side is double the angle opposite the smallest side

E None of these

Solution:

For a triangle with largest side 9,

$$9^2 = 81 < 6^2 + 8^2 = 100.$$

By the converse of the Pythagorean inequality, the angle opposite side 9 is acute. Since it is the largest angle, every angle is acute.

Thus, the correct answer is **C**.

16. If the base of a rectangle is increased by 10% and the area is unchanged, then the altitude is decreased by:

A 9%

B 10%

C 11%

D $11\frac{1}{9}\%$

E $9\frac{1}{11}\%$

Solution:

To keep the area fixed, the altitude is multiplied by $\frac{10}{11}$. Its fractional decrease is

$$1-\frac{10}{11}=\frac{1}{11},$$

which is $\frac{100}{11}\%=9\frac{1}{11}\%$.

Thus, the correct answer is **E**.

17. A merchant bought some goods at a discount of 20% of the list price. He wants to mark them at such a price that he can give a discount of 20% of the marked price and still make a profit of 20% of the selling price. The percent of the list price at which he should mark them is:

- A 20
- B 100
- C 125
- D 80
- E 120

Solution:

Let the list price be 1. The merchant's cost is 0.8. If the selling price is S, then $S - 0.8 = 0.2S$, so $S = 1$. If the marked price is M, the 20% discount gives $0.8M = S = 1$, hence $M = 1.25$.

The marked price must therefore be 125% of the list price, so the correct answer is **C**.

18. $\log p + \log q = \log(p+q)$ only if:

A $p = q = 0$

B $p = \dfrac{q^2}{1-q}$

C $p = q = 1$

D $p = \dfrac{q}{q-1}$

E $p = \dfrac{q}{q+1}$

Solution:

The logarithm product rule changes the equation to $\log(pq) = \log(p+q)$, so the positive arguments must satisfy

$$pq = p + q.$$

Therefore $p(q-1) = q$, giving $p = \frac{q}{q-1}$. Its admissible values have $q > 1$, which also keeps all logarithm arguments positive.

Thus, the correct answer is **D**.

19. Angle B of triangle ABC is trisected by BD and BE, which meet AC at D and E, respectively. Then:

A $\frac{AD}{EC} = \frac{AE}{DC}$

B $\frac{AD}{EC} = \frac{AB}{BC}$

C $\frac{AD}{EC} = \frac{BD}{BE}$

D $\frac{AD}{EC} = \frac{AB \cdot BD}{BE \cdot BC}$

E $\frac{AD}{EC} = \frac{AE \cdot BD}{DC \cdot BE}$

Solution:

Triangles ABD and EBC have bases AD and EC on the same line AC, so their common altitude from B gives

$$\frac{[ABD]}{[EBC]} = \frac{AD}{EC}.$$

Also, $\angle ABD = \angle EBC$ because both are one-third of angle B. Using two sides and the included angle for the same area ratio gives

$$\frac{[ABD]}{[EBC]} = \frac{AB \cdot BD}{BE \cdot BC}.$$

Thus, the correct answer is **D**.

20. If $\dfrac{x}{y} = \dfrac{3}{4}$, then the incorrect expression in the following is:

A $\dfrac{x+y}{y} = \dfrac{7}{4}$

B $\dfrac{y}{y-x} = \dfrac{4}{1}$

C $\dfrac{x+2y}{x} = \dfrac{11}{3}$

D $\dfrac{x}{2y} = \dfrac{3}{8}$

E $\dfrac{x-y}{y} = \dfrac{1}{4}$

Solution:

Set $x = 3k$ and $y = 4k$. The first four expressions become $\frac{7}{4}, 4, \frac{11}{3}$, and $\frac{3}{8}$, respectively. But

$$\frac{x-y}{y} = \frac{3k-4k}{4k} = -\frac{1}{4},$$

not $\frac{1}{4}$.

Thus, the correct answer is **E**.

21. The sides of a regular polygon of n sides, $n > 4$, are extended to form a star. The number of degrees at each point of the star is:

A $\frac{360}{n}$

B $\frac{(n-4)180}{n}$

C $\frac{(n-2)180}{n}$

D $180 - \frac{90}{n}$

E $\frac{180}{n}$

Solution:

Each exterior angle of a regular n-gon is $\frac{360}{n}$ degrees. The two sides forming a point of the star, together with the intervening polygon side, make a triangle whose two base angles are those exterior angles. Hence its point angle is

$$180 - 2\left(\frac{360}{n}\right) = \frac{(n-4)180}{n}.$$

Thus, the correct answer is **B**.

22. On hypotenuse AB of a right triangle ABC a second right triangle ABD is constructed with hypotenuse AB. If $BC = 1$, $AC = b$, and $AD = 2$, then BD equals:

A $\sqrt{b^2+1}$

B $\sqrt{b^2-3}$

C $\sqrt{b^2+1}+2$

D b^2+5

E $\sqrt{b^2+3}$

Solution:

In the first right triangle,

$$AB^2 = AC^2 + BC^2 = b^2 + 1.$$

In the second right triangle, AB is the hypotenuse and $AD = 2$, so

$$\begin{aligned} BD^2 &= AB^2 - AD^2 \\ &= b^2 + 1 - 4 = b^2 - 3. \end{aligned}$$

Therefore $BD = \sqrt{b^2-3}$.

Thus, the correct answer is **B**.

23. If $\dfrac{x^2 - bx}{ax - c} = \dfrac{m-1}{m+1}$ has roots which are numerically equal but of opposite signs, the value of m must be:

A $\dfrac{a-b}{a+b}$

B $\dfrac{a+b}{a-b}$

C c

D $\dfrac{1}{c}$

E 1

Solution:

Cross-multiplication gives

$$\begin{aligned}(m+1)x^2 &- (m+1)bx \\ &- (m-1)ax \\ &\quad + (m-1)c = 0.\end{aligned}$$

Roots that are equal in magnitude and opposite in sign have sum 0, so the coefficient of x is 0. Thus

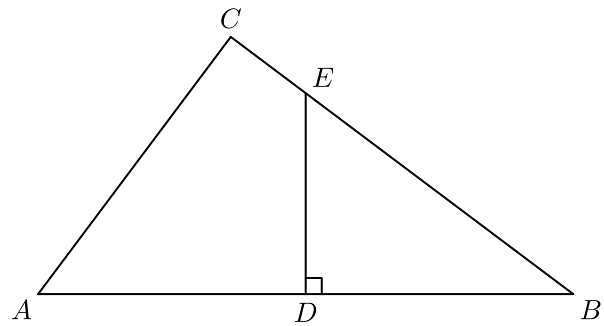
$$m(a+b) + (b-a) = 0,$$

and

$$m = \frac{a-b}{a+b}.$$

Thus, the correct answer is **A**.

24. In the figure, it is given that angle $C = 90°$, $AD = DB$, $DE \perp AB$, $AB = 20$, and $AC = 12$. The area of quadrilateral $ADEC$ is:



A 75

B $58\frac{1}{2}$

C 48

D $37\frac{1}{2}$

E None of these

Solution:

Since $AB = 20$, $AC = 12$, and angle C is right, $BC = 16$. Put $A = (0,0)$ and $B = (20,0)$. Solving

$$x_C^2 + y_C^2 = 144,$$
$$(x_C - 20)^2 + y_C^2 = 256$$

gives $C = (\frac{36}{5}, \frac{48}{5})$. Also $D = (10,0)$.

The line CB meets $x = 10$ at $E = (10, \frac{15}{2})$. The shoelace formula for A, D, E, C now gives

$$
\begin{aligned}
[ADEC] &= \frac{1}{2}\left(10 \cdot \frac{15}{2} + 10 \cdot \frac{48}{5}\right. \\
&\qquad \left. - \frac{15}{2} \cdot \frac{36}{5}\right) \\
&= \frac{117}{2} = 58\frac{1}{2}.
\end{aligned}
$$

Thus, the correct answer is **B**.

25. A powderman set a fuse for a blast to take place in 30 seconds. He ran away at a rate of 8 yards per second. Sound travels at the rate of 1080 feet per second. When the powderman heard the blast, he had run approximately:

A 200 yd.

B 352 yd.

C 300 yd.

D 245 yd.

E 512 yd.

Solution:

Let T be the total time in seconds. The powderman is $8T$ yards, or $24T$ feet, from the blast. Because the sound starts after 30 seconds,

$$1080(T-30) = 24T.$$

Thus $T = \frac{1350}{44}$ seconds, and the distance he ran is

$$8T = \frac{2700}{11} \approx 245.45 \text{ yards.}$$

Approximately 245 yards is the listed value, so the correct answer is **D**.

26. If $\left(r+\frac{1}{r}\right)^2=3$, then $r^3+\frac{1}{r^3}$ equals

A 1

B 2

C 0

D 3

E 6

Solution:

Let $u=r+\frac{1}{r}$. Expanding u^3 gives

$$r^3+\frac{1}{r^3}=u^3-3u.$$

Since $u^2=3$, this becomes $u(u^2-3)=0$.

Thus the value is 0, so the correct answer is **C**.

27. The ratio of the perimeter of an equilateral triangle having an altitude equal to the radius of a circle, to the perimeter of an equilateral triangle inscribed in the circle is:

A $1:2$

B $1:3$

C $1:\sqrt{3}$

D $\sqrt{3}:2$

E $2:3$

Solution:

An equilateral triangle with side s has altitude $\frac{s\sqrt{3}}{2}$. Thus the first triangle has side $\frac{2R}{\sqrt{3}}$ and perimeter $2\sqrt{3}R$. An equilateral triangle inscribed in a circle of radius R has side $\sqrt{3}R$ and perimeter $3\sqrt{3}R$.

The perimeter ratio is therefore $2:3$, so the correct answer is **E**.

28. In the table shown, the formula relating x and y is:

x	1	2	3	4	5
y	3	7	13	21	31

A $y = 4x - 1$

B $y = x^3 - x^2 + x + 2$

C $y = x^2 + x + 1$

D $y = (x^2 + x + 1)(x - 1)$

E None of these

Solution:

The successive differences of the y-values are $4, 6, 8, 10,$ whose successive differences are all $2.$ Thus the data fit a monic quadratic. Substitution shows

$$y = x^2 + x + 1$$

gives $3, 7, 13, 21, 31$ at $x = 1, 2, 3, 4, 5,$ respectively.

Thus, the correct answer is **C**.

29. In a circle of radius 5 units, CD and AB are perpendicular diameters. A chord CH cutting AB at K is 8 units long. The diameter AB is divided into two segments whose dimensions are:

A $1.25, 8.75$

B $2.75, 7.25$

C $2, 8$

D $4, 6$

E None of these

Solution:

Put the circle at the origin with $C = (0, 5)$ and let $H = (x, y)$. The equations

$$\begin{aligned} x^2 + y^2 &= 25, \\ x^2 + (y-5)^2 &= 64 \end{aligned}$$

give $y = -\frac{7}{5}$ and $x = \pm\frac{24}{5}$. Taking $x = \frac{24}{5}$ does not change the segment lengths. The line from $C = (0, 5)$ to $H = (\frac{24}{5}, -\frac{7}{5})$ crosses $y = 0$ at $x = \frac{15}{4}$. Its distances to the endpoints $(-5, 0)$ and $(5, 0)$ of diameter AB are therefore

$$\begin{aligned} 5 + \frac{15}{4} &= \frac{35}{4} = 8.75, \\ 5 - \frac{15}{4} &= \frac{5}{4} = 1.25. \end{aligned}$$

Thus, the correct answer is **A**.

30. When the sum of the first ten terms of an arithmetic progression is four times the sum of the first five terms, the ratio of the first term to the common difference is:

A $1:2$

B $2:1$

C $1:4$

D $4:1$

E $1:1$

Solution:

The sum formula gives

$$S_{10} = 5(2a + 9d),$$
$$S_5 = \frac{5}{2}(2a + 4d).$$

The equation $S_{10} = 4S_5$ becomes

$$2a + 9d = 2(2a + 4d),$$

so $d = 2a$. Hence $a : d = 1 : 2$.

Thus, the correct answer is **A**.

31. Given 12 points in a plane no three of which are collinear, the number of lines they determine is:

- A 24
- B 54
- C 120
- D 66
- E None of these

Solution:

Every pair of points determines one line, and no line is counted by more than one pair because no three points are collinear. Therefore the number of lines is

$$\binom{12}{2} = \frac{12 \cdot 11}{2} = 66.$$

Thus, the correct answer is **D**.

32. K takes 30 minutes less time than M to travel a distance of 30 miles. K travels $\dfrac{1}{3}$ mile per hour faster than M. If x is K's rate of speed in miles per hour, then K's time for the distance is:

A $\dfrac{x+\frac{1}{3}}{30}$

B $\dfrac{x-\frac{1}{3}}{30}$

C $\dfrac{30}{x+\frac{1}{3}}$

D $\dfrac{30}{x}$

E $\dfrac{x}{30}$

Solution:

For K, the distance is 30 miles and the rate is x miles per hour. Thus

$$\text{time} = \frac{\text{distance}}{\text{rate}} = \frac{30}{x} \text{ hours.}$$

The comparison with M is not needed for this expression.

Thus, the correct answer is **D**.

33. A circle and a square have the same perimeter. Then:

A Their areas are equal

B The area of the circle is the greater

C The area of the square is the greater

D The area of the circle is π times the area of the square

E None of these

Solution:

With common perimeter P, the circle's radius is $\frac{P}{2\pi}$, so its area is $\frac{P^2}{4\pi}$. The square's side is $\frac{P}{4}$, so its area is $\frac{P^2}{16}$. Since $4\pi < 16,$

$$\frac{P^2}{4\pi} > \frac{P^2}{16}.$$

The circle has the greater area, so the correct answer is **B**.

34. The price of an article was increased $p\%$. Later the new price was decreased $p\%$. If the last price was one dollar, the original price was:

A $\dfrac{1-p^2}{200}$

B $\dfrac{\sqrt{1-p^2}}{100}$

C One dollar

D $1-\dfrac{p^2}{10{,}000-p^2}$

E $\dfrac{10{,}000}{10{,}000-p^2}$

Solution:

If the original price is P dollars, then after both changes its price is

$$\begin{aligned} &P\left(1+\frac{p}{100}\right)\left(1-\frac{p}{100}\right) \\ &=P\left(1-\frac{p^2}{10{,}000}\right). \end{aligned}$$

Setting this equal to 1 gives

$$P=\frac{10{,}000}{10{,}000-p^2}.$$

Thus, the correct answer is **E**.

35. With a rational denominator, the expression $\frac{\sqrt{2}}{\sqrt{2}+\sqrt{3}-\sqrt{5}}$ is equivalent to:

A $\frac{3+\sqrt{6}+\sqrt{15}}{6}$

B $\frac{\sqrt{6}-2+\sqrt{10}}{6}$

C $\frac{2+\sqrt{6}+\sqrt{10}}{10}$

D $\frac{2+\sqrt{6}-\sqrt{10}}{6}$

E None of these

Solution:

Multiply first by the conjugate $\sqrt{2}+\sqrt{3}+\sqrt{5}$. This gives

$$\frac{\sqrt{2}(\sqrt{2}+\sqrt{3}+\sqrt{5})}{(\sqrt{2}+\sqrt{3})^2-5}$$
$$=\frac{2+\sqrt{6}+\sqrt{10}}{2\sqrt{6}}.$$

Multiplying numerator and denominator by $\sqrt{6}$ yields

$$\frac{2\sqrt{6}+6+\sqrt{60}}{12}$$
$$=\frac{3+\sqrt{6}+\sqrt{15}}{6}.$$

Thus, the correct answer is **A**.

36. To be continuous at $x = -1$, the value of $\frac{x^3+1}{x^2-1}$ is taken to be:

A -2

B 0

C $\frac{3}{2}$

D ∞

E $-\frac{3}{2}$

Solution:

For $x \neq -1$,

$$\frac{x^3+1}{x^2-1} = \frac{(x+1)(x^2-x+1)}{(x+1)(x-1)}$$
$$= \frac{x^2-x+1}{x-1}.$$

Its limit at $x = -1$ is

$$\frac{1+1+1}{-2} = -\frac{3}{2}.$$

Assigning this value removes the discontinuity.

Thus, the correct answer is **E**.

37. Two equal parallel chords are drawn 8 inches apart in a circle of radius 8 inches. The area of that part of the circle that lies between the chords is:

A $21\frac{1}{3}\pi - 32\sqrt{3}$

B $32\sqrt{3} + 21\frac{1}{3}\pi$

C $32\sqrt{3} + 42\frac{2}{3}\pi$

D $16\sqrt{3} + 42\frac{2}{3}\pi$

E $42\frac{2}{3}\pi$

Solution:

The equal chords are symmetrically 4 inches from the center. For either chord, the half-angle at the center satisfies $\cos\theta = \frac{4}{8} = \frac{1}{2}$, so $\theta = 60^\circ$. Each outer segment is a 120° sector minus the isosceles triangle:

$$\frac{120}{360}\pi(8)^2 - \frac{1}{2}(8)^2\sin 120^\circ$$
$$= \frac{64\pi}{3} - 16\sqrt{3}.$$

The area between the chords is therefore

$$64\pi - 2\left(\frac{64\pi}{3} - 16\sqrt{3}\right)$$
$$= \frac{64\pi}{3} + 32\sqrt{3}$$
$$= 21\frac{1}{3}\pi + 32\sqrt{3}.$$

Thus, the correct answer is **B**.

38. The area of a trapezoidal field is 1400 square yards. Its altitude is 50 yards. Find the two bases, if the number of yards in each base is an integer divisible by 8. The number of solutions to this problem is:

- A None
- B One
- C Two
- D Three
- E More than three

Solution:

If the bases are b_1, b_2, then

$$1400 = \frac{1}{2}(b_1 + b_2)(50),$$

so $b_1 + b_2 = 56$. The unordered positive pairs of multiples of 8 are

$$(8, 48), \quad (16, 40), \quad (24, 32).$$

There are three solutions.

Thus, the correct answer is **D**.

39. If the perimeter of a rectangle is p and its diagonal is d, the difference between the length and width of the rectangle is:

A $\dfrac{\sqrt{8d^2 - p^2}}{2}$

B $\dfrac{\sqrt{8d^2 + p^2}}{2}$

C $\dfrac{\sqrt{6d^2 - p^2}}{2}$

D $\dfrac{\sqrt{6d^2 + p^2}}{2}$

E $\dfrac{\sqrt{8d^2 - p^2}}{4}$

Solution:

The perimeter and diagonal give

$$\begin{aligned} L + W &= \frac{p}{2}, \\ L^2 + W^2 &= d^2. \end{aligned}$$

Hence

$$\begin{aligned} (L - W)^2 &= 2(L^2 + W^2) \\ &\quad - (L + W)^2 \\ &= 2d^2 - \frac{p^2}{4} \\ &= \frac{8d^2 - p^2}{4}. \end{aligned}$$

Taking the nonnegative square root gives

$$L - W = \frac{\sqrt{8d^2 - p^2}}{2}.$$

Thus, the correct answer is **A**.

40. In order to draw a graph of $f(x) = ax^2 + bx + c$, a table of values was constructed. These values of the function for a set of equally spaced increasing values of x were $3844, 3969, 4096, 4227, 4356, 4489, 4624,$ and 4761. The one which is incorrect is:

A 4096

B 4356

C 4489

D 4761

E None of these

Solution:

The values are intended to be the consecutive squares

$$62^2, 63^2, 64^2, 65^2,$$
$$66^2, 67^2, 68^2, 69^2.$$

These are $3844, 3969, 4096, 4225, 4356, 4489, 4624, 4761$. Thus the incorrect table entry is 4227, but it is not one of choices A-D.

Consequently none of the four listed numbers is incorrect, so the correct answer is **E**.

41. Increasing the radius of a cylinder by 6 units increases the volume by y cubic units. Increasing the altitude of the cylinder by 6 units also increases the volume by y cubic units. If the original altitude is $2,$ then the original radius is:

- A 2
- B 4
- C 6
- D 6π
- E 8

Solution:

With original height 2 and radius r, increasing the radius adds

$$2\pi\left((r+6)^2 - r^2\right) = 24\pi r + 72\pi$$

cubic units. Increasing the height by 6 adds $6\pi r^2$ cubic units. Equating these gives

$$6r^2 = 24r + 72,$$

or $r^2 - 4r - 12 = 0$. Its positive root is $r = 6$.

Thus, the correct answer is **C**.

42. Let D represent a repeating decimal. If P denotes the r figures of D which do not repeat themselves, and Q denotes the s figures which do repeat themselves, then the incorrect expression is:

A $D = 0.PQQQ\ldots$

B $10^r D = P.QQQ\ldots$

C $10^{r+s} D = PQ.QQQ\ldots$

D $10^r(10^s - 1)D = Q(P - 1)$

E $10^r \cdot 10^{2s} D = PQQ.QQQ\ldots$

Solution:

After shifting past the nonrepeating block,

$$10^r D = P.QQQ\ldots,$$

and after shifting one repeating block farther,

$$10^{r+s} D = PQ.QQQ\ldots.$$

Subtracting yields the standard relation

$$\begin{aligned} &10^r(10^s - 1)D \\ &= P(10^s - 1) + Q. \end{aligned}$$

This is not $Q(P-1)$, as stated in choice D.

Thus, the incorrect expression, and hence the correct answer, is **D**.

43. The diameter of a circle is divided into n equal parts. On each part a semicircle is constructed. As n becomes very large, the sum of the lengths of the arcs of the semicircles approaches a length:

A Equal to the semi-circumference of the original circle

B Equal to the diameter of the original circle

C Greater than the diameter but less than the semi-circumference of the original circle

D That is infinite

E Greater than the semi-circumference but finite

Solution:

Each small semicircle has diameter $\frac{d}{n}$, so its arc length is $\frac{\pi d}{2n}$. The sum of all n arc lengths is

$$n \cdot \frac{\pi d}{2n} = \frac{\pi d}{2},$$

exactly the semi-circumference of the original circle. This equality holds for every positive n, not merely in the limit.

Thus, the correct answer is **A**.

44. If an integer of two digits is k times the sum of its digits, the number formed by interchanging the digits is the sum of the digits multiplied by:

A $(9-k)$

B $(10-k)$

C $(11-k)$

D $(k-1)$

E $(k+1)$

Solution:

Let the original number be $10a+b=k(a+b)$. The reversed number is $10b+a$. Since

$$\begin{aligned}(10a+b)+(10b+a)\\ =11(a+b),\end{aligned}$$

the reversed number equals

$$\begin{aligned}11(a+b)-k(a+b)\\ =(11-k)(a+b).\end{aligned}$$

Thus, the correct answer is **C**.

45. If a and b are two unequal positive numbers, then:

A $\dfrac{2ab}{a+b} > \sqrt{ab} > \dfrac{a+b}{2}$

B $\sqrt{ab} > \dfrac{2ab}{a+b} > \dfrac{a+b}{2}$

C $\dfrac{2ab}{a+b} > \dfrac{a+b}{2} > \sqrt{ab}$

D $\dfrac{a+b}{2} > \dfrac{2ab}{a+b} > \sqrt{ab}$

E $\dfrac{a+b}{2} > \sqrt{ab} > \dfrac{2ab}{a+b}$

Solution:

The arithmetic-geometric mean inequality gives

$$\frac{a+b}{2} > \sqrt{ab}$$

for unequal positive a, b. Applying the same inequality to $\frac{1}{a}$ and $\frac{1}{b}$ gives

$$\sqrt{ab} > \frac{2ab}{a+b}.$$

Therefore the decreasing order is arithmetic mean, geometric mean, harmonic mean.

Thus, the correct answer is **E**.

46. The base of a new rectangle equals the sum of the diagonal and the greater side of a given rectangle, while the altitude of the new rectangle equals the difference of the diagonal and the greater side of the given rectangle. The area of the new rectangle is:

- A Greater than the area of the given rectangle
- B Equal to the area of the given rectangle
- C Equal to the area of a square with its side equal to the smaller side of the given rectangle
- D Equal to the area of a square with its side equal to the greater side of the given rectangle
- E Equal to the area of a rectangle whose dimensions are the diagonal and shorter side of the given rectangle

Solution:

Let the greater and smaller side lengths be L and W, and let the diagonal be d. The new rectangle's area is

$$(d + L)(d - L) = d^2 - L^2.$$

The Pythagorean theorem gives $d^2 = L^2 + W^2$, so the new area is W^2. This is the area of a square whose side equals the smaller side of the original rectangle.

Thus, the correct answer is **C**.

47. In the set of equations $z^x = y^{2x}, 2^z = 2 \cdot 4^x, x + y + z = 16,$ the integral roots in the order x, y, z are:

- A $3, 4, 9$
- B $9, -5, 12$
- C $12, -5, 9$
- D $4, 3, 9$
- E $4, 9, 3$

Solution:

For the intended positive integral solution, the first equation gives $z = y^2$. The second equation is

$$2^z = 2^{2x+1},$$

so $z = 2x + 1$ and $x = \frac{y^2-1}{2}$. Substituting into $x + y + z = 16$ gives

$$\frac{y^2 - 1}{2} + y + y^2 = 16.$$

The positive integral solution is $y = 3$, which gives $x = 4$ and $z = 9$. Direct substitution verifies all three displayed equations.

Thus, the intended listed answer is **D**.

48. Two cyclists, k miles apart, and starting at the same time, would be together in r hours if they traveled in the same direction, but would pass each other in t hours if they traveled in opposite directions. The ratio of the speed of the faster cyclist to that of the slower is:

A $\frac{r+t}{r-t}$

B $\frac{r}{r-t}$

C $\frac{r+t}{r}$

D $\frac{r}{t}$

E $\frac{r+k}{t-k}$

Solution:

Let the speeds be $u > v$. The two meeting conditions give

$$u - v = \frac{k}{r}, \qquad u + v = \frac{k}{t}.$$

Adding and subtracting these equations yields

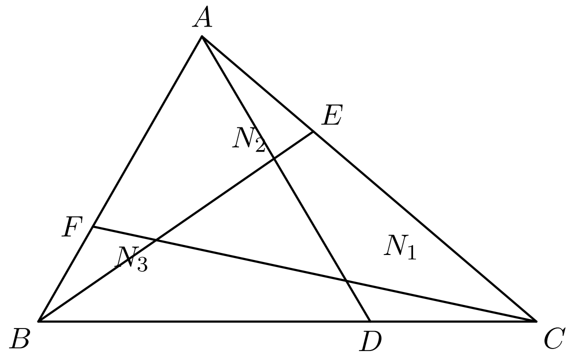
$$u = \frac{k}{2}\left(\frac{1}{r} + \frac{1}{t}\right),$$
$$v = \frac{k}{2}\left(\frac{1}{t} - \frac{1}{r}\right).$$

Therefore

$$\frac{u}{v} = \frac{r+t}{r-t}.$$

Thus, the correct answer is **A**.

49. In the figure, $CD, AE,$ and BF are one-third of their respective sides. It follows that $AN_2 : N_2N_1 : N_1D = 3 : 3 : 1$, and similarly for lines BE and CF. Then the area of triangle $N_1N_2N_3$ is:



A $\frac{1}{10}\triangle ABC$

B $\frac{1}{9}\triangle ABC$

C $\frac{1}{7}\triangle ABC$

D $\frac{1}{6}\triangle ABC$

E None of these

Solution:

Use $A = (0,1), B = (0,0), C = (1,0)$. Then

$$D = \left(\frac{2}{3}, 0\right),$$
$$E = \left(\frac{1}{3}, \frac{2}{3}\right),$$
$$F = \left(0, \frac{1}{3}\right).$$

Intersecting AD, BE, CF in pairs gives

$$N_1 = \left(\frac{4}{7}, \frac{1}{7}\right),$$
$$N_2 = \left(\frac{2}{7}, \frac{4}{7}\right),$$
$$N_3 = \left(\frac{1}{7}, \frac{2}{7}\right).$$

The determinant area formula gives

$$[N_1N_2N_3] = \frac{1}{14},$$
$$[ABC] = \frac{1}{2}.$$

Hence $\frac{[N_1N_2N_3]}{[ABC]} = \frac{1}{7}$.

Thus, the correct answer is **C**.

50. A line initially 1 inch long grows according to the following law, where the first term is the initial length.

$$1+\frac{1}{4}\sqrt{2}+\frac{1}{4}+\frac{1}{16}\sqrt{2}$$
$$+\frac{1}{16}+\frac{1}{64}\sqrt{2}+\frac{1}{64}+\cdots.$$

If the growth process continues forever, the limit of the length of the line is:

A ∞

B $\frac{4}{3}$

C $\frac{3}{8}$

D $\frac{1}{3}(4+\sqrt{2})$

E $\frac{2}{3}(4+\sqrt{2})$

Solution:

After the initial term, each power 4^{-k} appears once by itself and once multiplied by $\sqrt{2}$. Since $\sum_{k=1}^{\infty} 4^{-k}=\frac{1}{3}$, the limit is

$$1+\frac{1+\sqrt{2}}{3}=\frac{4+\sqrt{2}}{3}.$$

Thus, the correct answer is **D**.

Problems: https://live.poshenloh.com/past-contests/amc12/1952